# Utilizing cosmological post-Newtonian approximation for the PPN formalism

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## 1. How to accelerate the cosmic expansion?

Since the end of the 20th century,
when the observations of type Ia SNe were carried out,
one of the most important unsolved problems
in cosmology is:

**What is the cause of the acceleration of the cosmic expansion**

Theoretically possible candidates for the solution:

- Exotic matter or scalar field
  - quintessence, phantom, k-essence, DBI, Chaplygin gas, condensate, …
- Gravity modification
  - scalar-tensor theories, **f(R) gravity**, Horndeski, massive gravity, Proca, …
- Inhomogeneity
  - backreaction, void, ...

# 2. Post-Newtonian parameter(s)

- Defined in Parametrized Post-Newtonian (PPN) formalism
- Derived from weak-field approximation in generalized gravity theories
- Useful in exploring the deviation from Einstein's GR

$$\gamma,\ \beta,\ \alpha_1,\ \alpha_2,\ \alpha_3, \ldots$$

$$ds^2 = -(1+2\Phi)dt^2 + a(t)^2(1+2\Psi)\delta_{ij}dx^i dx^j$$

$$\gamma := -\Psi/\Phi$$

- Appears in the lowest order
- $\gamma = 1$ in Einstein's GR

## Experiment by Cassini

$$\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$$ (Bertotti et al 2003)

In modified gravity for small effective mass $M$ of a scalar field

$$\gamma \approx 1/2$$ (Chiba 2003; Olmo 2005; Chiba, Smith, Erickcek 2007; etc)

Design the theory in such a way that $M$ is small at the cosmological scales, and $M$ becomes large at the local scales

⇨ **Gravitational screening mechanism**

e.g. chameleon mechanism

$\gamma$ approaches unity at the local scales

(Capozziello & Tsujikawa 2008)

# Examples of chameleon f(R) models

$$f(R) = R - \lambda R_c \frac{(R/R_c)^{2n}}{(R/R_c)^{2n}+1}$$ (Hu & Sawicki 2007)

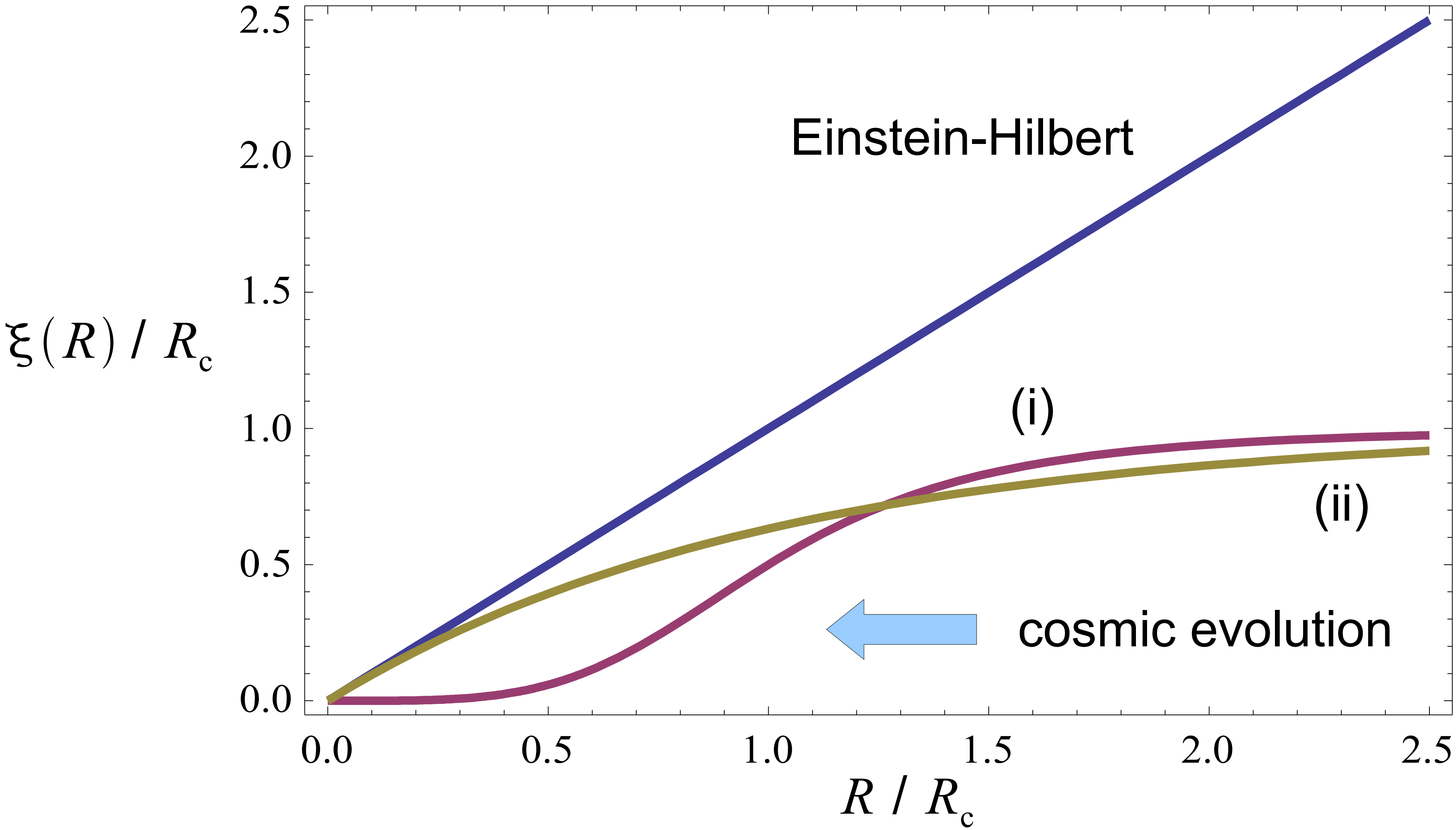
$$f(R) = R - \lambda R_c \left[1 - \left(1 + \frac{R^2}{R_c^2}\right)^{-n}\right]$$ (Starobinsky 2007)

