European Funds
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FNP Foundation for Polish Science

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How does torsion affect light propagation?

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15 July 2019

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Beyond General Relativity...

Why?

- Address the mysteries in the universe
  Dark energy, dark matter, inflation etc...
- Help constructing a quantum theory of gravity
- Better understand GR
  One of the best ways to understand something may be to break (modify) it and then to reconstruct it.
- Test GR

...beyond Riemannian Geometry
What does it mean?

- Riemannian manifold:
  smooth manifold with a positive defined metric tensor $(+ + ... +)$
- Pseudo-Riemannian manifold:
  smooth manifold with a nondegenerate metric tensor $(\pm \pm ... \pm)$
- Lorentzian manifold:
  smooth manifold with a pseudo-Riemannian metric of signature $(- + ... +)$

The most general manifolds equipped with a metric and a linear connection

$$g_{\alpha\beta},\ \Gamma^{\sigma}{}_{\alpha\beta},$$

are characterised by three geometrical entities:

- Non-metricity

$$Q_{\sigma\alpha\beta} \equiv \nabla_{\sigma} g_{\alpha\beta}, \quad (1)$$

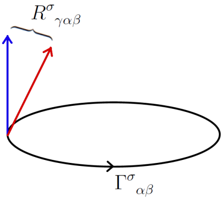
- Curvature

$$R^{\sigma}{}_{\gamma\alpha\beta} \equiv \partial_{\alpha}\Gamma^{\sigma}{}_{\gamma\beta} - \partial_{\beta}\Gamma^{\sigma}{}_{\gamma\alpha} + \Gamma^{\sigma}{}_{\rho\alpha}\Gamma^{\rho}_{\gamma\beta} - \Gamma^{\sigma}{}_{\rho\beta}\Gamma^{\rho}_{\gamma\alpha}, \quad (2)$$

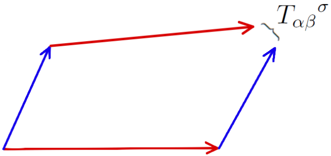
- Torsion

$$T_{\alpha\beta}{}^{\sigma} \equiv \Gamma^{\sigma}{}_{\alpha\beta} - \Gamma^{\sigma}{}_{\beta\alpha}. \quad (3)$$

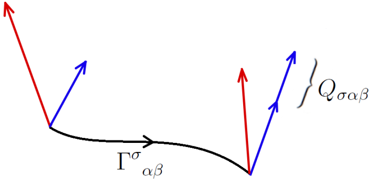
Curvature, Torsion, and Non-metricity
Geometrical interpretation



Curvature: Rotation of a vector transported along a closed curve



Torsion: Non-closure of parallelograms of vectors transported along each other



Non-metricity: Variation of lenghts and angles under transport along a curve

Classification of spaces in terms of Q, R, T.

Space	Q	R	T
Metric-affine	$\neq 0$	$\neq 0$	$\neq 0$
Riemann-Cartan	$= 0$	$\neq 0$	$\neq 0$
Weitzenbock-Weyl	$\neq 0$	$= 0$	$\neq 0$
Einstein-Weyl	$\neq 0$	$\neq 0$	$= 0$
Weitzenbock	$= 0$	$= 0$	$\neq 0$
Riemann	$= 0$	$\neq 0$	$= 0$
Minkowski-Weyl	$\neq 0$	$= 0$	$= 0$
Minkowski	$= 0$	$= 0$	$= 0$

Table inspired to Fig. 1 in [Y. Mao, M. Tegmark, A. H. Guth & S. Cabi, Phys. Rev. D **76**, 104029 (2007)]

The most general connection can be decomposed as

$$\Gamma^{\sigma}{}_{\alpha\beta} = \Lambda^{\sigma}{}_{\alpha\beta} + L^{\sigma}{}_{\alpha\beta} + K_{\alpha\beta}{}^{\sigma}, \tag{4}$$

where $\Lambda^{\sigma}{}_{\alpha\beta}$ is the Levi-Civita connection, the part

$$L^{\sigma}{}_{\alpha\beta} = \frac{1}{2}\left(Q_{\alpha}{}^{\sigma}{}_{\beta} + Q_{\beta}{}^{\sigma}{}_{\alpha} - Q^{\sigma}{}_{\alpha\beta}\right), \tag{5}$$

is a tensor symmetric in the two lower indices, and

$$K_{\alpha\beta}{}^{\sigma} = \frac{1}{2}\left(T^{\sigma}{}_{\alpha\beta} + T^{\sigma}{}_{\beta\alpha} + T_{\alpha\beta}{}^{\sigma}\right), \tag{6}$$

is the contorsion tensor, antisymmetric in the last two indices.

