Covariant formalism for light propagation in cosmological models

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Motivation

- The original motivation was the comparison of two approaches to the determination of the area distance: from the Sachs optical fields and from the Jacobi field. The comparison should cover the theoretical aspects of these approaches and also the numerical procedures.
- The quantities characterizing the given light beam which are measured by the observer lie in the projective screen space of his 4-velocity an the beam's wave vector.
- This space is effectively two-dimensional, thus in the standard description of the light beam propagation there is used the matrix-based formalism. This approach is necessary for adequate understanding of the problem and for proper physical interpretation of the particular quantities.
- When we have the interpretation, we can avoid the matrix formalism. The aim of this work is to provide the full four-dimensional description of the light beam propagation in cosmological models.

Definitions

- We consider some cosmological model given by a metric field of space-time g_{mn} and a 4-velocity field of cosmic fluid u_n.
- The 4-velocity is normalized $u^a u_a = -1$. We assume that the observer co-moves with the matter.
- We do not restrict the metric so the model could be inhomogeneous.
- We adopt the geometric approximation to the optics in space-time. The electromagnetic waves perceived by the observer are nearly plane, monochromatic and short compared with the typical radius of curvature of space-time. The light propagates along rays whose tangent vector k_n (the wave vector) is null, irrotational and obeys the geodesic equation:

$$k^a k_a = 0, \qquad \nabla_m k_n = \nabla_n k_m, \qquad \dot{k}_n = 0.$$

  where the dot denotes $\dot{X} \equiv k^a \nabla_a X$.
- This means that light rays are null geodesics.

Screen surface

- The circular frequency of the wave ω measured by the observer is defined as:

$$\omega = -u^a k_a.$$

- The screen S_{mn} is defined as a symmetric field projecting onto the surface orthogonal to the observer velocity and to the wave vector:

$$S_{mn} = S_{nm}, \qquad u^a S_{na} = 0, \qquad k^a S_{na} = 0.$$

- These conditions yield:

$$S_{mn} = -\frac{1}{\omega^2} k_m k_n + \frac{1}{\omega}\left(k_m u_n + u_m k_n\right) + g_{mn}.$$

- The screen surface is effectively two-dimensional $S^a{}_a = 2$.
- For a given wave vector, all quantities measurable by the observer are contained in the screen surface.

Screen surface

- Additionally, we define the area field A_{mn} as a totally antisymmetric field on the screen surface:

$$A_{mn} = \frac{1}{\omega} k^b u^a A_{mnba},$$

where A_{klmn} is the alternating (totally antisymmetric) field of space-time.

- It represents the effective area element on the screen surface.
- It has a property that:

$$A_{kl} A_{mn} = S_{km} S_{ln} - S_{kn} S_{lm},$$

which is useful for simplifying formulas.

Optical fields

- Let us consider an infinitesimal light beam consisting of close geodesics with a wave vector k_n. The change rate of morphology of the beam's screen-section is described by the optical deformation rate field D_{mn}:

$$D_{mn} = S_m{}^b S_n{}^a \nabla_b k_a.$$

- It is an on-screen gradient of the wave vector. This field is symmetric since the wave vector is irrotational. It could be decomposed into its trace-free and pure-trace parts as:

$$D_{mn} = \Sigma_{mn} + \frac{1}{2} S_{mn}\Theta, \qquad \Sigma^a{}_a = 0.$$

- The field Σ_{mn} is the optical shear rate and it represents the change rate of shape of the beam's screen-section which could evolve from circular to elliptical one.
- The scalar Θ is the optical expansion rate and it represents the change rate of size of the beam's screen-section which could isotropically expand or contract.
- These two fields are called the Sachs optical fields.
- From the above definitions, we have:

$$\Sigma_{mn} = D_{mn} - \frac{1}{2} S_{mn}\Theta, \qquad \Theta = D^a{}_a.$$

Optical fields

- The transport equations for optical fields are obtained from the Ricci identity for the wave vector:
$$\nabla_l \nabla_m k_n - \nabla_m \nabla_l k_n = R_{lmn}{}^{a} k_a,$$
where R_{klmn} is the Riemann tensor.
- After suitable projections, we get two coupled equations:
$$S_m{}^b S_n{}^a \dot{\Sigma}_{ba} = -\left(\Sigma_m{}^a \Sigma_{na} - \frac{1}{2} S_{mn} \Sigma^{ba} \Sigma_{ba}\right) - \Sigma_{mn} \Theta - S_m{}^d k^c S_n{}^b k^a C_{dcba},$$
$$\dot{\Theta} = -\Sigma^{ba} \Sigma_{ba} - \frac{1}{2} \Theta^2 - k^b k^a R_{ba},$$
where C_{klmn} is the Weyl tensor and R_{mn} is the Ricci tensor.
- These equations are called the Sachs optical equations. Usually, they are presented using the (Sachs) basis vectors spanning the screen surface. Here, we present them using the screen field itself.
- We see that focusing is directly produced by the trace-free part of Ricci curvature, but also is indirectly influenced by the conformal curvature through shearing.
- In practice, we do not solve these equations because shear and expansion are singular at the observation event (vertex, where all beam's rays intersect) so we can not give initial conditions for them.

