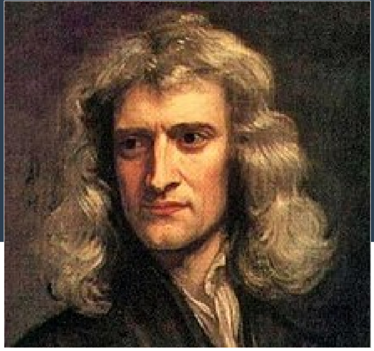
GRAMSES: a new route to general relativistic *N*-body simulations in cosmology

Cristian Barrera-Hinojosa
PhD Supervisor: Baojiu Li
Institute for Computational Cosmology
Durham University

Talk Layout

- Introduction and motivation
- A fully constrained formulation of GR
- GRAMSES overview
- Results from ΛCDM simulations
- On the generation of Initial Conditions
- Conclusions and outlooks

Cosmic Web à la Newton



In standard cosmological *N*-body simulations we solve a Newtonian system of equations:

Gravity sector

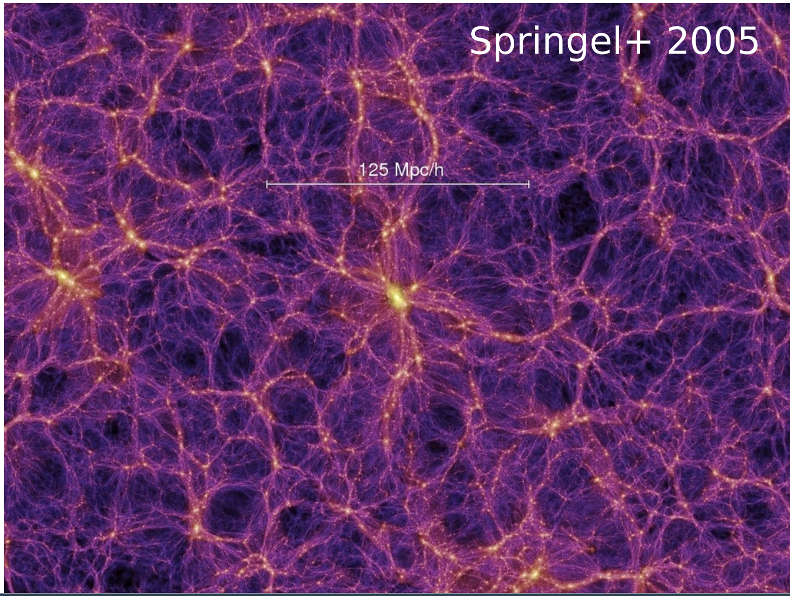
$$\nabla^2\Phi = 4\pi G a^2 \delta\rho$$

Matter sector

$$\dot{v} + Hv = -\nabla\Phi$$

where $a = a(\Omega_m, \Omega_\Lambda)$

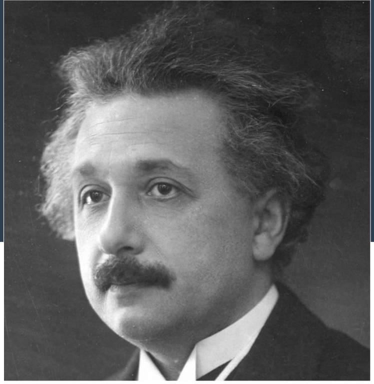
- ✔ Great success to explain our observable Universe.
- ✔ (Deceivingly?) Intuitive picture.
- ✖ Fixed FLRW background expansion.
- ✖ Structures do not back-react on spacetime.



Motivation for GR simulations

- In the era of precision cosmology, is Newtonian approximation going to remain sufficient for the upcoming years?
- Test whether **GR** makes any measurable difference on observables.
- How good is the **FLRW** background? Can we shed some light on the current 5.3σ tension in H_0 ? (Wong+ 2019).
- Cosmology with 'exotic objects' (beyond ΛCDM) such as relativistic particles, scalar fields, cosmic strings, gravitational waves, etc.

The Ultimate Cosmic Web



In principle, we would like to solve the full theory of gravity without any approximation:

Gravity sector	Matter sector
$G_{\mu\nu} = T_{\mu\nu}$	$\nabla_\mu T^{\mu\nu} = 0$

- ✔ Fully nonlinear and background-independent model for the Universe.
- ✔ Structures do interact with spacetime.
- ✖ Computationally expensive.
- ✖ Not very intuitive.

Known routes for GR cosmological simulations

- Perturbative Expansions route:
  - gevolution (Adamek+ 2016)

✓ Shell-crossing for DM.
● Linearisation procedure.
● No AMR (fixed resolution).

