Symmetry in Szekeres Models

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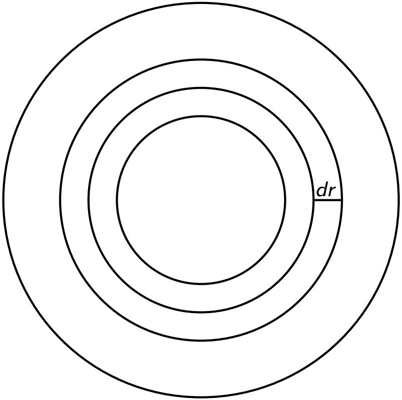
05 July 2017

LT metric

$$ds^2 = -dt^2 + \frac{R'^2}{1+f} dr^2 + R^2 d\Omega^2$$



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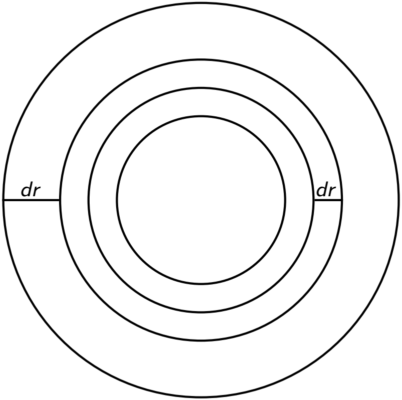


- radial proper distance $\frac{R'(r,t)}{\sqrt{1+f(r)}}dr$
- depends on r, t
- This creates the inhomogeneity
- Shell-crossing

Szekeres metric

LT-metric:

$$ds^2 = -dt^2 + \frac{R'^2}{1+f}dr^2 + R^2 d\Omega^2$$

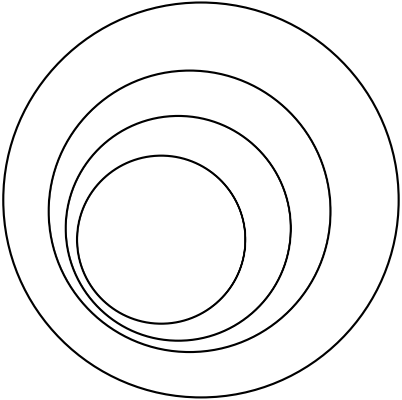
Szekeres metric:

$$ds^2 = -dt^2 + \frac{(R' - R\mathcal{E}'/\mathcal{E})^2}{\epsilon + f}dr^2 + \frac{R^2}{\mathcal{E}^2}(dp^2 + dq^2)$$

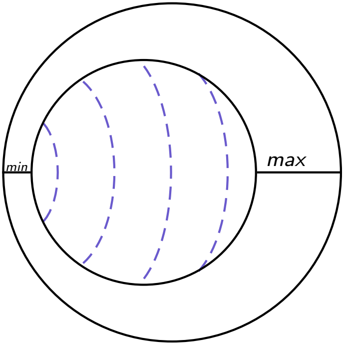
where S, P, Q, f are functions of r only, and $R = R(t, r)$

$$\mathcal{E} = \frac{S}{2}\left[\epsilon + \left(\frac{p-P}{S}\right)^2 + \left(\frac{q-Q}{S}\right)^2\right]$$

Szekeres metric



- "radial" proper distance
  $$\frac{R' - R\mathcal{E}'/\mathcal{E}}{\sqrt{1+f(r)}}dr$$
- r merely labels shells
- shells cannot agree on a radial direction
- proper distance depends on all coordinates r, p, q, t
- shell-crossing



- there is a minimum and a maximum distance between two “neighbouring“ shells
- points on a curve of latitude have same distance to the next shell
- points on curves of latitude are constant in $\mathcal{E}'/\mathcal{E}$

$\Rightarrow$ Dipole structure on shells.

BST Theorem

after Bonnor, Sulaiman, Tomimura, Gen. Relativ. Gravit. **8**, 549 (1977),
"Szekere's Space-Times Have No Killing Vectors"

short:
Szekeres solutions ($\epsilon = 1$) in their most general form have no Killing vector fields, except possibly on isolated submanifolds of the space time.

longer:
Consider a Szekeres space time that is not singular and not FLRW. If it satisfies the following conditions

- $\epsilon = 1$
- $M' \neq 0, f' \neq 0$
- P', Q' are linearly independent

then there is no Killing vector field except possibly on isolated submanifolds of the space time.

BST Theorem

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longer:
Consider a Szekeres space time that is not singular and not FLRW. If it satisfies the following conditions

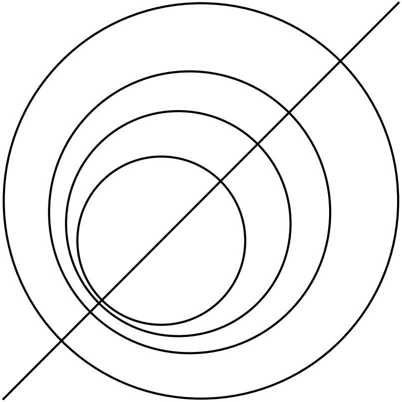
- $M' \neq 0, f' \neq 0$ for $\epsilon = \pm 1$ and $\frac{f}{M^{2/3}} \neq$ const. for $\epsilon = 0$
- P', Q' are linearly independent

then there is no Killing vector field except possibly on isolated submanifolds of the space time.

Motivation

- We started with a simple question: what models are axisymmetric? If the most general Szekeres models have no symmetry what restrictions on the free functions then lead to a symmetry?
  BST: "It would be interesting to enumerate the types of symmetry that arise for special choices of the arbitrary fuctions in Szekeres solution."
- Models with one symmetry can be considered as stepping stones between full symmetry and no symmetry.