Area distance

- The actual morphology of the beam's screen-section is characterized by the Jacobi field J_{mn} which is defined by the equation:

$$S_m{}^b S_n{}^a \dot{J}_{ba} = D_m{}^a J_{an}, \qquad u^a J_{na} = 0, \qquad k^a J_{na} = 0.$$

- As expected, its logarithmic on-screen derivative gives the optical deformation rate field. It is an on-screen field and in general, it is not symmetric.
- The Jacobi field encodes the Jacobi matrix of the map relating the physical separations of rays within the beam with the angular separations of these rays seen on the observer's celestial sphere.
- In particular, the determinant of the Jacobi field J is the Jacobian of the mentioned map which is the ratio of the physical area of the beam's screen-section to its observed solid angle. This enables us to define the area distance Δ from the observer as the square root of the determinant of the Jacobi field:

$$\Delta^2 = J.$$

- The determinant of the Jacobi field can be calculated with the help of the area field:

$$J = \frac{1}{A^{fe}A_{fe}} A^{db}A^{ca}J_{dc}J_{ba} = \frac{1}{2}\left(J^{b}{}_{b}J^{a}{}_{a} - J^{ab}J_{ba}\right).$$

- We can not use the definition for direct calculation of the Jacobi field because the optical deformation rate field is singular at the vertex. Instead, the propagation equation for the Jacobi field is obtained from its definition by differentiation:

$$S_{m}{}^{b}S_{n}{}^{a}\ddot{J}_{ba} = -S_{m}{}^{d}k^{c}S^{eb}k^{a}R_{dcba}J_{en}.$$

- This equation is a reminiscence of the geodesic deviation equation which holds for the (Jacobi) vectors connecting nearby rays in the beam. The factor on the right hand side of the equation is called the optical tidal field. Once we solve this equation, we can find the area distance.

Area distance

- Alternatively, we can return to the Sachs optical equations. The optical expansion rate is expressed by the determinant of the Jacobi field and the area distance as follows:
$$\Theta = \frac{1}{J}\dot{J} = \frac{2}{\Delta}\dot{\Delta}.$$
- Hence, we rewrite the Sachs optical equations into the form:
$$S_m{}^b S_n{}^a \dot{\Xi}_{ba} = -\frac{1}{\Delta^2}\left(\Xi_m{}^a \Xi_{na} - \frac{1}{2} S_{mn} \Xi^{ba} \Xi_{ba}\right) - \Delta^2 S_m{}^d k^c S_n{}^b k^a C_{dcba},$$
$$\ddot{\Delta} = -\frac{1}{2}\frac{1}{\Delta^3}\Xi^{ba}\Xi_{ba} - \frac{1}{2}\Delta k^b k^a R_{ba},$$
where we have introduced the non-singular (at the vertex) optical shear rate $\Xi_{mn} = \Delta^2 \Sigma_{mn}$.
- This system of equations can be solved directly to obtain the area distance.

Initial conditions

- In order to impose the initial conditions for the considered equations, one needs to know the relation between the optical shear rate and the Jacobi field:

$$\Xi_{mn} = -\Delta S_m{}^b S_n{}^a \left(J_{ba} \left(\frac{J^c{}_c}{\Delta} \right)^{\cdot} - J_{bc} \left(\frac{J^c{}_a}{\Delta} \right)^{\cdot} \right).$$

- Since the observation event is a vertex point for the beam's rays, the Jacobi field vanishes there:

$$J_{mn}\Big|_0 = 0.$$

- By the relationships between the respective fields, this implies that:

$$\Delta\Big|_0 = 0, \qquad \Xi_{mn}\Big|_0 = 0, \qquad \frac{\Xi_{mn}}{\Delta}\Big|_0 = 0, \qquad \frac{J_{mn}}{\Delta}\Big|_0 = \frac{\dot{J}_{mn}}{\dot{\Delta}}\Big|_0,$$

and additionally we get the identity for initial conditions for derivatives:

$$\dot{\Delta}^2\Big|_0 = \frac{1}{2}\left(\dot{J}^b{}_b \dot{J}^a{}_a - \dot{J}^{ab} \dot{J}_{ba} \right)\Big|_0.$$

Initial conditions

- Because of its properties, we shall impose the initial conditions for only two of the components of the optical shear rate. Likewise, we give the initial conditions for four of the components of the Jacobi field.
- The initial condition for the derivative of the area distance comes from the physical requirement that in the vicinity of the vertex the distance should correspond to the path traveled by the photon with respect to the observer. This gives:

$$\dot{\Delta}\Big|_0 = -\omega\Big|_0 \equiv -\omega_0.$$

- The minus sign is due to the choice that the wave vector is future oriented.
- The initial conditions for the components of the derivative of the Jacobi field are subjected only to the mentioned identity and otherwise, they are unrestricted.