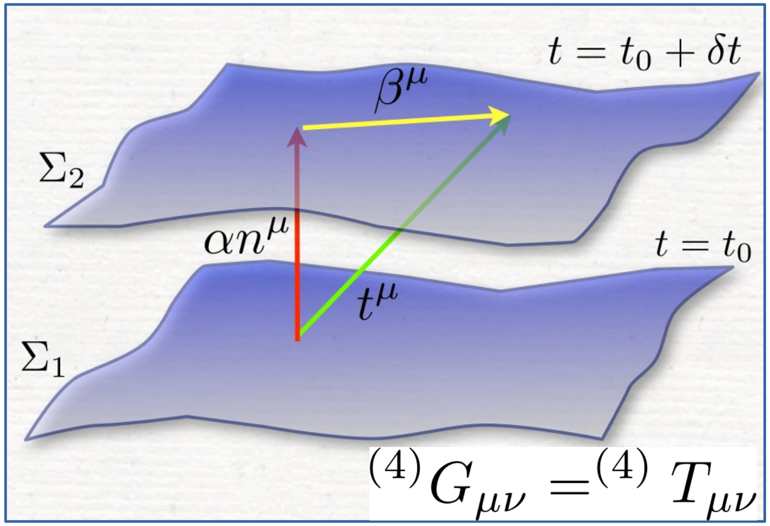
- Numerical Relativity route:
  - Bentivegna & Bruni 2016
  - Mertens+ 2017
  - Macpherson+ 2017

✓ Non-perturbative GR.
● No shell-crossing for DM.
● No Λ.

Let's try to have the best of both:

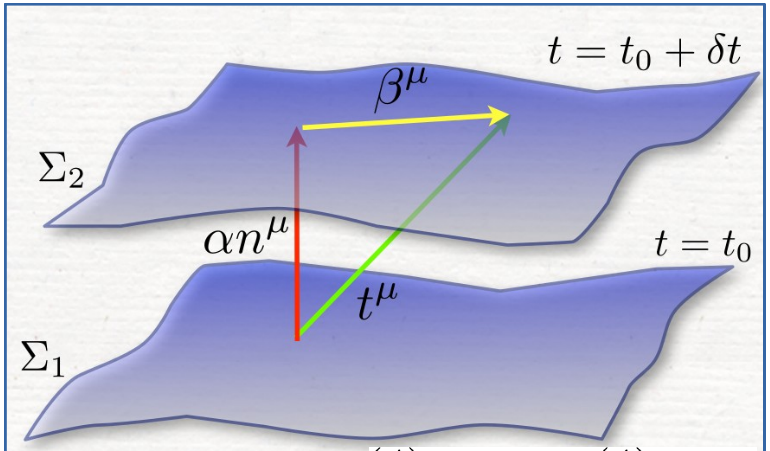
colisionless particles in non-perturbative GR.

Numerical Relativity: the 3+1 Formalism



$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(\beta^i dt + dx^i)(\beta^j dt + dx^j)$$

Numerical Relativity: the 3+1 Formalism



$${}^{(4)}G_{\mu\nu} = {}^{(4)}T_{\mu\nu}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(\beta^i dt + dx^i)(\beta^j dt + dx^j)$$

Evolution Equations:

$$(\partial_t - \mathcal{L}_\beta)\gamma_{ij} = -2\alpha K_{ij}$$

$$(\partial_t - \mathcal{L}_\beta)K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^k_j + KK_{ij}) - 8\pi\alpha\left[S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)\right]$$

Constraint Equations:

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^i$$

CTT decomposition:

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij},$$

$$\bar{A}^{ij} = \bar{A}^{ij}_{TT} + \bar{A}^{ij}_L$$

A Fully Constrained Formulation of GR

- However, the Einstein Equations in 3+1 are **weakly hyperbolic**, a problem that is fixed by introducing auxiliary variables as in the BSSN formalism (Shibata and Nakamura 1995, Baumgarte and Shapiro 1999).

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- For structure formation we might argue that tensor modes are subdominant.
- In order to kill tensor modes, we might take the approximation:

$$\bar{\gamma}_{ij} = \delta_{ij}, \quad \bar{A}^{ij}_{TT} = 0 \qquad \forall t$$

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- In principle the conformal flatness approximation is very strong. However, we can use the gauge freedom so that we don't lose information about 1 scalar + 2 vector modes in the metric.