$$f(R) = R - \lambda R_c \left(1 - e^{-R/R_c}\right)$$

$$\left( R_c \sim H_0^2 \right)$$

$$f(R) = R - \xi(R), \quad \xi(0) = 0$$
$$\xi(R \gg R_c) \to \text{const}$$

(i) $\xi(R) = \lambda R_c \frac{(R/R_c)^{2n}}{(R/R_c)^{2n}+1} \quad (n=2)$ (ii) $\xi(R) = \lambda R_c \left(1 - e^{-R/R_c}\right)$



# 3. Cosmological post-Newtonian approximation

(Futamase 1988, 1996; Shibata & Asada 1995; Takada & Futamase 1998, 1999)

Two small parameters $\epsilon := \frac{V_{\rm pec}}{c}, \quad \kappa := \frac{al}{l_{\rm H}}$

At the local scales $\kappa^2 \ll \epsilon^2 \ll 1$

e.g. the galactic scale $\epsilon \sim 10^{-3}, \quad \kappa \sim 10^{-5}$

$$\frac{1}{a^2}\nabla^2\Phi = O\left(H^2 \epsilon^2/\kappa^2\right) = O\left(H^2 \cdot 10^4\right)$$

Amplitude of density perturbations

**It can be applied even if curvature and density is large compared to the background**

## Metric

$$ds^2 = -(1+2\Phi)\,dt^2 + a(t)^2(1+2\Psi)\,\delta_{ij}\,dx^i\,dx^j$$

## Curvature $R \approx R_{\rm H} + \mathcal{R}$

$$R_{\rm H} := 6\left(2H^2 + \partial_t H\right) \quad \ll \quad \mathcal{R} := -\frac{2}{a^2}\nabla^2\left(\Phi + 2\Psi\right)$$

Cf In conventional approaches, Taylor expansion has been used for $f(R)$ gravity:

$$f(R) \approx f(R_0) + f'(R_0)R_1 \qquad \text{for} \quad R = R_0 + R_1$$

$$R_0 \gg R_1$$

# 4. Demonstration in $f(R)$ gravity

Action: $$S = \frac{1}{16\pi G}\int f(R)\sqrt{-g}\, d^4x \quad + S_{\mathrm{m}}$$

Field equations:

$$F(R)\,R_{\mu\nu} - \frac{1}{2} f(R)\, g_{\mu\nu} - F(R)_{;\mu\nu} + g_{\mu\nu} F(R)^{;\alpha}_{\ ;\alpha} = 8\pi G\, T^{(\mathrm{m})}_{\mu\nu}$$

$$F(R) := f'(R)$$

In the case of Einstein's general relativity with $\Lambda$

$$f(R) = R - 2\Lambda\,, \quad F(R) = 1$$

In the case of $f(R)$ gravity

$$f(R) = R - \xi(R)\,, \quad F(R) = 1 - \xi'(R)$$

The field equations

$$\frac{F}{a^2}\nabla^2(\Phi-\Psi)+\frac{1}{a^2}\delta^{ij}F_{,i}(\Phi-\Psi)_{,j}=8\pi G\rho\,,$$

$$\frac{F}{a^2}(\Phi+\Psi)_{,ij}+\frac{1}{a^2}(F_{,ij}-F_{,i}\Psi_{,j}-F_{,j}\Psi_{,i})=0\,,$$

(ignored the terms including the time derivative)

For the spatial derivatives of $F$

$$F_{,i}=R_{,i}\partial_R F\approx\mathcal{R}_{,i}\partial_R F\,,$$

$$F_{,ij}=R_{,ij}\partial_R F+R_{,i}R_{,j}\partial_R^2 F\approx\mathcal{R}_{,ij}\partial_R F+\mathcal{R}_{,i}\mathcal{R}_{,j}\partial_R^2 F\,.$$

