Inhomogeneous Cosmologies IV
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The dynamics of General Relativity in the Geodesic Light-Cone coordinates

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Outlook

1. GLC gauge a brief review
2. Exploiting its properties to identify coordinates with "observables"
3. The observational gauge
4. Dynamics along the observer past light-cone: an attempt

Encoding the past-light cone in FRW

- Let us start from FRW

$$ds^2 = a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 d\Omega^2\right]$$

with $k_\mu = -a(\eta)\,\partial_\mu\,(\eta + r)$ and $u_\mu = -a(\eta)\,\partial_\mu \eta$ and redefine coordinates such that they are adapted to the observed past light-cone

$$(\eta, r, \theta^a) \to (\eta, \eta_+, \theta^a)$$

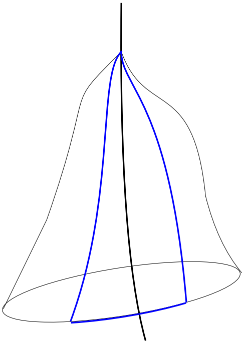
with $\eta_+ = \eta + r$ so $k_\mu = -a(\eta)\,\partial_\mu \eta_+$ and $u_\mu = -a(\eta)\,\partial_\mu \eta$, hence

$$ds^2 = a^2(\eta)\left[d\eta_+^2 - 2\,d\eta\,d\eta_+ + (\eta_+ - \eta)^2\,d\Omega^2\right]$$

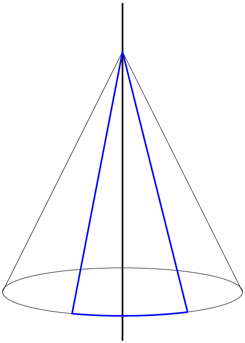
Encoding the past light-cone beyond FRW: the GLC gauge (1)

The Geodesic Light-Cone coordinates consist of a timelike coordinate τ (which can always be identified with the proper time of the synchronous gauge), of a null coordinate w and of two angular coordinates $\tilde{\theta}^a$ ($a = 1, 2$):

$$ds^2 = \Upsilon^2 dw^2 - 2\,\Upsilon d\tau\, dw + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw)$$



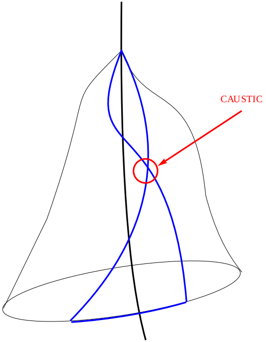
Past light-cone in FRW coordinates



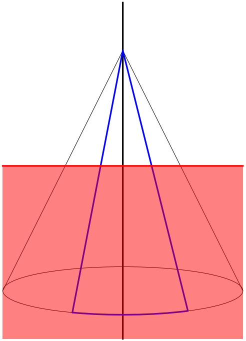
Past light-cone in GLC coordinates

Encoding the past light-cone beyond FRW: the GLC gauge (2)

This identification holds if no caustics appear in light-propagation. Otherwise, an intrinsic limit breaks our description



Caustic on the inhomogeneous light-cone



Past light-cone in GLC coordinates

Encoding the past light-cone beyond FRW: the GLC gauge (3)

Fundamental properties:

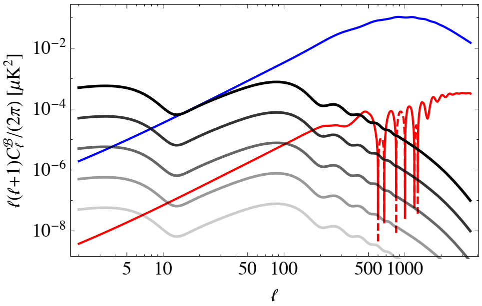
- $w =$ constant defines the past light-cone of the observer (ourselves)
- $u_\mu = -\partial_\mu \tau$ describes a geodesic flow (related to SG)
- $k^\mu = \Upsilon^{-1}\delta^\mu_\tau$ is the quadri-momentum of the photon (constant w and $\tilde{\theta}^a$), or equivalently $k_\mu = -\partial_\mu w$
- These properties are really useful: they allow us to express easily photon's redshift only as function of τ:

$$1 + z_s = \frac{(k^\mu u_\mu)_s}{(k^\mu u_\mu)_o} = \frac{\Upsilon\left(\tau_o, w, \tilde{\theta}^a\right)}{\Upsilon\left(\tau_s, w, \tilde{\theta}^a\right)}$$

- Really similar to FRW metric, but exact and non perturbative!!!