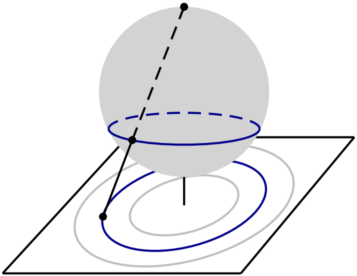
Guess



- dipoles align
- looking down the axis: we find rotational symmetry
- the axis must be a "straight line", i.e. a geodesic
- along the axis all shells agree on this radial direction

What about $\epsilon \neq 1$?

Stereographic Projection



$\mathcal{E}'/\mathcal{E}$ contour lines

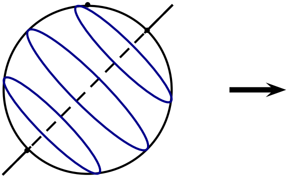


Figure: Dipole on sphere

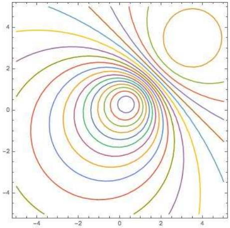


Figure: Dipole on p, q plane

Killing equation

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \quad \Rightarrow \; \xi_{\nu;\mu} + \xi_{\mu;\nu} = 0 \quad \Rightarrow$$

1. $hP' = \text{const.}$
2. $hQ' = \text{const.}$
3. $h(\epsilon SS' + PP' + QQ') = \text{const.}$

where $h(r)$ is a function of integration.

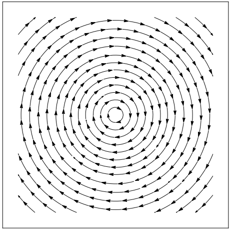
Case 1 $P' = 0 = Q'$. No constraint on S. This is the well known case.

Case 2 Let one of P', Q' be zero and the other one nonzero, say $P' \neq 0$, $Q' = 0$. Then

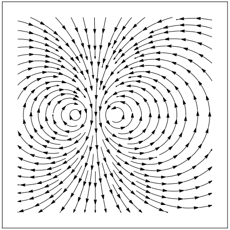
$$\epsilon S^2 = -P^2 + 2c_2 P + c_3 \ .$$

Case 3 Let both $P', Q' \neq 0$. Then

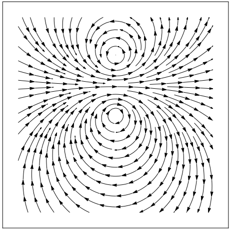
$$Q = cP + c_Q \ ,$$
$$\epsilon S^2 = -(1 + c^2)P^2 + 2c_2 P + c_3 \ .$$



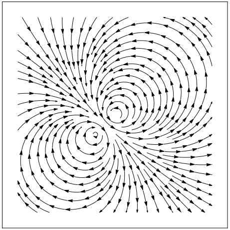
(a) Case 1, $P' = 0 = Q'$



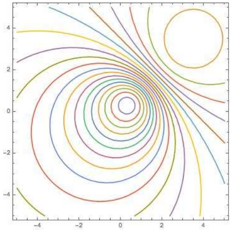
(b) Case 2, $Q' = 0$



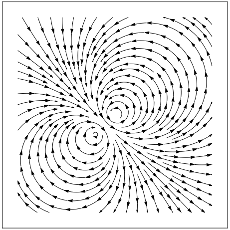
(c) Case 2, $P' = 0$



(d) Case 3, $P' \neq 0, Q' \neq 0$



(e) contour lines of $\mathcal{E}'/\mathcal{E}$

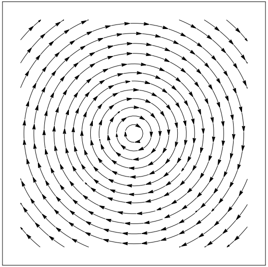


(f) Case 3

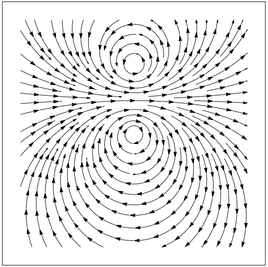
extreme points = fixed points if extreme points exists but there are KVF with fixed points even though the extreme points do not exists (hyperbolic geometry)

$$\mathcal{E}'/\mathcal{E}|_e \overset{\text{case 3 cond.}}{=} \pm \frac{P'}{\epsilon S^2}\sqrt{d}$$

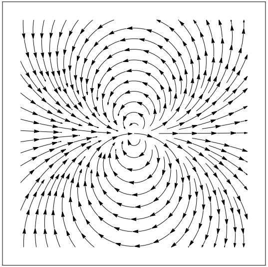
$$d = c_2^2 + (1 + c^2)c_3$$



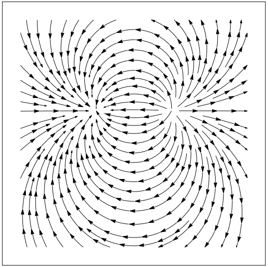
(g) type 1, $d = \infty$



(h) type 1, $d > 0$



(i) type 2, $d = 0$



(j) type 3, $d < 0$

Conclusion

quasi-spherical: The well known axisymmetric models are all there is except for conformal coordinate transformations

quasi-planar: There is either full symmetry or no symmetry.

quasi-hyperbolic: There are rotational symmetric models but also models with a single symmetry (horolation, h-translation) that are not rotational. However the latter then suffer from shell-crossings.

- BST is generalisable to all ϵ
- in the case of axial symmetry $\mathcal{E}'/\mathcal{E}$ coincides with the Killing Vector field streamlines.
- axis of symmetry is geodesic

Thank You!