A Fully Constrained Formulation of GR

- Following Cordero-Carrion+ 2009, we can have a consistent fully constrained formulation by picking the Dirac gauge condition:

$$\partial_i \bar{\gamma}^{ij} = 0$$

- This gauge condition translates into a condition for the shift vector and makes the conformal flatness approximation ‘safe’.
- This is in the same spirit as the ‘waveless theories of gravity’ first explored by Isenberg in 1978 and later by Wilson and Mathews 1989.
- In this approach time evolution is due to particles (solving the geodesic equation).
- Even if no GW are originally present, these still can be added under certain approximations (Cordero-Carrion+ 2012).

System of equations for the gravity sector

In this approach the elliptic system of equations is decoupled into:

1. Momentum Constraint:

$$(\bar{\Delta}_L W)_i = 8\pi s_i \qquad \bar{A}^{ij}_L = (\bar{L}W)^{ij} \equiv \bar{A}^{ij}$$

2. Hamiltonian Constraint:

$$\bar{\nabla}^2 \psi = -2\pi\psi^{-1} s_0 - \frac{1}{8}\psi^{-7}\bar{A}_{ij}\bar{A}^{ij} + 2\pi\psi^5 \hat{\rho}_m \, ,$$

3. Constant Mean Curvature (CMC) slicing condition:

$$\bar{\nabla}^2(\alpha\psi) = \alpha\left[2\pi\psi^{-1}(s_0 + 2s) + \frac{7}{8}\psi^{-7}\bar{A}_{ij}\bar{A}^{ij} + \psi^5\left(\frac{5K^2}{12} + 2\pi\rho_\Lambda\right)\right] - \psi^5 \dot{K}$$

4. Determine shift vector associated to Minimal Distortion gauge (in CFA):

$$(\bar{\Delta}_L \beta)^i = 2\partial_j\left(\alpha\psi^{-6}\bar{A}^{ij}_L\right)$$