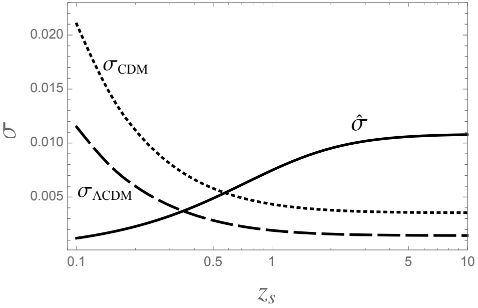
Applications and results within the GLC framework (1)

- A well posed prescription for averaging physical observables (Gasperini, Marozzi, Nugier, Veneziano 2011)
- Exact solution for the Jacobi map (F, Gasperini, Marozzi, Veneziano 2013)
- Perturbative evaluation of non-linear correction to angular distance (Ben-Dayan, Gasperini, Marozzi, Nugier, Veneziano 2012-2013)
- Non-linear lensing corrections in particular to CMB spectra (Marozzi, F, Di Dio, Durrer 2017-2019)



Applications and results within the GLC framework (2)

- Estimation of non-linear statistics for Galaxy number counts (Di Dio, Durrer, Marozzi, Montanari 2015-2016)
- Weak lensing and strongly inhomogeneous and anisotropic cosmologies (F, Nugier 2014 and Fleury, Nugier, F 2016)
- Perspectives in multi-messenger cosmology, involving ultra-relativistic particles (F, Gasperini, Marozzi, Veneziano 2015)



Applications and results within the GLC framework (3)

- The reason why these coordinates are particularly useful to describe observations involving (almost) the past light-cone of the observer is that they can naturally be related to the observed coordinates at which sources are view, namely their redshift and observed angles
- However, even if constant along the past light-cone, $\tilde{\theta}^a$ have to be properly defined, in order to be identified with the direction of the incoming photons in the observer rest frame
- Even if geometry can be fully non linearly described within this coordinates system, in order to solve the dynamics of $g_{\mu\nu}$ one always needs to refer to a solvable metric

Identifying the angular coordinates with the observed angles (1)

- In order to properly identify the angular coordinates with the observed angles, we first exploit the residual gauge freedom that these coordinates still have (F, Gasperini, Marozzi, Veneziano 2019)
- Indeed we are allowed to redefine the coordinates as

$$w \to \hat{w}(w)$$

$$\tilde{\theta}^a \to \hat{\theta}^a(w, \theta)$$

and still remain within the GLC gauge

Identifying the angular coordinates with the observed angles (2)

- Because this reparametrization does not depend on τ, it allows us to redefine the angles and the past light-cone at a given time, let us say the observer time
- The question is now about how to define the observer directions: our choice involves the angular coordinates as defined around the tip of the cone where the metric is locally flat
- In this sense we adopt Fermi Normal Coordinates (FNC) where the observer is always located at the center of the coordinates

$$g'_{\alpha\beta}(x') = \eta_{\alpha\beta} + f_{\alpha A \beta B}(t') x'^A x'^B + \cdots$$

where $f_{0A0B} = (R'_{0A0B})_{x'^A=0}$; $f_{0ACB} = \frac{2}{3}(R'_{0ACB})_{x'^A=0}$; $f_{CADB} = \frac{1}{3}(R'_{CADB})_{x'^A=0}$

Identifying the angular coordinates with the observed angles (3)

