Monograph: Shape of the Universe

B.F. Roukema

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  - ◆ verbal averaging: can we do better?

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  - standard model: density perturbations (anisotropy)
  - scalar (GR) averaging: statistically homogeneous spatial slices
- within this model, what is the shape of the Universe?

verbal averaging

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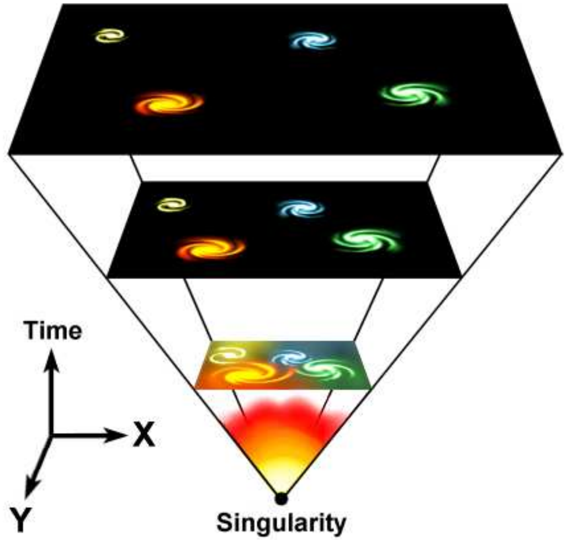
  3. assume that $(M, \mathbf{g})$ remains unchanged if we add density perturbations to an early time slice

verbal averaging

- w:Comoving coordinates

verbal averaging



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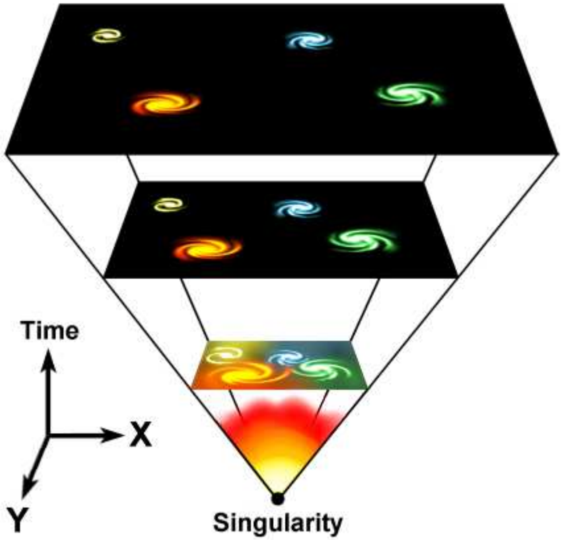

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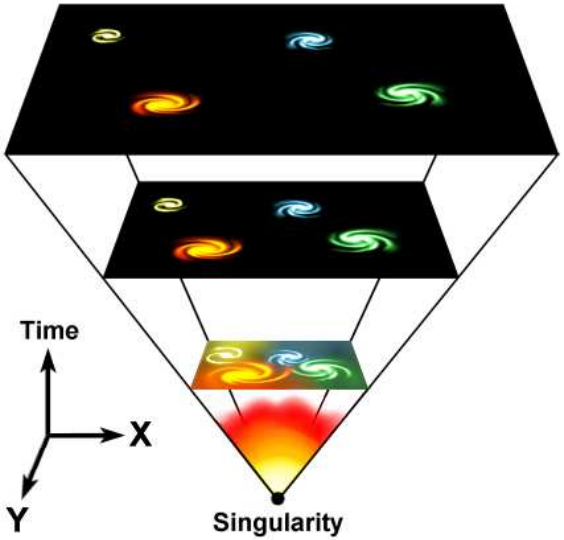
verbal averaging



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- spherical coordinates for spatial slice

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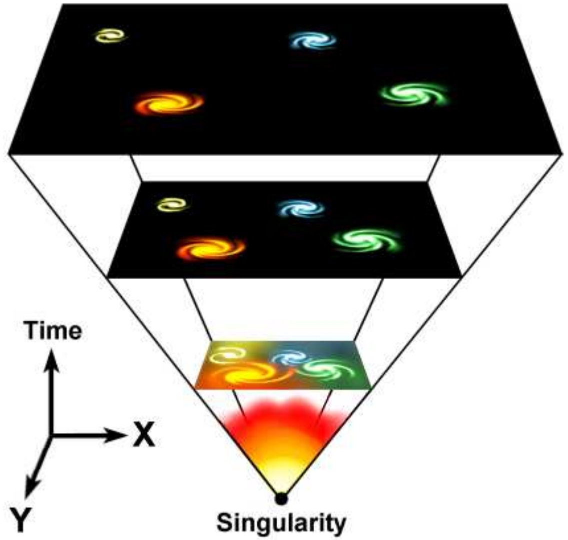

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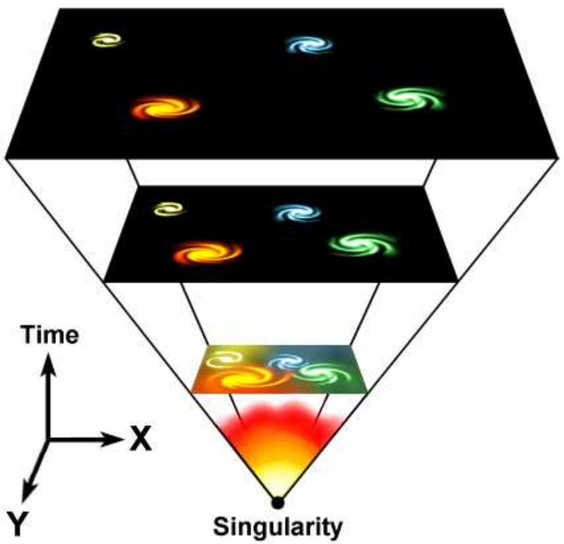

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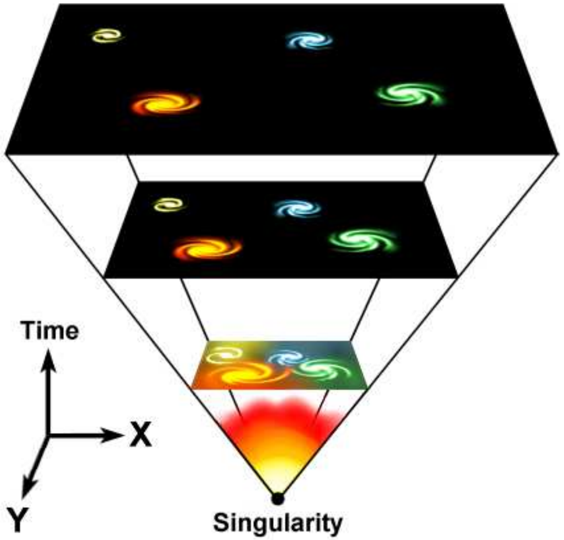

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- universe is static in comoving coordinates (r, θ, ϕ)

FLRW metric

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- w:  w:
  w:Howard Percy Robertson
  w:Arthur Geoffrey Walker

FLRW metric

$$ds^2 = -dt^2 + \dots$$

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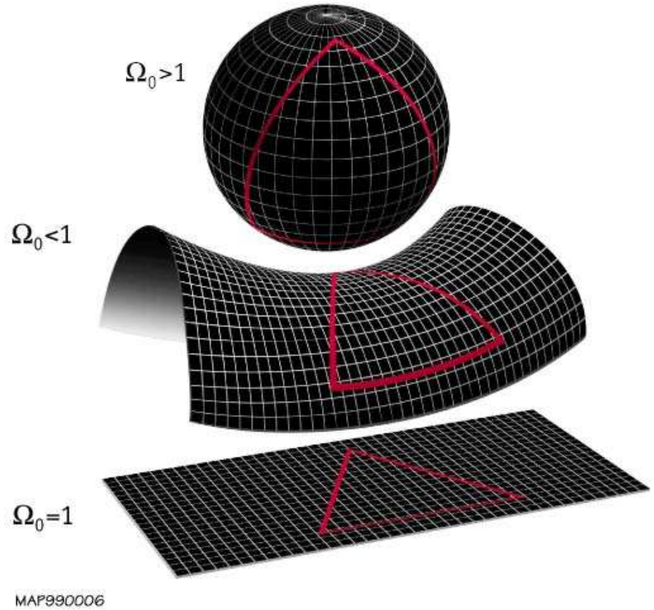