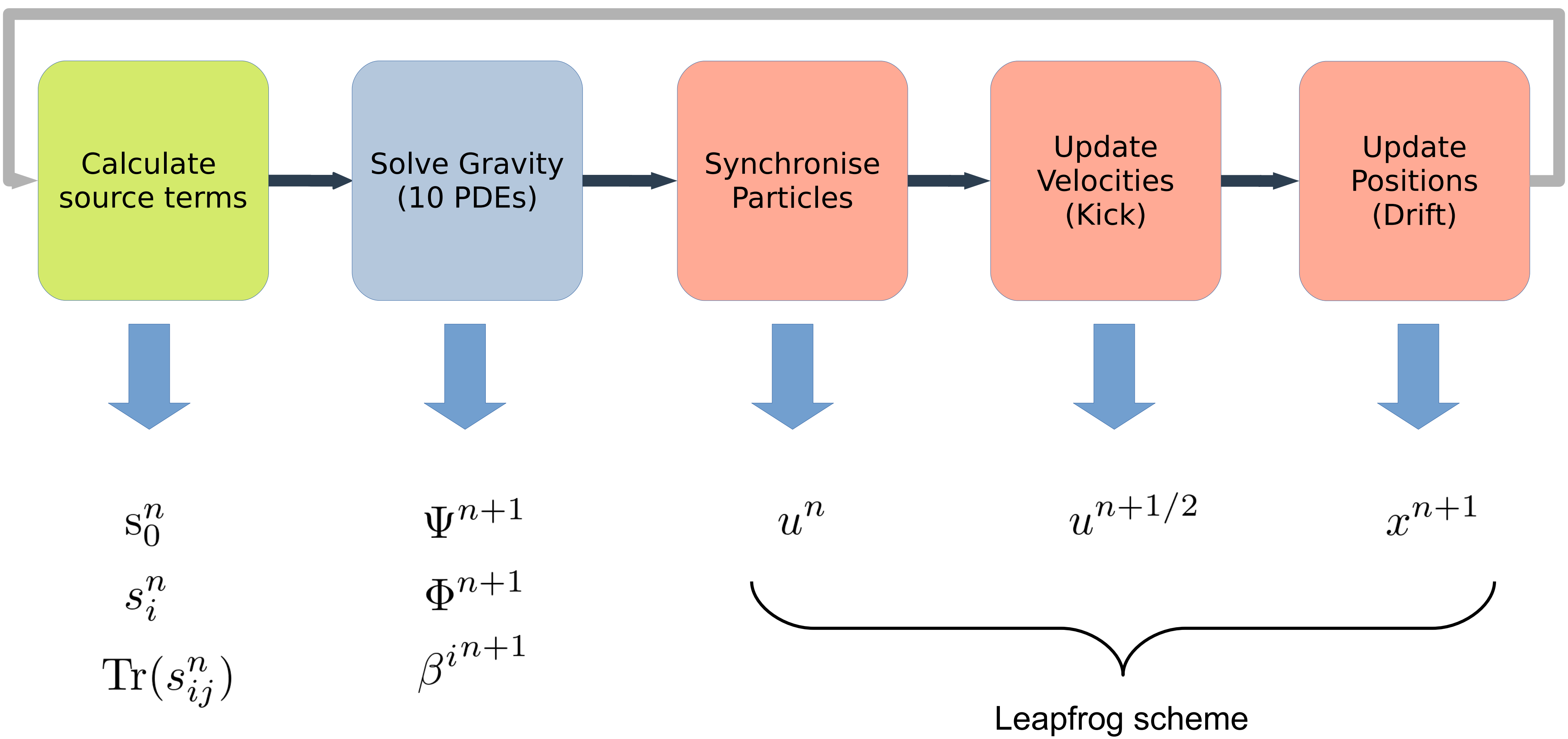
The GRAMSES code

- Based on (Newtonian) **RAMSES** code (Teyssier ‘02)
  - Grid-based code.
  - Parallelisation (MPI).
  - Multigrid + Adaptive Mesh Refinement (AMR).
  - Cloud In Cell (CIC) scheme for particles.

- The implementation/modification of several subroutines are needed for these GR simulations, including:
  - Calculation of relativistic source terms:
    - Matter source terms.
    - Geometric terms.
  - Multigrid Gauss-Seidel Solver for 8 Poisson-like equations.
  - Multigrid Gauss-Seidel Solver for 2 non-Poisson-like equations.