↓

$$\frac{F}{a^2}\nabla^2(\Phi+\Psi)+\frac{1}{a^2}\left(\nabla^2\mathcal{R}\partial_R F+(\nabla\mathcal{R})^2\partial_R^2 F\right)-\frac{2}{a^2}\nabla\mathcal{R}\cdot\nabla\Psi\partial_R F=0\,.$$

# Order-of-magnitude estimation

$$\frac{F}{a^2}\nabla^2(\Phi+\Psi)+\frac{1}{a^2}\left(\nabla^2\mathcal{R}\partial_R F+(\nabla\mathcal{R})^2\partial_R^2 F\right)-\frac{2}{a^2}\nabla\mathcal{R}\cdot\nabla\Psi\partial_R F=0\,.$$

$$(1-\zeta_s)H_0^2\frac{\epsilon^2}{\kappa^2}|1-\gamma|,\quad H_0^2\frac{\epsilon^2}{\kappa^4}|(1-2\gamma)\zeta_{ss}|,\quad H_0^2\frac{\epsilon^4}{\kappa^6}(1-2\gamma)^2|\zeta_{sss}|,\quad H_0^2\frac{\epsilon^4}{\kappa^4}|(1-2\gamma)\zeta_{ss}|,$$

$$f(R)=R-\lambda R_c\zeta(R)\,,\quad F(R)=1-\lambda\zeta_s$$

$$s:=R/R_c\qquad \zeta_s:=\partial\zeta/\partial s$$

$$Ml\gg 1$$

$$|\zeta_{ss}|\le\kappa^2|\gamma-1|\sim 10^{-14},\qquad |\zeta_{sss}|\le\frac{\kappa^4}{\epsilon^2}|\gamma-1|\sim 10^{-18}$$

# Expression for the post-Newtonian parameter:

$$\gamma := -\frac{\Psi}{\Phi} = \frac{\displaystyle\int \frac{\mathrm{d}^3\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}\left[16\pi G\rho(\boldsymbol{x}') + F\mathcal{R}(\boldsymbol{x}')\right]}{\displaystyle\int \frac{\mathrm{d}^3\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}\left[32\pi G\rho(\boldsymbol{x}') - F\mathcal{R}(\boldsymbol{x}')\right]}$$

$$\mathcal{R} := -\frac{2}{a^2}\nabla^2\left(\Phi + 2\Psi\right)$$

$$\nabla^2\mathcal{R} - M^2\mathcal{R} + \mu\left(\nabla\mathcal{R}\right)^2 = -\frac{8\pi G M^2\rho}{F}$$

$$M^2 := F a^2/3F', \qquad \mu := F''/F'$$

(effective mass of the scalaron squared)

# Spherically symmetric case

Top-hat model $\rho(r) = \begin{cases} \rho_\star & (0 \leq r < r_\star) \\ 0 & (r \geq r_\star) \end{cases}$

$$\frac{\partial^2 \mathcal{R}}{\partial r^2} + \frac{2}{r}\frac{\partial \mathcal{R}}{\partial r} - M^2 \mathcal{R} + \mu \left( \frac{\partial \mathcal{R}}{\partial r} \right)^2 = -\frac{8\pi G M^2}{F} \rho(r)$$

Solution:

$$\mathcal{R}(r) = \begin{cases} \mathcal{R}_\star + \frac{1}{\mu} \ln(1 + \frac{2A_1}{r} \sinh Mr) & (0 \leq r < r_\star) \\ \frac{1}{\mu} \ln(1 + \frac{B_2}{r} \mathrm{e}^{-Mr}) & (r \geq r_\star) \end{cases}$$

$$A_1 = \frac{1 + Mr_\star}{2M} \left( \mathrm{e}^{-\mu \mathcal{R}_\star} - 1 \right) \mathrm{e}^{-Mr_\star}$$

$$B_2 = \frac{1}{M} \left( \mathrm{e}^{\mu \mathcal{R}_\star} - 1 \right) \left( Mr_\star \cosh Mr_\star - \sinh Mr_\star \right)$$

Taking the leading order in $Mr_* >> 1$ ,

$$\gamma \approx \frac{1 + \dfrac{2}{Mr_\star}\dfrac{\cosh \mu\mathcal{R}_\star - 1}{\mu\mathcal{R}_\star}}{1 - \dfrac{2}{Mr_\star}\dfrac{\cosh \mu\mathcal{R}_\star - 1}{\mu\mathcal{R}_\star}}$$

$$\mu := \frac{\partial_R^2 F}{\partial_R F} = \frac{\zeta_{sss}}{R_c \zeta_{ss}} \ ; \qquad \mu\mathcal{R}_\star = \frac{\mathcal{R}_\star \zeta_{sss}}{R_c \zeta_{ss}}$$

can be huge, depending on models

# 5. Conclusion and outlook

- Improved method for obtaining PPN parameters has been proposed using ***cosmological*** post-Newtonian approximation

- *Cosmological* post-Newtonian approximation can be applied to high-curvature and -density regions, and thus useful for the case in which the gravitational screeing mechanism would take place

- More stringent constraints obtained than known ones

- Higher-order approximation for other post-Newtonian parameters