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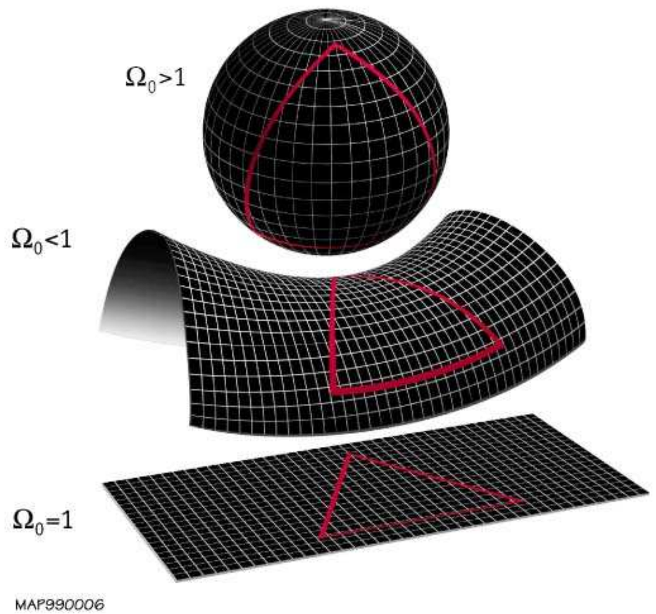

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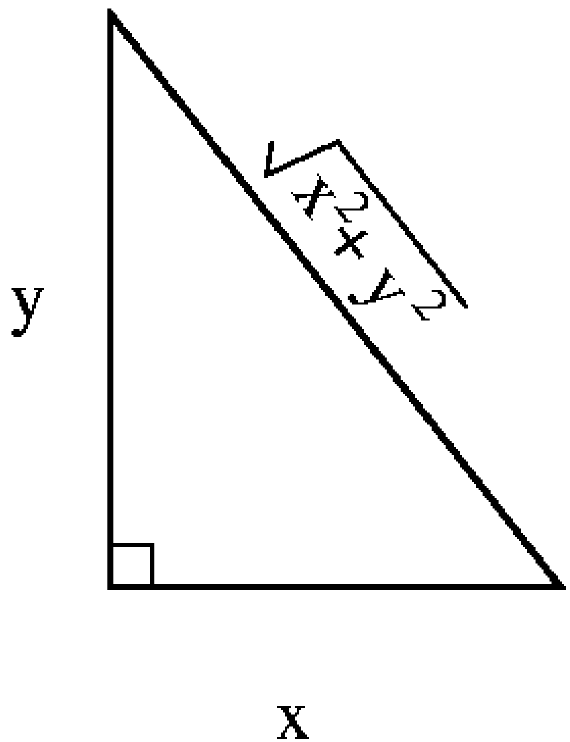
for a comoving radius of curvature R_C and curvature of sign k

curvature

- on a spatial slice (fixed value of t):



$$k = 0$$

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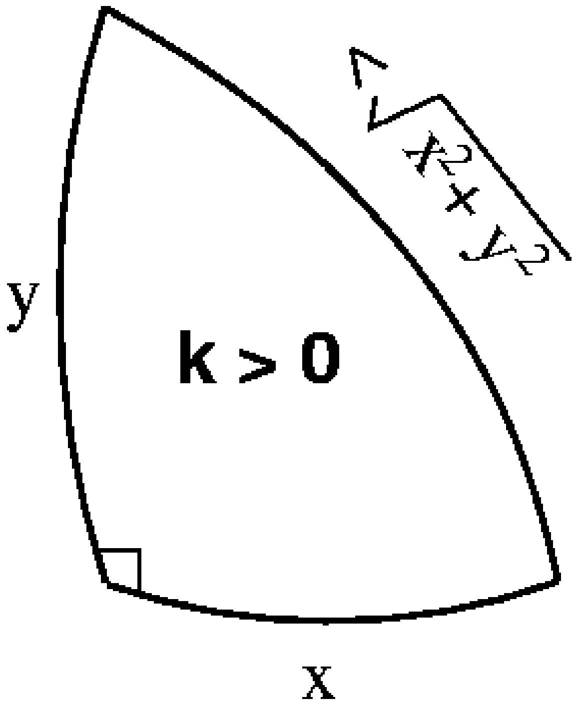

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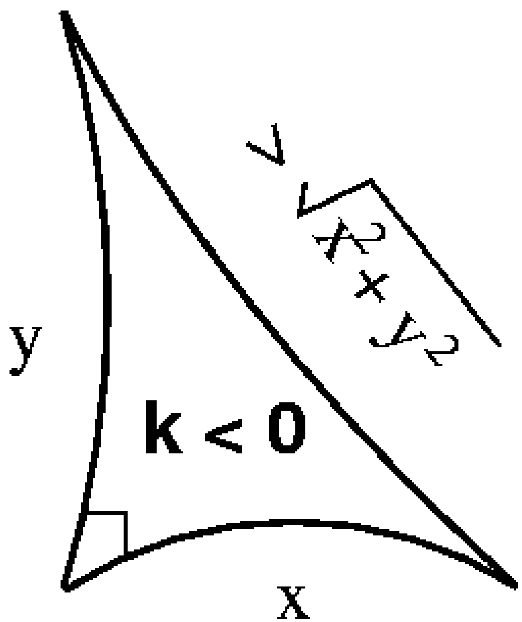

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2D curvature intuition: $k > 0$

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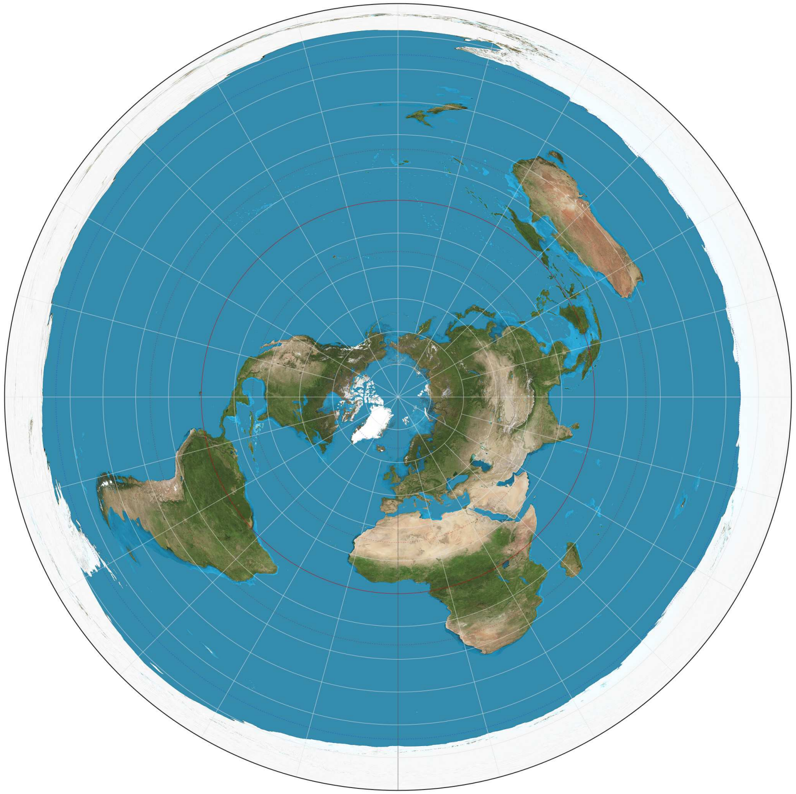