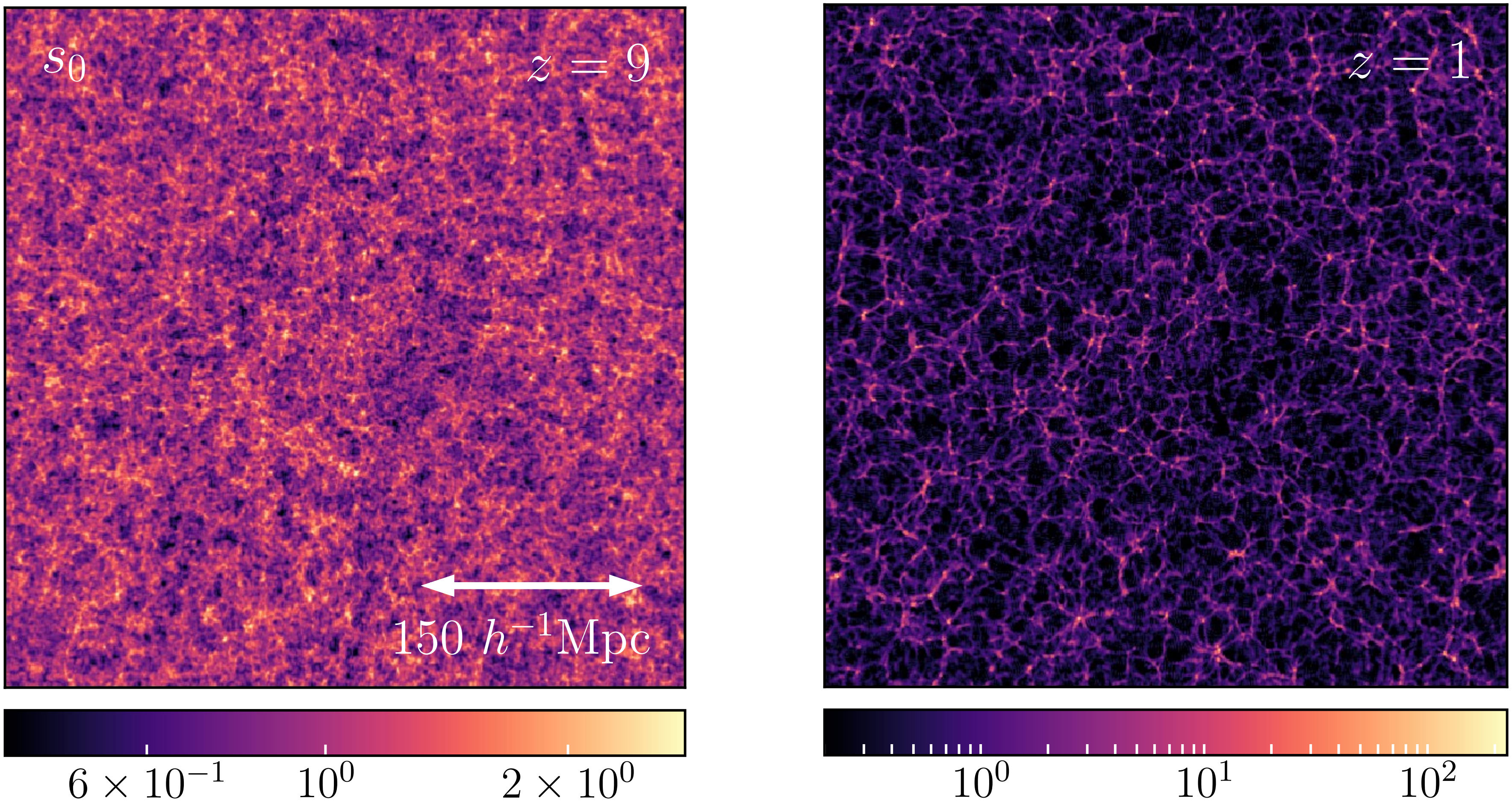
Flowchart of GRAMSES Solver

Given a snapshot of the system at a given time-step *n*, the solver follows the next logical sequence:



Cosmic web in a ΛCDM Universe

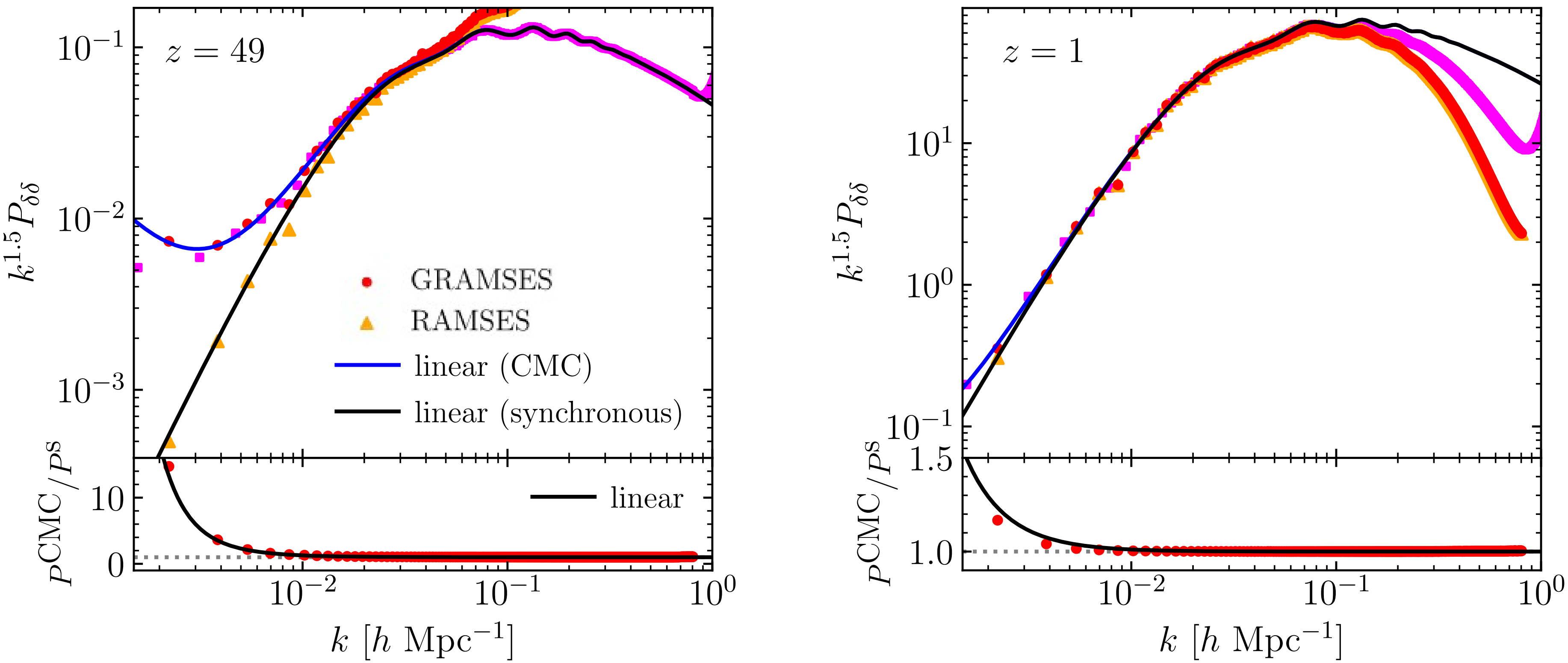
Setup: 512 Mpc/h box with 512^3 DM particles