- Working in the so-called temporal gauge (Fleury, Nugier, F 2015) allows to easily define a non perturbative radius $w - \tau$ needed for the displacement-from-the-geodesic expansion: in particular we identify $w = \tau_o$, being τ_o the time as measured by the observer along its whole geodesic
- To implement the identification we consider solve

$$g_{\mu\nu}^{GLC}(y) = \frac{\partial x'^{\alpha}}{\partial y^{\mu}} \frac{\partial x'^{\beta}}{\partial y^{\nu}} g'_{\alpha\beta}(x')$$

Identifying the angular coordinates with the observed angles (4)

- This is solved by the coordinates transformation

$$t' = \tau - \frac{\dot{\Upsilon}}{2}(w-\tau)^2 + \cdots$$

$$x'^A = N_0^A(w,\theta)(w-\tau) + \frac{1}{2}\dot{\Upsilon} N_0^A(w,\theta)(w-\tau)^2 + \cdots$$

where $\delta_{AB} N_0^A N_0^B = 1$ and

$$\Upsilon = 1 + \dot{\Upsilon}(w,\theta)(w-\tau) + \cdots$$

$$U_a = -\left(\frac{1}{2}\partial_a \dot{\Upsilon} + \delta_{AB}\, \partial_w N_0^A \partial_a N_0^B\right)(w-\tau)^2 + \cdots$$

$$\gamma_{ab} = \delta_{AB}\, \partial_a N_0^A \partial_b N_0^B\, (w-\tau)^2 + \cdots$$

Identifying the angular coordinates with the observed angles (5)

- By identifying the angular coordinates given by the vector N_0^A with $\tilde{\theta}^a$, we get that

$$\theta^1 \to \theta^1(w,\theta) = \arccos\left\{N_0^3\right\}$$
$$\theta^2 \to \theta^2(w,\theta) = \arcsin\left\{N_0^2\left[1-(N_0^3)^2\right]^{-1/2}\right\}$$

- This implies that

$$U_a = -\frac{1}{2}\partial_a\dot{\Upsilon}(w-\tau)^2 + \cdots$$
$$\gamma_{ab} = \mathrm{diag}\left(1, \sin\tilde{\theta}^1\right)(w-\tau)^2 + \cdots$$

and this completely fixes the coordinates in the so-called observational gauge

The residual gauge freedom from a geometrical viewpoint (1)

- The residual gauge transformation as found in the GLC gauge looks very similar to the ones one have when a D-dimensional manifold is sliced in hypersurfaces like

$$ds^2 = s\alpha^2 dy^2 + \gamma_{ij}\left(dx^i - s\beta^i dy\right)\left(dx^j - s\beta^j dy\right) , \qquad s = \pm 1$$

we get not all the coordinates transformations allows the normal unit vector n^μ to the $y =$ constant hypersurfaces given by α and β

$$n = \alpha^{-1}\left(\partial_y + s\beta^i \partial_i\right) , \qquad g(n, n) \equiv s$$

to transfrom as a vector

- This is the case only under the reparametrization

$$y \to \widetilde{y}(y) , \qquad x^i \to \widetilde{x}^i(y, x)$$

The residual gauge freedom from a geometrical viewpoint (2)

- This reparametrization looks exactly as the one exploited in the GLC gauge in order to fix the angular coordinates on the sub-manifold defined by $\{w, \tilde{\theta}^a\}$
- With a bottom-top approach we realize that the GLC coordinate system can be written as a foliation of a given manifold applied twice, such that the line element at τ constant hypersurfaces reads as

$$dl^2 = \Upsilon^2 dw^2 + \gamma_{ab}\left(d\tilde{\theta}^a - U^a dw\right)\left(d\tilde{\theta}^b - U^b dw\right)$$

with the unit vector normal to τ-and-w-constant hypersurfaces given by

$$\nu = \Upsilon^{-1}\left(\partial_w + U^a \partial_a\right)$$

The Einstein equations in the GLC gauge (1)