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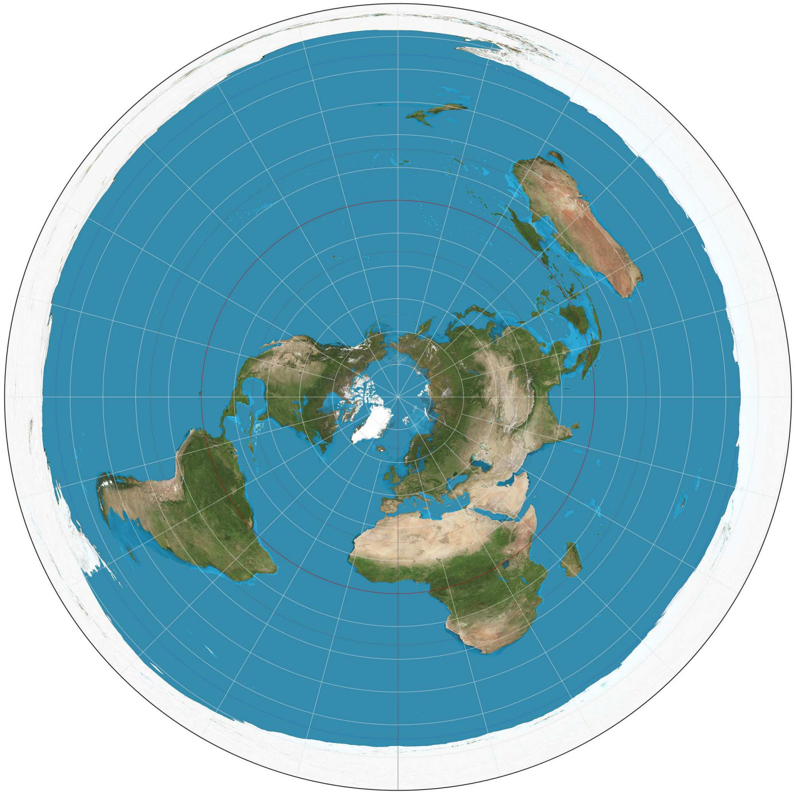

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- intuition switch: S^2 easier vs S^3 more physical

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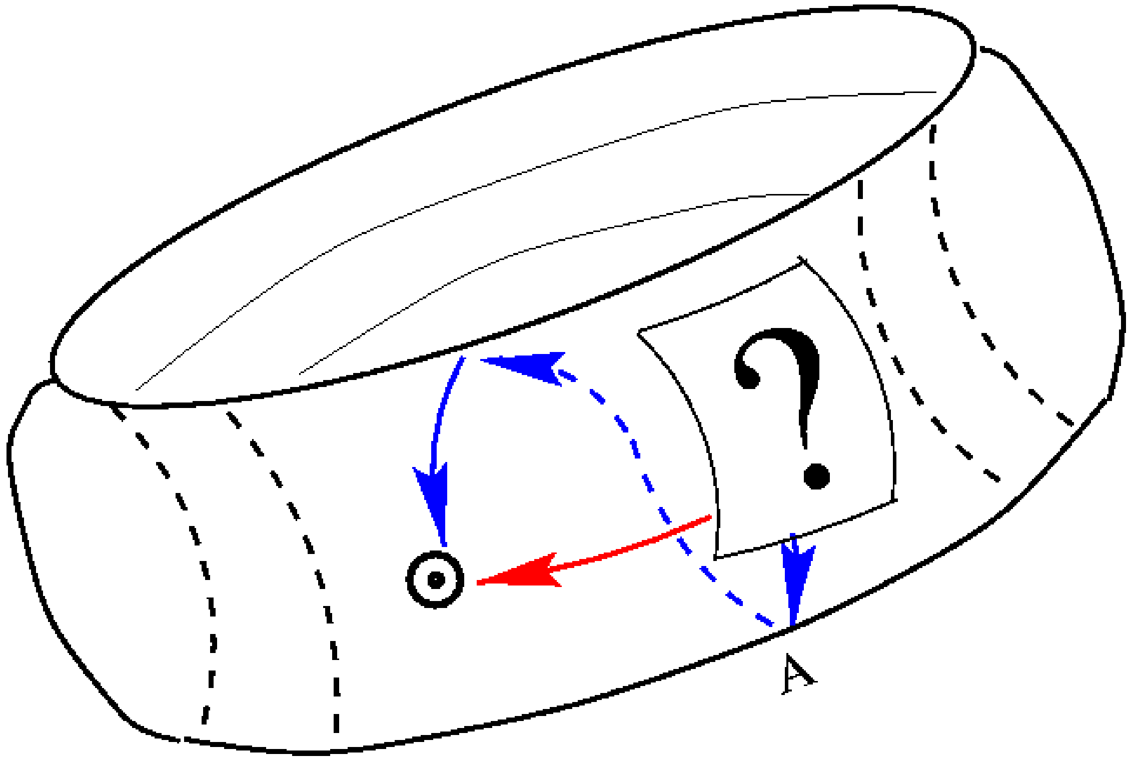

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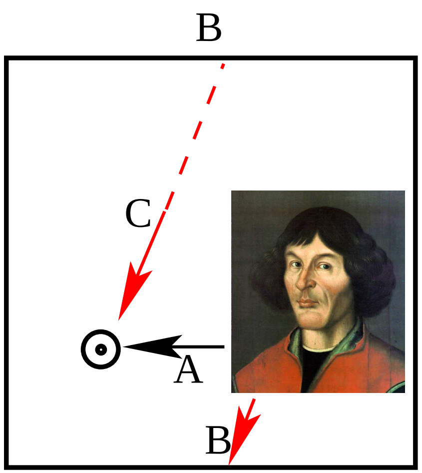

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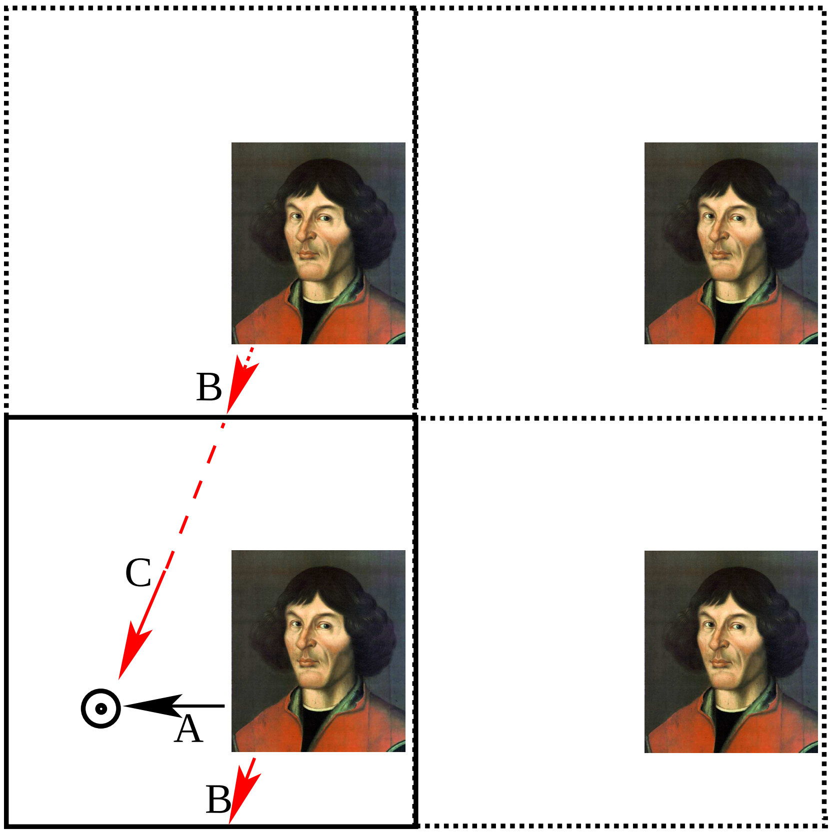

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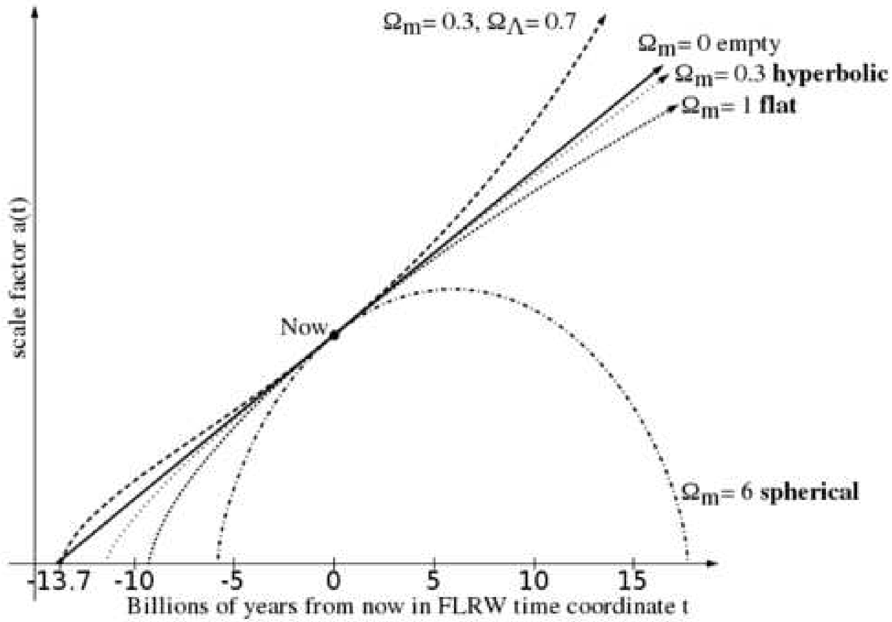