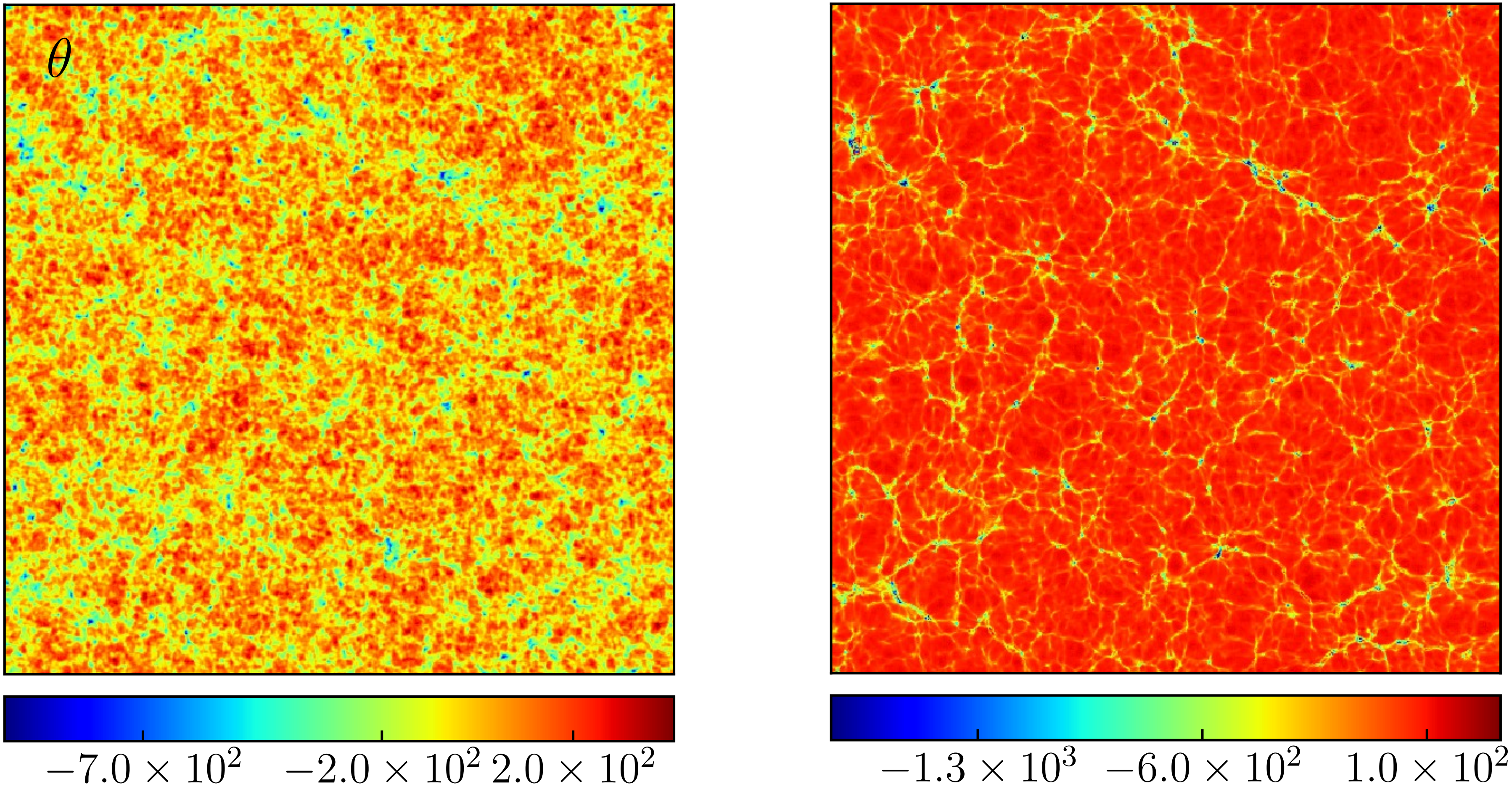
Matter Power Spectrum

Setup: 4 Gpc/h box with 1024^3 DM particles.

$$\langle \delta(k)\delta(k')\rangle = \frac{2\pi^2}{k^3} P_{\delta\delta}\delta^{(3)}(k-k')$$ ← gauge-dependent