Redshift dependence

- The area distance is determined by the derived equations as a function of the affine parameter along the given geodesic x^n. Since the affine parameter is not observable, it is useful to introduce the redshift Z as a new independent variable:

$$Z = \frac{\omega}{\omega_0} - 1.$$

- Its differential connection with the affine parameter reads:

$$k^a \partial_a = -\omega_0 \frac{1}{\mathfrak{B}} l^a \partial_a, \qquad \mathfrak{B} = l^b l^a \nabla_b u_a,$$

  where we have introduced the vector $l^n = \frac{dx^n}{dZ}$. This relation could be obtained by the calculation of the derivative $\dot{Z}$.

- Accordingly, the second derivative reads:

$$k^b \partial_b \left(k^a \partial_a \right) = \omega_0^2 \frac{1}{\mathfrak{B}^2} l^b \partial_b \left(l^a \partial_a \right) + \omega_0^2 \frac{\mathfrak{C}}{\mathfrak{B}^3} l^a \partial_a, \qquad \mathfrak{C} = l^c l^b l^a \nabla_c \nabla_b u_a.$$

Redshift dependence

- The geodesic equation in the redshift dependent form can be written as:

$$l'_n = -\frac{\mathfrak{C}}{\mathfrak{B}} l_n,$$

where the prime denotes $X' \equiv l^a \nabla_a X$.

- This equation enables us to find the geodesic directly as a function of the redshift.
- In this approach the circular frequency is absent from the equations but can be calculated as:

$$\omega = \omega_0 \frac{\mathfrak{A}}{\mathfrak{B}}, \qquad \mathfrak{A} = l^a u_a.$$

- To be consistent with the derivation, while specifying the initial conditions for the above geodesic equation, one should assure that:

$$\mathfrak{B}\Big|_0 = \mathfrak{A}\Big|_0,$$

which sets the initial normalization for the vector l_n.

Redshift dependence

- The system of equations for the area distance in the redshift dependent form reads:

$$S_m{}^b S_n{}^a X'{}_{ba} = -\mathfrak{B}\frac{1}{\Delta^2}\left(X_m{}^a X_{na} - \frac{1}{2}S_{mn}X^{ba}X_{ba}\right) - \frac{1}{\mathfrak{B}}\Delta^2 S_m{}^d l^c S_n{}^b l^a C_{dcba},$$

$$\Delta'' = -\frac{\mathfrak{C}}{\mathfrak{B}}\Delta' - \frac{\mathfrak{B}^2}{2}\frac{1}{\Delta^3}X^{ba}X_{ba} - \frac{1}{2}\Delta l^b l^a R_{ba},$$

  where we have introduced the scaled optical shear rate $X_{mn} = -\frac{1}{\omega_0}\Xi_{mn}$.
- The initial condition for the derivative of the area distance with respect to the redshift takes the form:

$$\Delta'\Big|_0 = \mathfrak{B}\Big|_0.$$

- Finally, the equation for the Jacobi field as a function of the redshift reads:

$$S_m{}^b S_n{}^a J''{}_{ba} = -\frac{\mathfrak{C}}{\mathfrak{B}}S_m{}^b S_n{}^a J'{}_{ba} - S_m{}^d l^c S^{eb} l^a R_{dcba} J_{en},$$

  where we have used that $\mathfrak{B}' = -\mathfrak{C}$.
- The relation between initial derivatives holds in the form:

$$\Delta'^2\Big|_0 = \frac{1}{2}\left(J'^b{}_b J'^a{}_a - J'^{ab}J'{}_{ba}\right)\Big|_0.$$

- These equations could be easily expanded and implemented.

Summary

- We have presented two complementary approaches for determining the area distance in cosmology: the optical fields approach and the Jacobi field approach.
- The two approaches offer two theoretically equivalent but numerically distinct and independent methods for calculating the area distance. They could be chosen depending on applications.
- We have also developed a consistent four-dimensional formulation for both of these approaches. Additionally, we have given this formulation in a covariant redshift-dependent form.
- The matrix-based formulation is undoubtedly more efficient numerically but the four-dimensional formulation is more appealing theoretically.