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  (Defn: $a_0 := 1$)

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- $$1 + z = \frac{1}{a_{\rm em}}$$

  (Defn of redshift z)

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- radiation density: $E = h\nu \Rightarrow \rho_{\mathrm{r}} \propto a^{-4} = (1+z)^4$

Black body: COBE (~ 1992)

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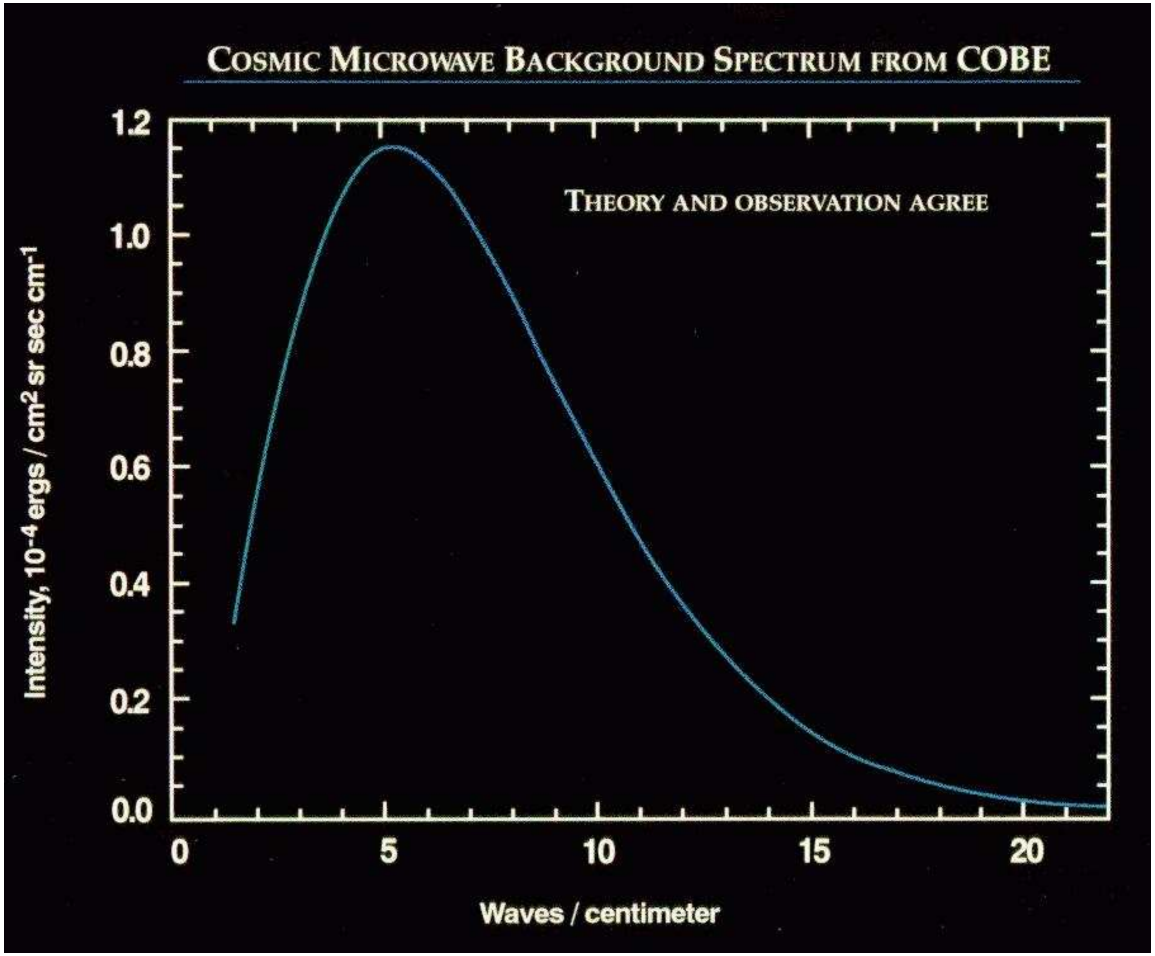

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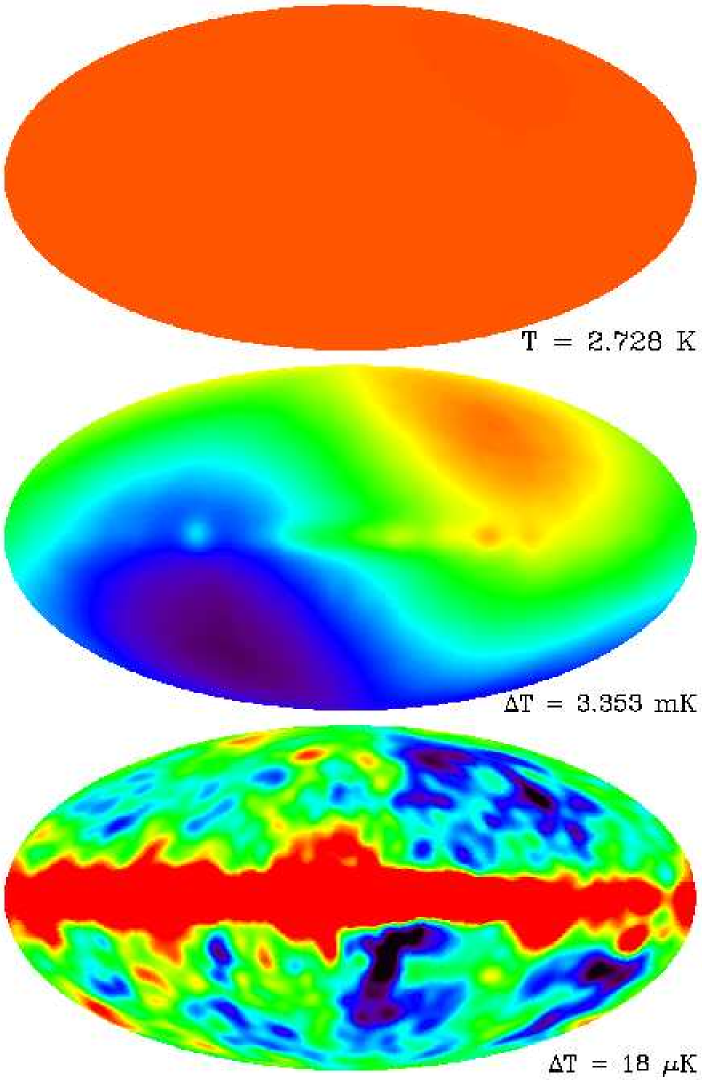
- COBE /DMR (Differential Microwave Radiometer)



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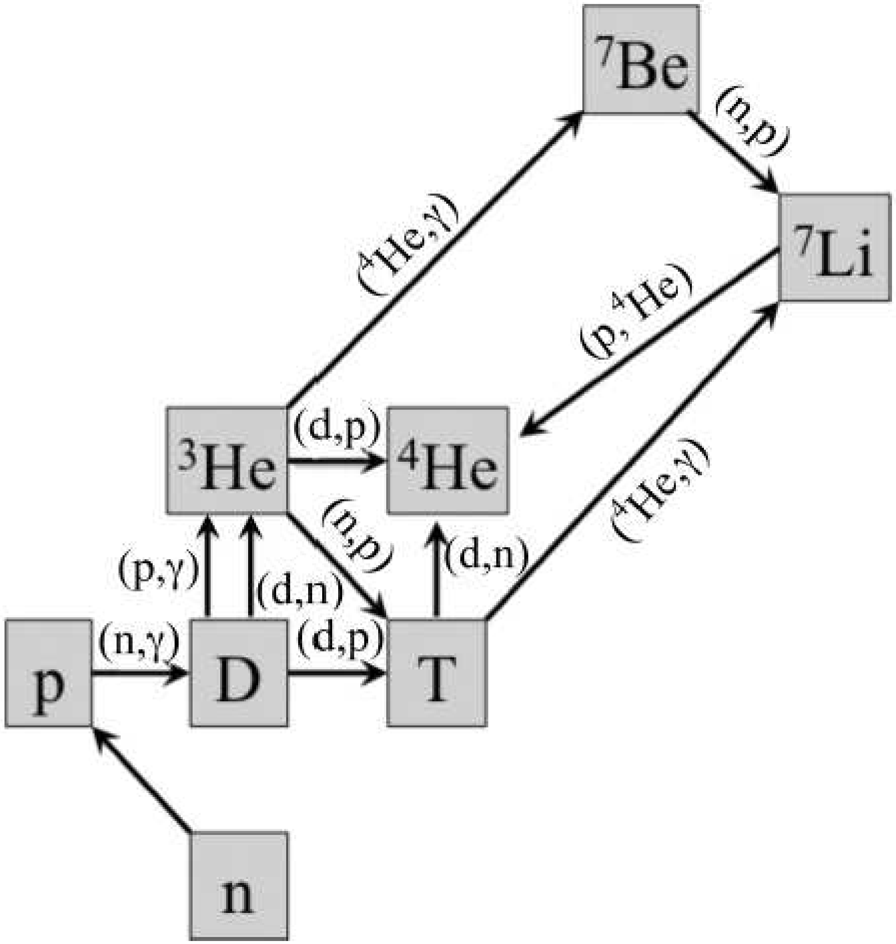