Velocity divergence $\theta = \nabla \cdot u$



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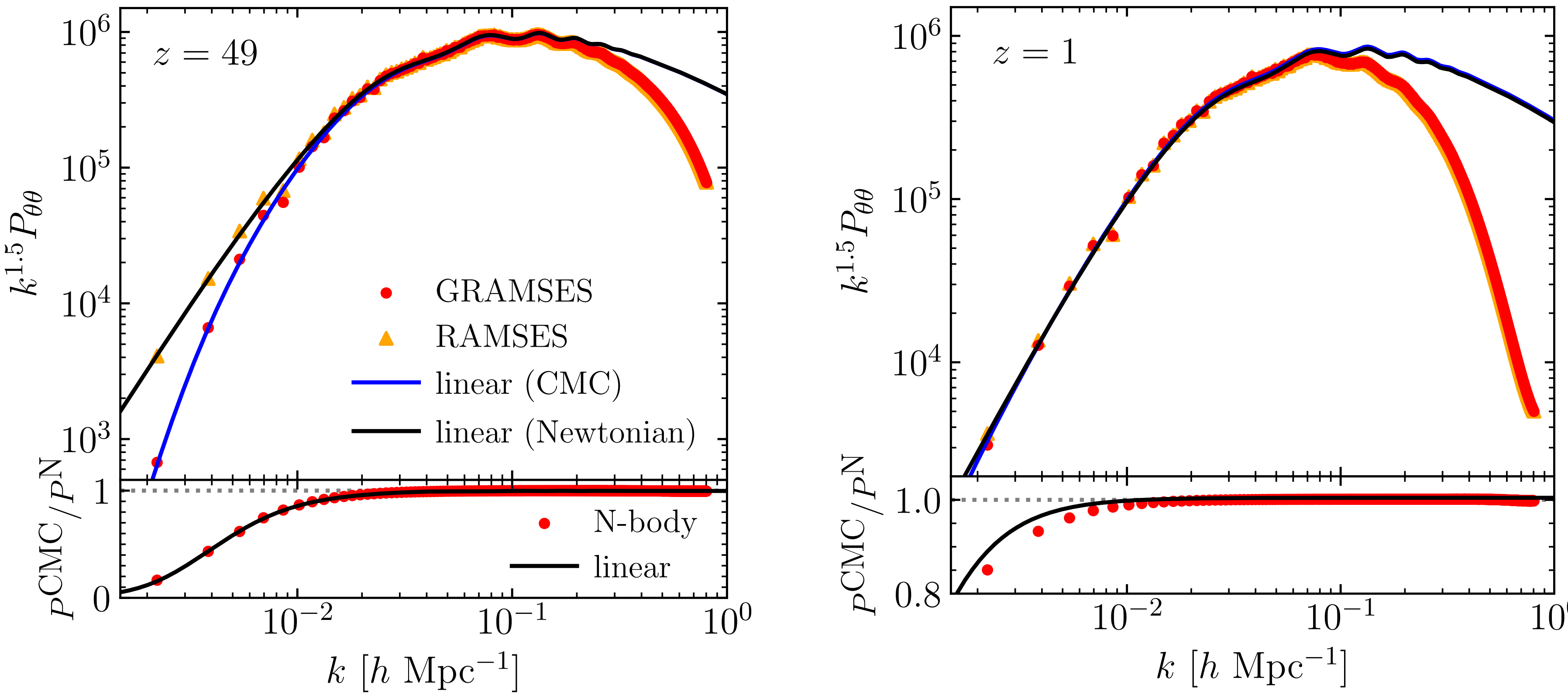


Vector Potential

- The vector modes in the metric are responsible for a “frame dragging” effect which is absent in Newtonian gravity.

- In perturbation theory, β_i^V is a 2nd order quantity that is sourced by the rotational part of the momentum field.