When the metricity is conserved

$$Q_{\sigma\alpha\beta} = 0,$$

the connection can be rewritten as

$$\Gamma^{\sigma}{}_{\alpha\beta} = \Lambda^{\sigma}{}_{\alpha\beta} + \left(\frac{T^{\sigma}{}_{\alpha\beta} + T^{\sigma}{}_{\beta\alpha}}{2} \right) + \frac{1}{2} T_{\alpha\beta}{}^{\sigma}, \qquad (7)$$

where the symmetric part of the connection is necessarily the Levi-Civita connection

$$\widetilde{\Gamma}^{\sigma}{}_{\alpha\beta} = \Lambda^{\sigma}{}_{\alpha\beta} + \left(\frac{T^{\sigma}{}_{\alpha\beta} + T^{\sigma}{}_{\beta\alpha}}{2} \right), \qquad \widetilde{\Gamma}^{\sigma}{}_{\alpha\beta} \equiv \Lambda^{\sigma}{}_{\alpha\beta}, \qquad (8)$$

so, the torsion tensor, antisymmetric in the first two indices, is also antisymmetric in the last two [L. Fabbri 2007]

$$T_{\sigma\alpha\beta} + T_{\sigma\beta\alpha} = 0, \qquad T_{\alpha\beta\sigma} = T_{[\alpha\beta\sigma]} \qquad (9)$$

- The autoparallel equation

$$k^{\alpha}{}_{;\beta}k^{\beta} = 0, \tag{10}$$

can be written in component form as

$$\frac{\mathrm{d}^2 x^{\alpha}}{\mathrm{d}v^2} + \Gamma^{\alpha}{}_{\beta\gamma}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}v}\frac{\mathrm{d}x^{\gamma}}{\mathrm{d}v} = 0. \tag{11}$$

- The curves that minimise the arc length $ds^2 = g_{\alpha\beta}\,\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta}$ are described by the equation

$$\frac{\mathrm{d}^2 x^{\rho}}{\mathrm{d}v^2} + \frac{1}{2}g^{\rho\alpha}\left(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}\right)\frac{\mathrm{d}x^{\beta}}{\mathrm{d}v}\frac{\mathrm{d}x^{\gamma}}{\mathrm{d}v} = 0. \tag{12}$$

$$\frac{\mathrm{d}^2 x^{\rho}}{\mathrm{d}v^2} + \Lambda^{\rho}{}_{\beta\gamma}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}v}\frac{\mathrm{d}x^{\gamma}}{\mathrm{d}v} = 0. \tag{13}$$

Riemann-Cartan space
...are still autoparallel curves

- The geodesic equation in presence of torsion takes the form

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}v^2} + \Gamma^\rho{}_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}v}\frac{\mathrm{d}x^\gamma}{\mathrm{d}v} = \frac{1}{2}T_{\beta\gamma}{}^\rho\frac{\mathrm{d}x^\beta}{\mathrm{d}v}\frac{\mathrm{d}x^\gamma}{\mathrm{d}v}. \quad (14)$$

- Since torsion is by definition antisymmetric in the two lower indices the last equation reduces to

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}v^2} + \Gamma^\rho{}_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}v}\frac{\mathrm{d}x^\gamma}{\mathrm{d}v} = 0, \quad (15)$$

In presence of torsion, geodesics are still autoparallel curves.

In the geometrical optics approximation,

- the electromagnetic wave can be approximated as

$$F = a\left(x^{\alpha}\right) f\left(\psi\right), \tag{16}$$

  where a is the amplitude of the wave and f is an arbitrary function of the phase ψ;

- the propagation vector k^{α} is defined in terms of the phase ψ as

$$k_{\alpha} = \nabla_{\alpha}\psi \quad \Rightarrow \quad \nabla_{\beta}k_{\alpha} - \nabla_{\alpha}k_{\beta} = -T_{\alpha\beta}{}^{\gamma}k_{\gamma}; \tag{17}$$