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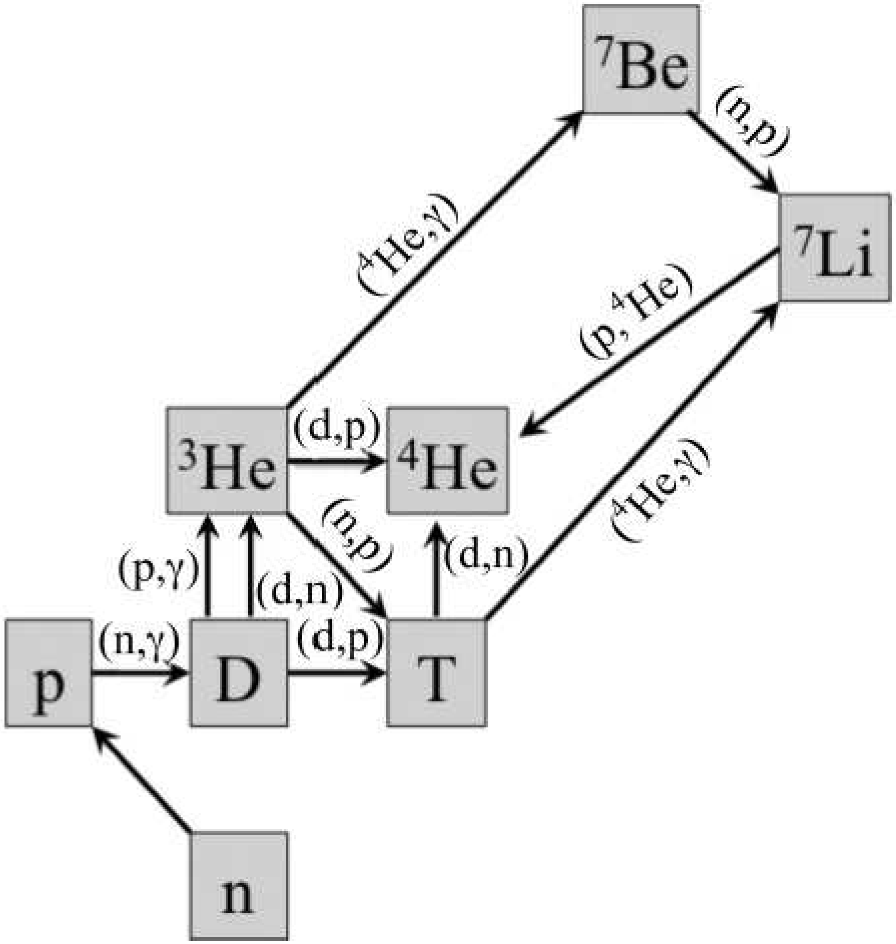

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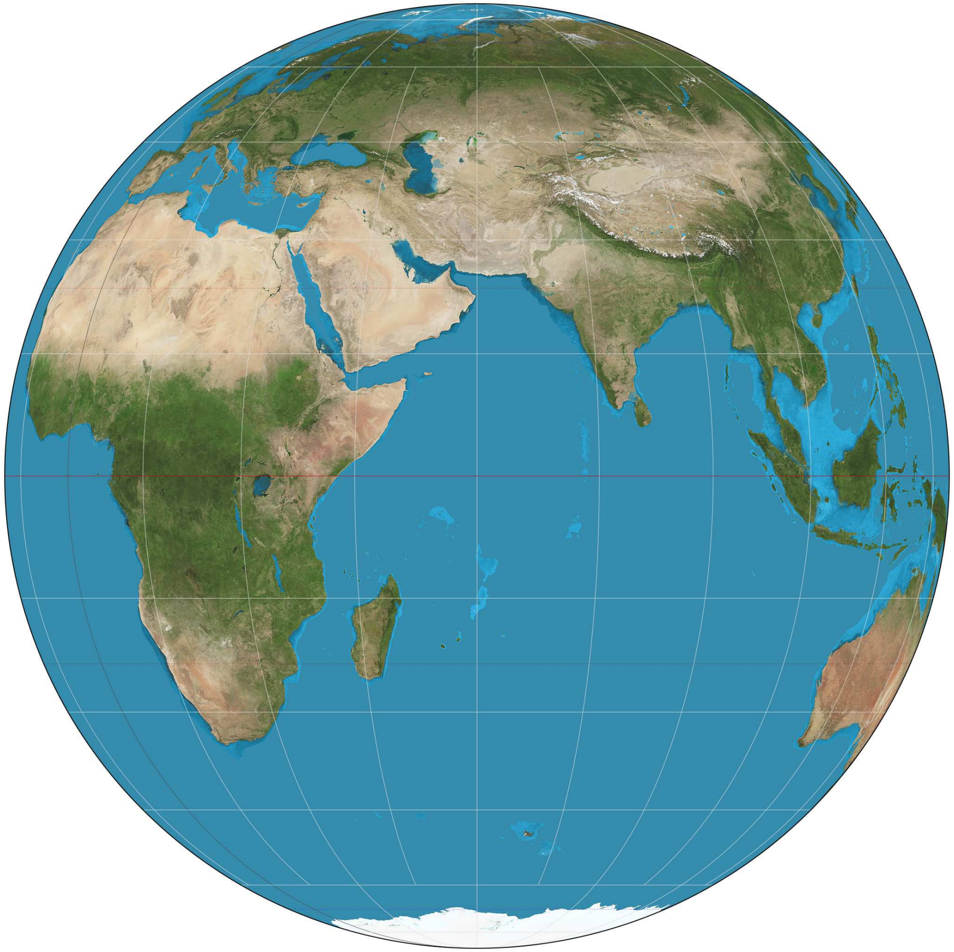

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} = 8\pi\mathbf{T}$ and simplify: http://cosmo.torun.pl/Cosmo/FLRWEquationsGR

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- Friedmann Eqn: $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3}$$

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- $$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \cos^2\theta d\phi^2)\right]$$
- universe content: $\mathrm{diag}(\mathbf{T}) = (-\rho, p, p, p)$
- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} = 8\pi\mathbf{T}$ and simplify: http://cosmo.torun.pl/Cosmo/FLRWEquationsGR
- Friedmann Eqn: $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3}$$
- acceleration Eqn: $$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G\,(\rho + 3\,p/c^2)}{3}$$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}}$$

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- Friedmann Eqn: $\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a^3} - \frac{c^2\,k}{a^2}}$

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- $k = 0$ case: $\dot{a}^2 \propto a^{-1}$

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- $k = 0$ case: $(2/3)a^{3/2} \propto t$

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- convenient conversion: 1 km/s ≈ 1.04 kpc/Gyr ≈ 1 kpc/Gyr

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- 1980's: $H_0 \approx 50$ or 100 km/s/Mpc

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- 1980's: $H_0 \approx 0.05$ or 0.1 Gyr^{-1}

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- 1980's: $H_0 \approx 0.05$ or 0.1 Gyr^{-1} $\Rightarrow t_0(\mathrm{EdS}) \approx 13.0$ or 6.5 Gyr, resp.