- Thanks to this geometrical properties, we can apply twice the Gauss-Codazzi-Mainardi formalism and the obtain a $2+1+1$ decomposition of the Einstein equations
- A long but straightforward calculation gives for the Einstein equations

$$\dot{\Theta} = \Upsilon^{-1}\left[-\left(\partial_w + \mathcal{L}_U\right)(\Theta - C) + \nabla^2\Upsilon + A^a\nabla_a\Upsilon\right]$$
$$- \Theta\left[\Theta + \mathcal{K}\right] - \frac{1}{2}A_aA^a + C_{ab}C^{ab} + \frac{1}{2}\left[\Sigma - \mathcal{S} + E\right]$$
$$\dot{A}_a = -\Upsilon^{-1}\left(\partial_w + \mathcal{L}_U\right)A_a + 2\left(\mathcal{K}_a^b - \Theta\delta_a^b\right)\nabla_b \log\Upsilon - \mathcal{K}A_a$$
$$+ 2\left[\nabla_b C_a^b - \nabla_a C - \mathcal{S}_a\right]$$
$$\dot{\mathcal{K}}_{ab} = -\Upsilon^{-1}\left(\partial_w + \mathcal{L}_U\right)\left(\mathcal{K}_{ab} - C_{ab}\right) - \Theta\mathcal{K}_{ab} + 2\mathcal{K}_{ac}\mathcal{K}_b^c - \mathcal{K}_{ab}\mathcal{K}$$
$$- 2C_{ac}C_b^c + C_{ab}C + \frac{1}{2}A_aA_b$$
$$- A_{(a}\nabla_{b)}\log\Upsilon + \Upsilon^{-1}\nabla_a\nabla_b\Upsilon + \mathcal{S}_{ab} - \frac{1}{2}\gamma_{ab}\left[R + \Sigma + \mathcal{S} - E\right]$$

The Einstein equations in the GLC gauge (2)

- Where the conjugated momenta are

$$\dot{\Upsilon} = \Upsilon\Theta \quad , \quad \dot{U}^a = A^a - \nabla^a\Upsilon \quad , \quad \dot{\gamma}_{ab} = 2\left(\mathcal{K}_{ab} - C_{ab}\right)$$

- and the constraint equations are

$$\begin{aligned}
E &= \frac{1}{2}\left[R - C^2 - C_{ab}C^{ab} + (\mathcal{K} + 2\Theta)\,\mathcal{K} - \mathcal{K}_{ab}\mathcal{K}^{ab} - \frac{1}{2}\,A_aA^a\right] \\
&\quad - \Upsilon^{-1}\left[\nabla^2\Upsilon + (\partial_w + \mathcal{L}_U)\,C\right] \\
\mathcal{P} &= \mathcal{K}^{ab}C_{ab} - \Theta C + \frac{1}{2}\,\nabla_a A^a + A^a\nabla_a \log\Upsilon + \Upsilon^{-1}\,(\partial_w + \mathcal{L}_U)\,\mathcal{K} \\
\mathcal{P}_a &= \nabla_a\Theta + \mathcal{K} \text{ - } \nabla_b\mathcal{K}^b_a - \left(\mathcal{K}^b_a - \Theta\delta^b_a\right)\nabla_b \log\Upsilon + \tfrac{1}{2}\,CA_a + \tfrac{1}{2}\,{}^{-1}\,(\partial_w + \mathcal{L}_U)\,A_a
\end{aligned}$$

The Einstein equations in the GLC gauge (3)

- The geometrical structure of the GLC gauge allows to write down the dynamics in a straightforward way, thanks to the Gauss-Codazzi-Mainardi formalism
- The great achievement in this regard consists of having control of the fully non-linear dynamics along the inhomogeneous past light-cone of the observer
- This has to be added to the fact that GLC gauge is naturally adapted to evaluate observables along the photon path and there are well defined prescription for the averages which became trivial in these coordinates
- Unfortunately there seems not such a great hope to find analytical solutions. Maybe some numerical attempts should be explored

Summary and conclusion

- GLC gauge has been successfully used to provide exact analytical expression for several observables involving light-like signals
- One the concern about the possibility to complete the gauge fixing in order to directly identify the angular coordinates with the observed angles has been addressed thanks to the FNC
- The problem about how to study caustics still remains
- Despite the fact that dynamics looks quite complicated to be solved, having formal control on it may allow to construct some generalized consistency relation in a full inhomogeneous contest