- the propagation vector is null, $k^{\alpha}k_{\alpha} = 0$, and so

$$\nabla_{\beta}\left(k_{\alpha}k^{\alpha}\right) = 0 \quad \Rightarrow \quad k_{\alpha}\,\nabla_{\beta}k^{\alpha} = -\frac{1}{2}Q_{\beta\alpha\gamma}k^{\alpha}k^{\gamma}. \tag{18}$$

- The formula for the propagation of k^α is given by

$$g_{\gamma\alpha} k^\mu \nabla_\mu k^\alpha = B_{\gamma\alpha\beta}\, k^\alpha k^\beta, \tag{19}$$

where the propagation tensor $B_{\gamma\alpha\beta}$ is defined as

$$B_{\gamma\alpha\beta} \equiv T_{\gamma\alpha\beta} + \frac{1}{2}\left(Q_{\gamma\alpha\beta} - Q_{\alpha\gamma\beta}\right) - \frac{1}{2} Q_{\alpha\beta\gamma}. \tag{20}$$

- In general null curves are not autoparallel
- When metricity is conserved

$$B_{\gamma\alpha\beta} k^\alpha k^\beta = T_{\gamma\alpha\beta} k^\alpha k^\beta = T_{[\gamma\alpha\beta]} k^\alpha k^\beta = 0, \tag{21}$$

In presence of torsion, light propagates on null geodesics

When metricity is not conserved,

- if null vector is aligned with the propagation tensor

$$B_{\gamma\alpha\beta}\, k^{\alpha} k^{\beta} = c\, k_{\gamma} \tag{22}$$

$$k^{\mu} \nabla_{\mu} k^{\alpha} = c\, k^{\alpha}, \tag{23}$$

  the null curve is a geodesic with a non-affine parametrisation: non-metricity induces inaffinity acceleration but keeps photons on autoparallel curves.

- in the most general case

$$B_{\gamma\alpha\beta}\, k^{\alpha} k^{\beta} \neq c\, k_{\gamma}, \tag{24}$$

Light curves are no longer autoparallel

- The redshift can be inferred from the change in $u^\alpha k_\alpha$

$$\frac{\mathrm{D}\nu}{\mathrm{D}s} = k^\mu \nabla_\mu (u_\alpha k^\alpha) = k^\alpha k^\mu \nabla_\mu u_\alpha + u^\sigma g_{\sigma\alpha} k^\mu \nabla_\mu k^\alpha \tag{25}$$

where the term

$$k^\alpha k^\mu \nabla_\mu u_\alpha \tag{26}$$

gives the standard formula for the redshift, and

$$u^\sigma g_{\sigma\alpha} k^\mu \nabla_\mu k^\alpha = u^\sigma B_{\sigma\alpha\beta}\, k^\alpha k^\beta, \tag{27}$$

vanishes when metricity is conserved

Torsion does not affect redshift

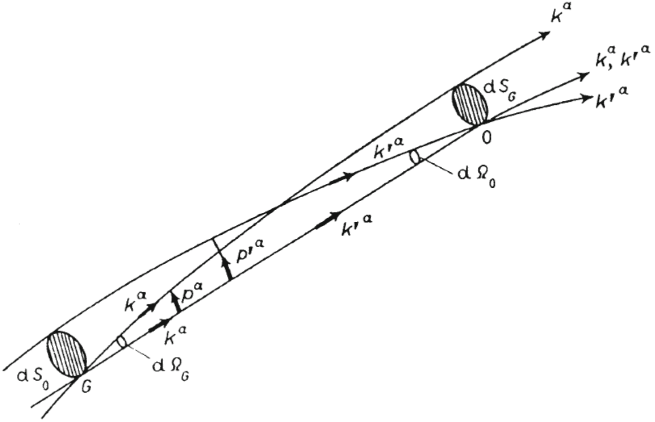
- The distance duality relation is a fundamental law of cosmology that relates area and luminosity distances

$$D_L = (1 + z)^2 \, D_A. \tag{28}$$

- Alternative names: reciprocity relation, reciprocity theorem, Etherington theorem
- Its first derivation dates back to 1933 [I. M. H. Etherington 1933], while a more modern version of it was proposed by Ellis [G. F. R. Ellis 1971].

Distance-duality relation
Assumptions

- According to some... [S. More+ 2016]
  - conservation law for light rays in geometrical optics
  - Lorentzian geometry
- ...according to others [X. Fu+ 2019]
  - conservation of the number of photons
  - light travels along null geodesics in a Riemannian geometry



Areas and angles at observer O and galaxy G [Ellis 1971].