FLRW: ρ_{crit}

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  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$

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  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$ flat
  - $\rho_{\rm m0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0$

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Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

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Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter

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Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm m} + \Omega_k}$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
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  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm tot} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

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  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\rm tot} :=$

FLRW: $\rho_{\rm crit}$

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- $\Omega_{\rm tot} := \Omega_{\rm m} + \Omega_{\rm r} +$

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  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\rm tot} := \Omega_{\rm m} + \Omega_{\rm r} + \Omega_\Lambda$

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\mathrm{tot}} + \Omega_k}$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\mathrm{m}0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\mathrm{m}0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\mathrm{m}0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\mathrm{tot}} := \Omega_{\mathrm{b}} + \Omega_{\mathrm{nbDM}} + \Omega_{\mathrm{r}} + \Omega_{\Lambda}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_C

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_C
  - orthographic coords: k

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
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- orthographic: $1 - kr^2 = 0$ coord singularity at equator

FLRW curvature constant

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- $\Rightarrow kR_{\mathrm{C}}^2 = 1$

FLRW curvature constant

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FLRW curvature constant

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  - azimuthal equidistant coords: $R_{\rm C}$
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\rm C}^2 = 1 \Rightarrow k = 1/R_{\rm C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; \Omega_{k0} = -\frac{c^2 k}{H_0^2}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; k = -\frac{\Omega_{k\,0} H_0^2}{c^2}$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; R_{\mathrm{C}}^2 = -\frac{c^2}{\Omega_{k0} H_0^2}$

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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; R_{\mathrm{C}}^2 = -\frac{c^2}{H_0^2}\frac{1}{\Omega_{k0}}$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; R_{\mathrm{C}}^2 = -\frac{c^2}{H_0^2}\frac{1}{1-\Omega_{\mathrm{tot}\,0}}$

FLRW curvature constant

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  - orthographic coords: k
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- $\Rightarrow kR_\mathrm{C}^2 = 1 \Rightarrow k = 1/R_\mathrm{C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_\mathrm{C} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$

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- $\Omega_{\mathrm{tot}0} > 1$

FLRW curvature constant

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical*

FLRW curvature constant

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  - azimuthal equidistant coords: R_C
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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $$R_\mathrm{C} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}$$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_C real

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: $R_{\rm C}$
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- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\rm C}^2 = 1 \Rightarrow k = 1/R_{\rm C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\rm C} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\rm tot0} - 1}}}$
- $\Omega_{\rm tot0} > 1$ *spherical* $R_{\rm C}$ real
- $\Omega_{\rm tot0} = 1$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat*

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined

FLRW curvature constant

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_C undefined
- $\Omega_{\mathrm{tot}0} < 1$

FLRW curvature constant

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- $\Rightarrow kR_{\rm C}^2 = 1 \Rightarrow k = 1/R_{\rm C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\rm C} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\rm tot0} - 1}}}$
- $\Omega_{\rm tot0} > 1$ *spherical* $R_{\rm C}$ real
- $\Omega_{\rm tot0} = 1$ *flat* $R_{\rm C}$ undefined
- $\Omega_{\rm tot0} < 1$ *hyperbolic*

FLRW curvature constant

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined
- $\Omega_{\mathrm{tot}0} < 1$ *hyperbolic* R_{C} imaginary (or use $|R_{\mathrm{C}}|$)

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} - \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
  http://cosmo.torun.pl/Cosmo/FLRWEquationsGR

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} - \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
  http://cosmo.torun.pl/Cosmo/FLRWEquationsGR
- *hint:* mixed index form of $\mathbf{g}$ is easy

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\, k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\, G\, \rho}{3} + \frac{c^2\, \Lambda}{3}$$

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$$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G\,(\rho + 3\,p/c^2)}{3} + \frac{c^2\,\Lambda}{3}$$

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Defn: "dust solution": $p(t) = 0$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{H^2}{2}\frac{\rho}{\rho_{\mathrm{crit}}} + \Omega_\Lambda\,H^2$$

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{H^2\,\Omega_{\mathrm{m}}}{2} + \Omega_\Lambda\,H^2$$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2} = -\frac{\Omega_{\mathrm{m}}}{2} + \Omega_\Lambda$$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2} = -\frac{\Omega_{\mathrm{m}}}{2} + \Omega_\Lambda$$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

acceleration Eqn ($\Lambda \neq 0$): $$q = \frac{\Omega_{\mathrm{m}}}{2} - \Omega_\Lambda$$

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

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Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

- $\boxed{q = \frac{\Omega_\mathrm{m}}{2} - \Omega_\Lambda}$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

- $\boxed{q = \frac{\Omega_{\mathrm{m}}}{2} - \Omega_\Lambda}$ acceleration equation
- if $\Lambda = 0$ and $\Omega_{\mathrm{m}} > 0$ then $\frac{\ddot{a}}{a} < 0$, i.e. $q > 0$

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

- $\boxed{q = \frac{\Omega_\mathrm{m}}{2} - \Omega_\Lambda}$ acceleration equation
- if $\Lambda = 0$ and $\Omega_\mathrm{m} > 0$ then $\frac{\ddot{a}}{a} < 0$, i.e. $q > 0$ *deceleration*

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

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- $\boxed{q = \frac{\Omega_{\rm m}}{2} - \Omega_\Lambda}$ acceleration equation
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distances in FLRW cosmology

- azimuthal equidistant r:

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Low z limit Hubble law

- use $r =$ comoving radial distance

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$\Rightarrow$ when $0 < z \ll 1$, $\beta := v/c$, $r \approx \dfrac{z\,c}{H_0}$

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EdS: radial comoving distance

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- GPL numerical package: `cosmdist`
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- standard install to `/usr/local`:
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- command line pipe:
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- high-level frontends (e.g. python) should be easy to write

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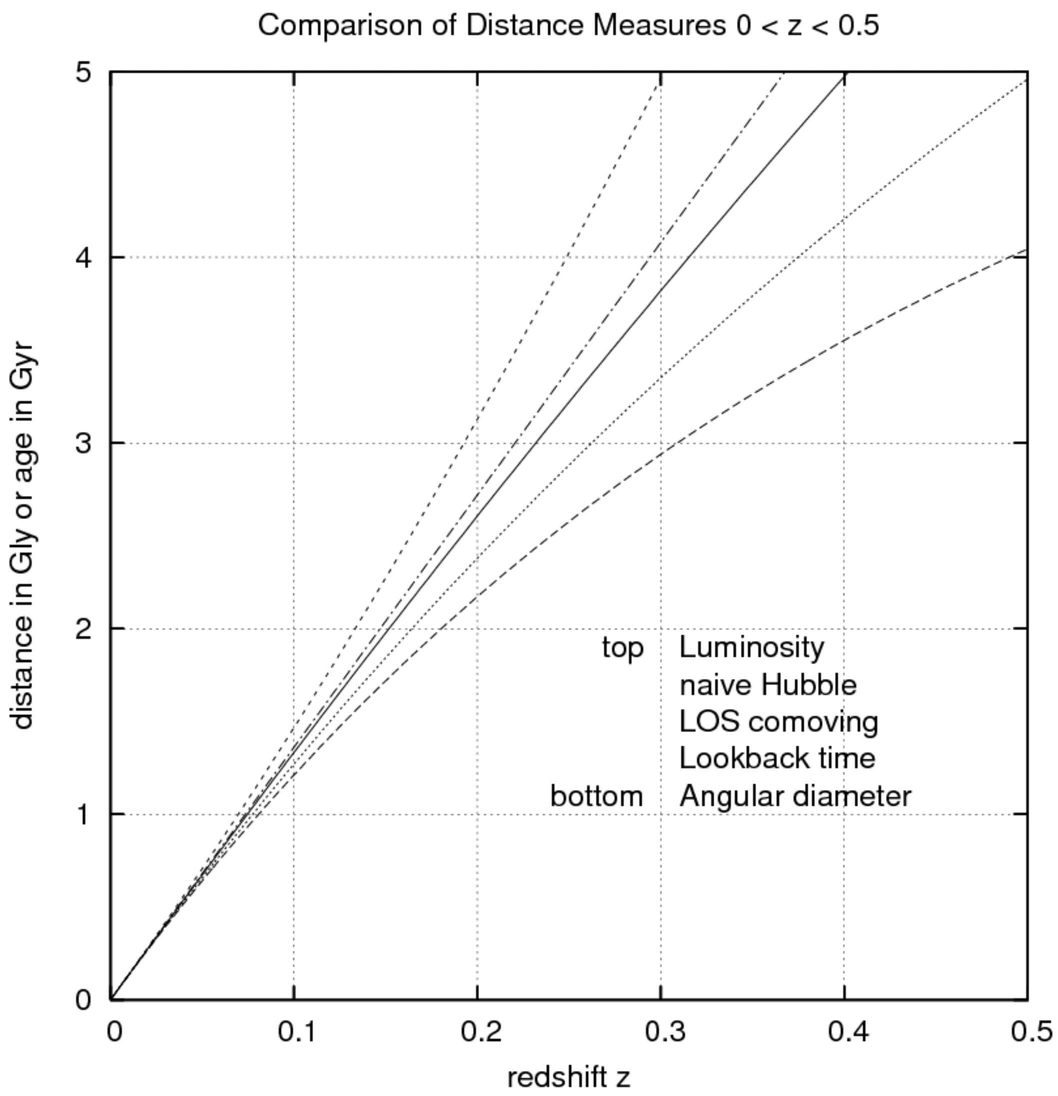
- w:Distance measures (cosmology)

