Einstein clusters as models of inhomogeneous spacetimes

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Task 1: find a toy-model to study the effect of small scale inhomogeneities

- an exact 'inhomogeneous' solution to Einstein equations
- reasonable matter content and an equation of state
- freedom in a choice of the density profile $\rho(x^\mu)$

Task 2: Gedankenexperiment

- place observer in the inhomogeneous spacetime
- let him/her to construct a simplified homogeneous model based on limited observational knowledge
- explore possible artefacts of simplification

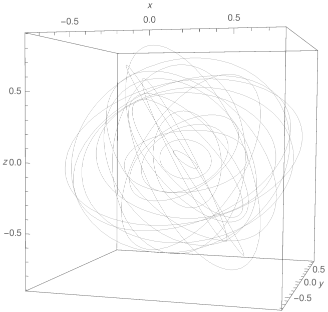
Problems:

- dust: Lemaître–Tolman-Bondi — develops shell crossing singularities
- ‘perturbed’ inner Schwarzschild solution — in general unstable and ‘strange’ equation of state

The Great Hercules Cluster

credit: ESA/NASA/HST

Einstein cluster (centrifugally supported)



Albert Einstein,
On a Stationary System With Spherical Symmetry Consisting of Many Gravitating Masses, Annals of Mathematics, 40(4):922–936, 1939

The essential result of this investigation is a clear understanding as to why the “Schwarzschild singularities” do not exist in physical reality.

metric

- stationary spherically symmetric solution

$$g = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \ ,$$

  where ν, λ are functions of r only

- the nonvanishing energy-momentum tensor components (‘averaged’ $T^{\mu}_{\nu} = \frac{n(x)}{m} p^{\mu} p_{\nu}$, where $n(x)$ is a number density and m is a particle mass)

$$T^t_{\ t} = -\rho \ , \ \ T^{\theta}_{\ \theta} = T^{\varphi}_{\ \varphi} = p \ ,$$

  where $\rho = \rho(r)$ is the energy density and $p = p(r)$ is a tangential pressure

metric

- metric functions

$$\nu = \int \frac{dr}{r}\left(\frac{1}{\mu} - 1\right) ,$$
$$\lambda = \ln\frac{1}{\mu} ,$$

- auxilarly function

$$\mu = 1 - \frac{8\pi}{r}\int \rho r^2 dr ,$$

metric

- tangential pressure

$$p = \frac{r\nu'}{4}\rho$$

- radial stability condition

$$\begin{aligned} &0 < r\nu'/2 < 1 \ , \\ &r\nu'' - r(\nu')^2 + 3\nu' > 0 \ . \end{aligned}$$

models

- homogeneous (effective) model — constant density $\rho(r) = \rho_A$

$$\begin{aligned} \nu_A &= -\ln\sqrt{1-a_A r^2} + 3\ln\sqrt{1-a_A R_A^2}\,, \\ \lambda_A &= -\ln\left(1-a_A r^2\right)\,, \end{aligned}$$

$$g_A = -\frac{\sqrt{1-a_A R_A^2}^{\,3}}{\sqrt{1-a_A r^2}}dt^2 + \frac{1}{1-a_A r^2}dr^2 + r^2 d\Omega^2\,,$$

models

- inhomogeneous model — $\rho(r) = 2\rho_0 \cos^2(2\pi r/l + \pi/4)$

$$\nu_l = -\int \frac{dr}{r} \frac{\mu_l}{\mu_0} \frac{1}{(\mu_0 + \mu_l)}$$

$$\begin{aligned} \mu_0 &= 1 - ar^2 \,, \\ \mu_l &= \frac{3al}{32\pi^3} \left[-\frac{l^2}{r} + 4\pi l \sin(\frac{4\pi r}{l}) + (\frac{l^2}{r} - 8\pi^2 r) \cos(\frac{4\pi r}{l}) \right] \end{aligned}$$

$$g = -\frac{\sqrt{1 - aR^2}^3}{\sqrt{1 - ar^2}} e^{\nu_l} dt^2 + \frac{1}{1 - ar^2 + \mu_l} dr^2 + r^2 d\Omega^2 \,.$$

no backreaction in the Green–Wald framework

- we define $h(l) = g(l) - g^{(0)}$
- the non-zero components of $h_{\alpha\beta}$ for small l are

$$h_{tt} \approx \frac{\nu_l}{\sqrt{1 - ar^2}} \; , \quad h_{rr} \approx -\frac{\mu_l}{(1 - ar^2)^2} \; .$$

- ν_l and μ_l and the first derivatives of ν_l vanish in the high-frequency limit $l \to 0$
- $\partial_r \mu_l$ is not pointwise convergent, but remains bounded
- the effective energy-momentum tensor vanishes
- this probably remains true for the whole class: one of the Einstein equations depends on $(\nu')^2$

$$r\nu'' + r(\nu')^2 + 2\nu' = 8\pi r(1 + r\nu')^2 \rho \; .$$

  non-zero weak limit of this terms implies that λ is not pointwise convergent which contradicts the Green-Wald assumptions