$$\nabla^2 \beta_i^V = \frac{9\Omega_m^2 H_0^4}{2a^2} (\delta u_i)^V$$

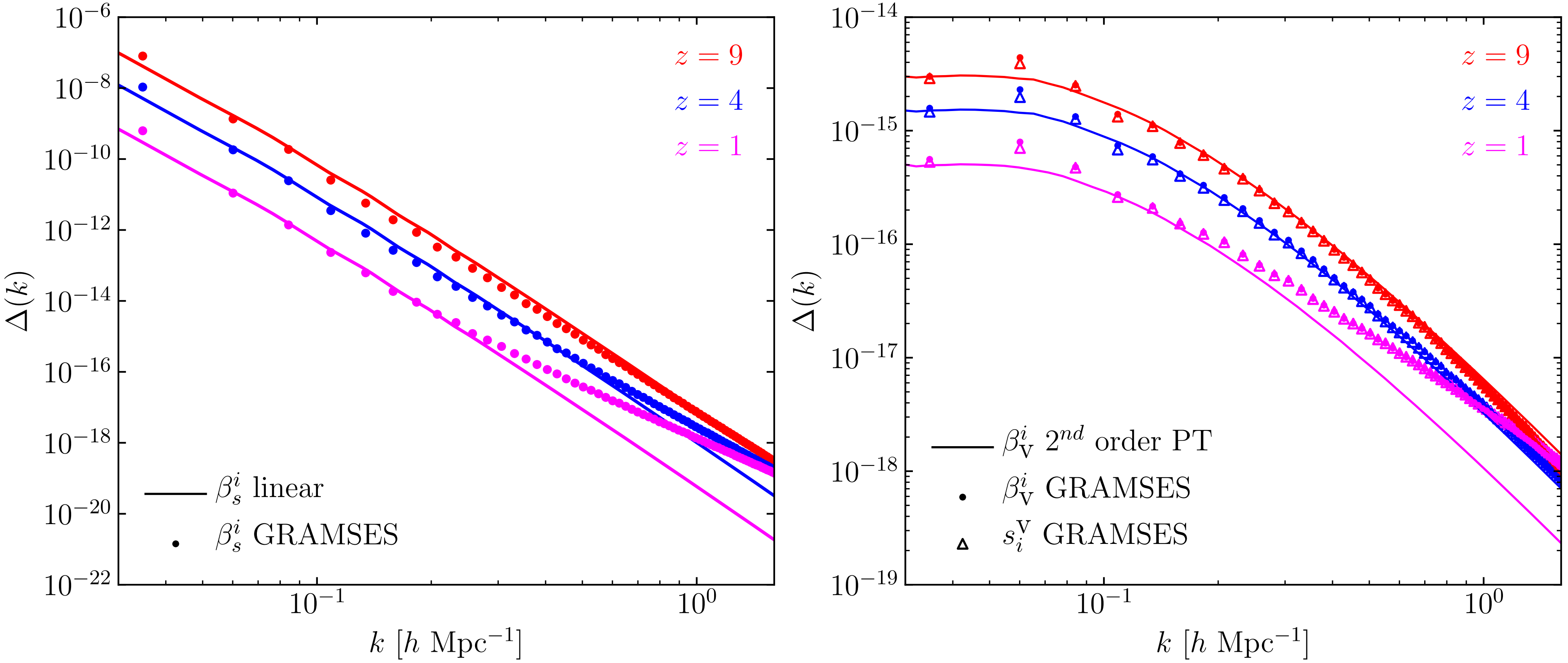
Using Wick’s theorem, the power spectrum of the vector mode is given by (Lu+ 2009):

$$\Delta_{\beta^V}(k) = \frac{9\Omega_m^2 H_0^4}{2a^2 c^2 k^2} \int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \Pi \left[\Delta_{\delta\delta}(ku)\Delta_{vv}(kv) - \frac{v}{u}\Delta_{\delta v}(ku)\Delta_{\delta v}(kv) \right]$$

Vector Potential Power Spectrum

In GRAMSES the shift vector has both scalar and vector modes

$$\beta^i = \beta^i_s + \beta^i_V$$



On the generation of Initial Conditions

- Given a density field from a linear code (CAMB, CLASS), we need to translate this into particle information.
- This can be approached as a density/displacement duality:

Positions: $x^i(q) = q^i + \Psi^i(q) :$ $\delta(x) = -\partial_i \Psi^i$

Velocities: $v^i := \frac{dx^i}{dt} = \dot{\Psi}^i$ $\partial_i v^i = -\dot{\delta}$

- In order to generate velocities we need some information about the growth rate of density perturbations (both theory and gauge-dependent).

On the generation of Initial Conditions

- In order to avoid explicit parametrizations, we can calculate the initial velocities via finite differences of 2 neighbouring density fields:

$$\delta\Big|_{z+\Delta z} \qquad \delta\Big|_{z-\Delta z}$$

$$v^i \approx \frac{\Delta x^i}{\Delta t} = -\frac{H}{a}\frac{\Psi^i(q_+) - \Psi^i(q_-)}{2\Delta z}$$

- We implemented this into `2LPTic` code (Crocce+ 2006).
- Since we want u_i rather than v^i we still need to correct inside GRAMSES by solving:

$$(\bar{\Delta}_L \beta_{\rm ini})^i - 6\frac{\Omega_m}{c^2}\beta^i_{\rm ini} = 6\frac{\Omega_m}{ac^2}v^i_{\rm ini}$$

Conclusions

- We have implemented a fully non-linear **GR** N-body system on **RAMSES.**
- We cast **GR** into 10 elliptic PDEs.
- Results so far look promising.
- Due to gauge issues comparisons are difficult and the generation of IC is non-trivial.
- Upcoming/future work:
  - High resolution simulations.
  - Study back-reaction and GR effects.
  - Implement Scalar fields (dark matter models, early universe)
  - Add GW content following Cordero-Carrion 2012+.
  - Observables? (ray tracing).