FLRW distances e.g. ΛCDM



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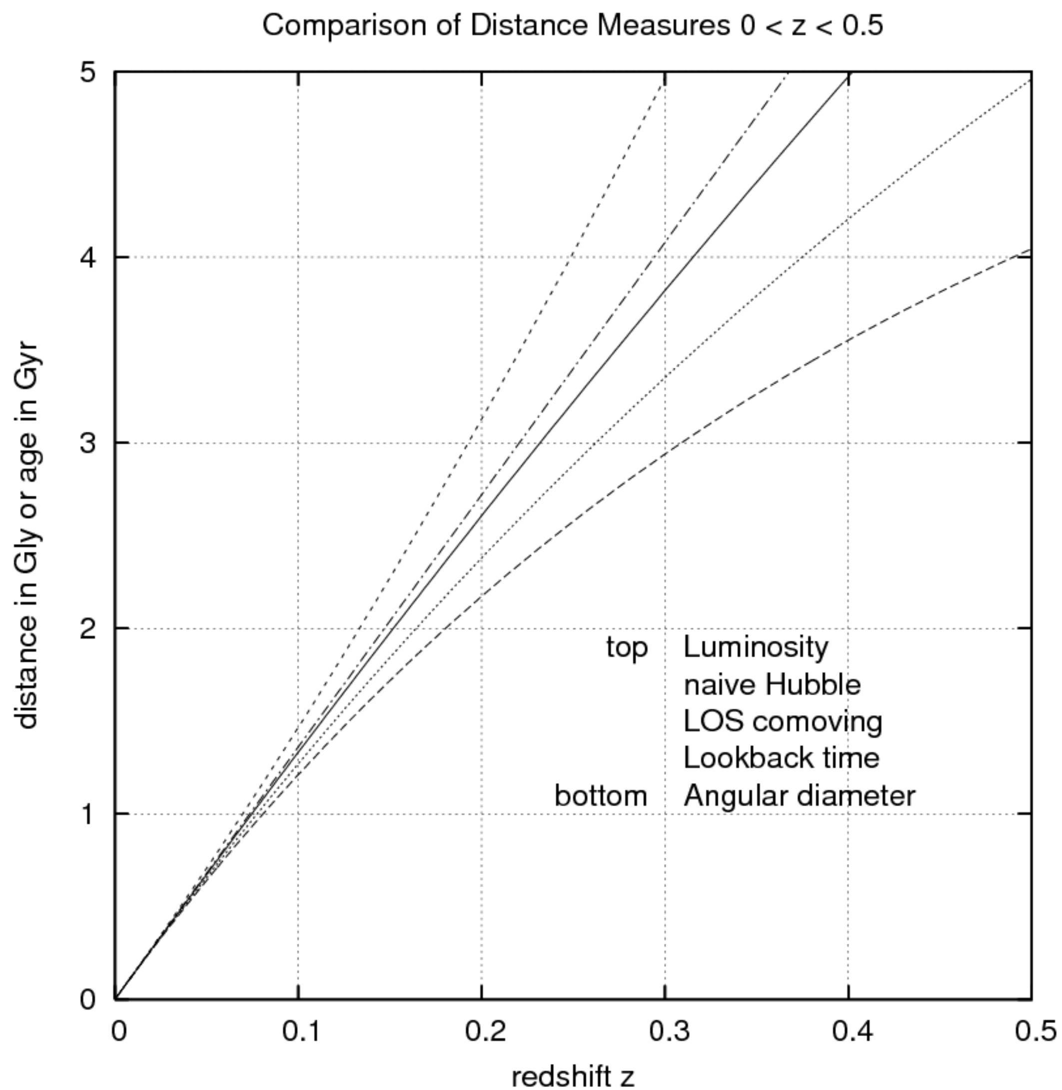

Defn: $\boxed{h := H_0/100 \text{ km/s/Mpc}}$ (without a "0" subscript on h)

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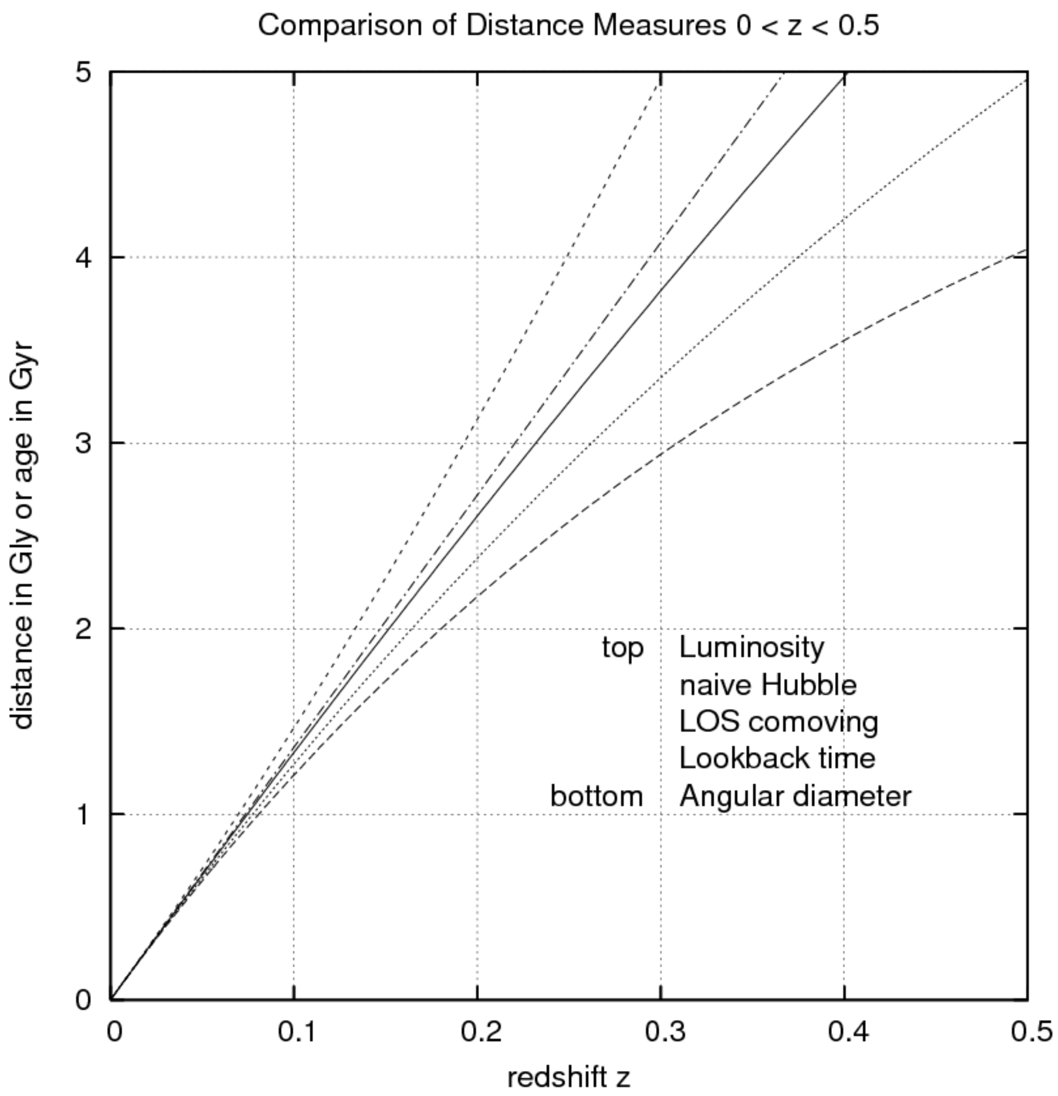