Gedankenexperiment

- the inhomogeneous spacetime is defined by three parameters
  - an average energy density ρ_0,
  - a size — an area radius R,
  - a frequency of inhomogeneities l or a number n of inhomogeneous regions (such that $nl = R$).
- the effective spacetime depends on analogous two parameters ρ_A, R_A
- we assume that 'observations' allow to determine
  - active graviational mass of the system M (satellites!)
  - blueshift z of stars at the boundary of configuration

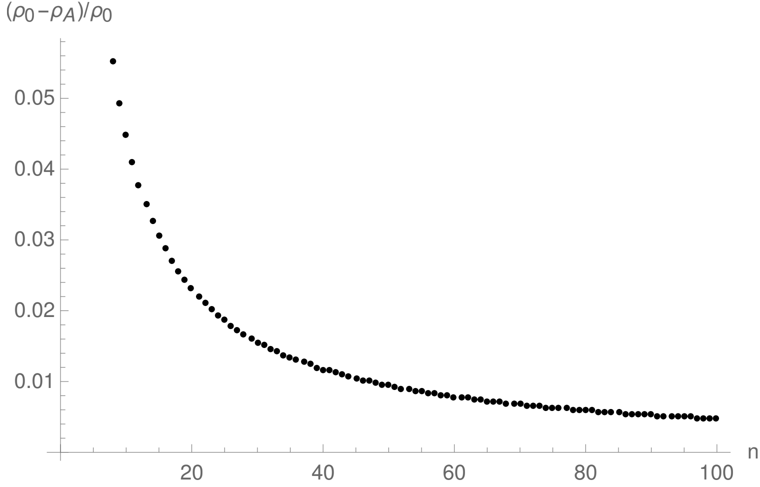
Gedankenexperiment

- unknown parameters of the effective spacetime, namely, ρ_A, R_A in terms of 'observational parameters' M, z and parameters of the inhomogeneous spacetime ρ_0, R, n are given by

$$
\begin{aligned}
\rho_A &= \frac{3}{32\pi}\frac{\left((-z)(2+z)[(2+z)z+2]\right)^3}{M^2} = \frac{3}{32\pi}\frac{(1-e^{-2\nu(R)})^3}{\left[(4/3\pi+1/n)R^3\rho_0\right]^2} \\
R_A &= \frac{2M}{-z(2+z)[(2+z)z+2]} = 2\frac{(4/3\pi+1/n)R^3\rho_0}{1-e^{-2\nu(R)}} \ .
\end{aligned}
$$

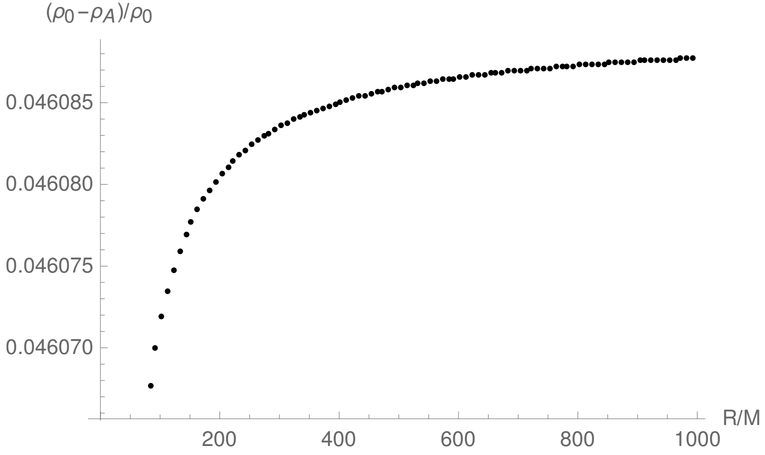
Gedankenexperiment

- the most compact metastable configuration $R = 3M$ ($z \simeq -0.23926$) — a relativistic system



Gedankenexperiment

- dependence of the effect on the compactness parameter R/M (for $n = 10$)



Gedankenexperiment

- Milky Way (for fun!)
  - the mass is $M = 10^{12} M_\odot = 1.477 \times 10^{15} m$
  - the radius $R = 400000 ly = 3.784 \times 10^{21} m$
  - the compactness parameter $R/M = 2.563 \times 10^6$
  - the Schwarzschild radius is one order smaller than stellar distances $2M = 0.312 ly$
  - the energy density for the system compressed million times to the minimal configuration $R = 3M$ would be $5.45 \times 10^{-6} kg/m^3$ which qualifies as a high vacuum for Earth standards
  - if the local clustering scale is assumed to be $l \approx 1 kpc$ (the size of satellite dwarf galaxies), then $n \approx 40$
- for these parameters the inhomogeneity effect is small $(\rho_0 - \rho_A)/\rho_0 \approx 1\%$

Summary

- a new stationary toy-model to study the inhomogeneity effect
- a large freedom in setting the density $\rho(r)$
- an exact solution (one of metric functions is given in the integral form)