- From the definition of connecting vector p^α (and p'^α as well) we have

$$\frac{Dp}{Dv} = p^\alpha{}_{;\beta} k^\beta = k^\alpha{}_{;\beta} p^\beta \tag{29}$$

from which follows the geodesic deviation equation

$$\frac{D^2 p^\alpha}{Dv^2} = R^\alpha{}_{\beta\gamma\rho} k^\beta k^\gamma p^\rho, \tag{30}$$

and the corresponding primed equation.

- Since $R^\alpha{}_{\beta\gamma\rho} k^\beta k^\gamma = R^\alpha{}_{\beta\gamma\rho} k'^\beta k'^\gamma$ and $v = v'$ along OG, we have

$$p'^\alpha \frac{D^2 p_\alpha}{Dv^2} - p^\alpha \frac{D^2 p'_\alpha}{Dv^2} = R_{\alpha\beta\gamma\rho}\, p'^\alpha k^\beta k^\gamma p^\rho - R_{\alpha\beta\gamma\rho}\, p^\alpha k^\beta k^\gamma p'^\rho = 0$$

- Hence we have

$$p'^{\alpha}\frac{Dp_{\alpha}}{Dv} - p^{\alpha}\frac{Dp'_{\alpha}}{Dv} = \mathrm{constant\ along}\, OG \tag{31}$$

- Evaluating the constant at O and at G we find

$$\left(p'^{\alpha}\frac{Dp_{\alpha}}{Dv}\right)_G = \left(p^{\alpha}\frac{Dp'_{\alpha}}{Dv}\right)_O \tag{32}$$

  and the corresponding primed equation.
- Choosing two pairs of vectors orthogonal at both O and G we obtain

$$\mathrm{d}S_G\,\mathrm{d}\Omega_O\,(k^{\alpha}u_{\alpha})^2\,|_O = \mathrm{d}S_O\,\mathrm{d}\Omega_G\,(k^{\alpha}u_{\alpha})^2\,|_G \tag{33}$$

  from which the distance duality relation follows

- From the definition of connecting vector p^α (and p'^α as well) we have

$$\frac{Dp^\alpha}{Dv} = p^\alpha{}_{;\beta} k^\beta = k^\alpha{}_{;\beta} p^\beta + T_{\beta\gamma}{}^\alpha p^\gamma k^\beta. \quad (34)$$

from which follows the geodesic deviation equation

$$\frac{D^2 p^\alpha}{Dv^2} = R^\alpha{}_{\beta\gamma\rho} k^\beta k^\gamma p^\rho - k^\beta \frac{D}{Dv}\left(T_{\beta\gamma}{}^\alpha p^\gamma\right), \quad (35)$$

with

$$R^\alpha{}_{\beta\gamma\rho} = \widetilde{R}^\alpha{}_{\beta\gamma\rho} + \frac{1}{2}\left(\nabla_\gamma T_{\beta\rho}{}^\alpha - \nabla_\rho T_{\beta\gamma}{}^\alpha\right) + \frac{1}{4}\left(T_{\sigma\gamma}{}^\alpha T_{\beta\rho}{}^\sigma - T_{\sigma\rho}{}^\alpha T_{\beta\gamma}{}^\sigma\right),$$

where $\widetilde{R}^\alpha{}_{\beta\gamma\rho}$ is the Riemann tensor of GR.

- So we have

$$p'^{\alpha}\frac{D^2 p_{\alpha}}{Dv^2} - p^{\alpha}\frac{D^2 p'_{\alpha}}{Dv^2} \propto T$$

and we expect the distance duality relation to be modified accordingly

$$D_L = (1+z)^2 D_A \left(1 + \xi(T)\right),$$

- Work in progress...

$$\xi(T) = ...$$

- The DDR is a powerful tool to test GR and various extensions [B. A. Bassett & M. Kunz 2004]
- Observational tests based on Supernovae and the baryon acoustic oscillations do not show any violation of the DDR [R. F. L. Holanda+ 2017]
- An additional test could be possible using gravitational waves [X. Fu+ 2019]
- Theoretical studies of possible modifications of the DDR almost only consider non-metric theories of gravity [S. More+ 2016]
- It is still not clear whether the DDR is torsionproof...

Thanks for your attention