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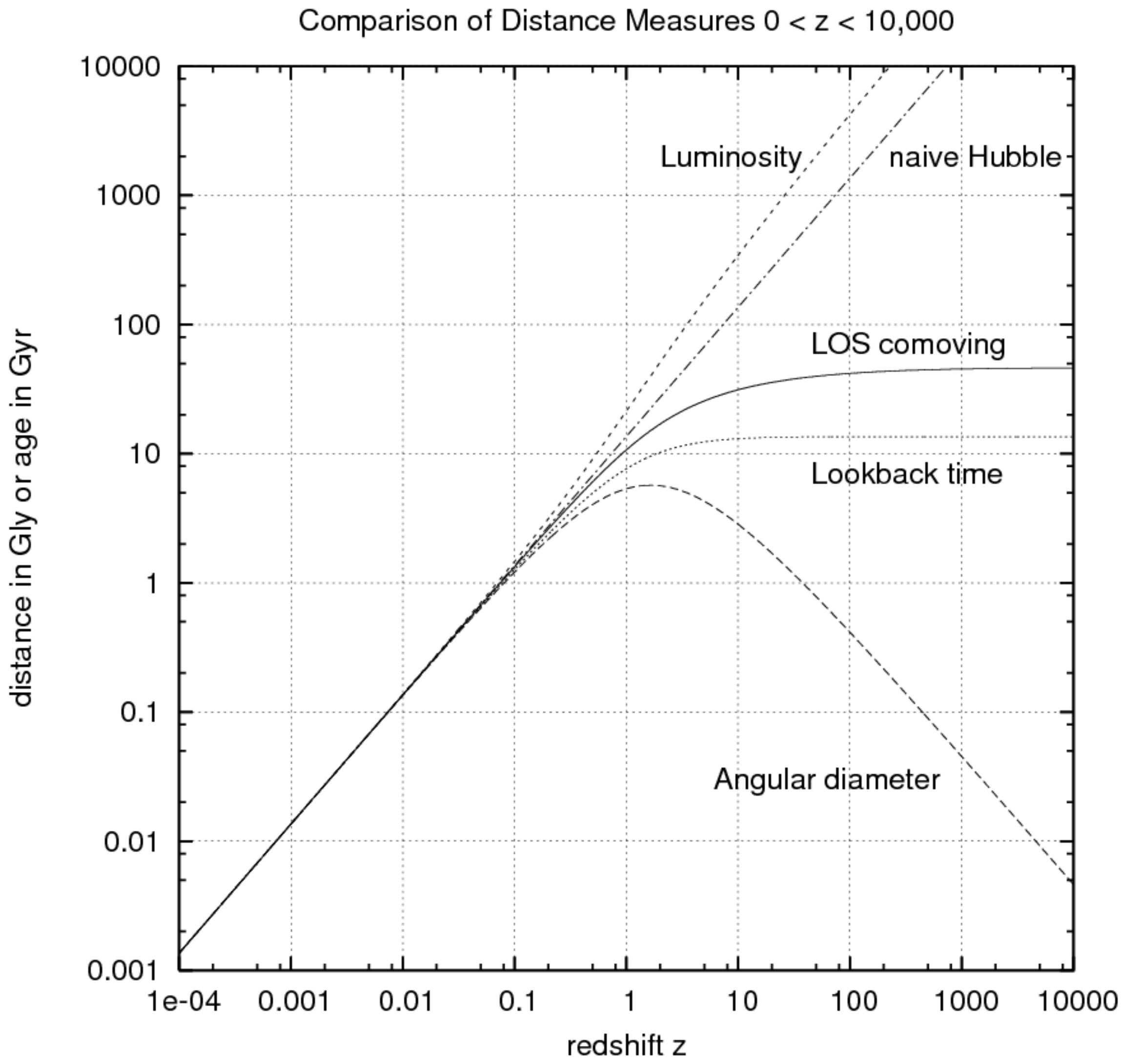

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- $v_{\mathrm{gal}}^{\mathrm{glob}}$, $\langle v \rangle_{\mathrm{gal}}^{\mathrm{glob}}$ = theoretical, global $\neq$ velocity in tangent spacetime

$\Rightarrow$ no conflict with locally Lorentzian (SR) spacetime

Non-radial spatial geodesics

- What is the comoving distance between two objects at different celestial positions and different redshifts, for an arbitrary curvature (+, 0, $-$)?

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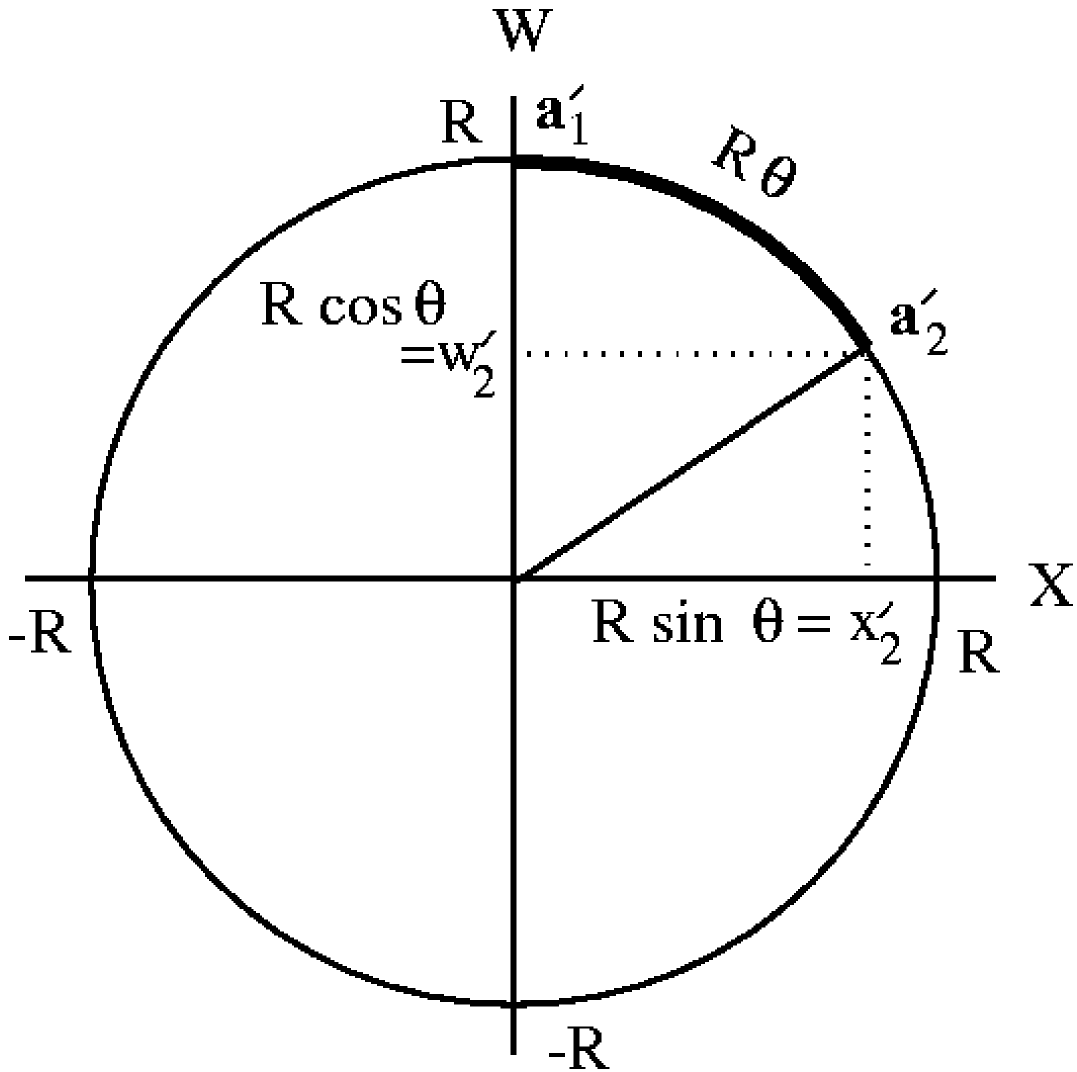

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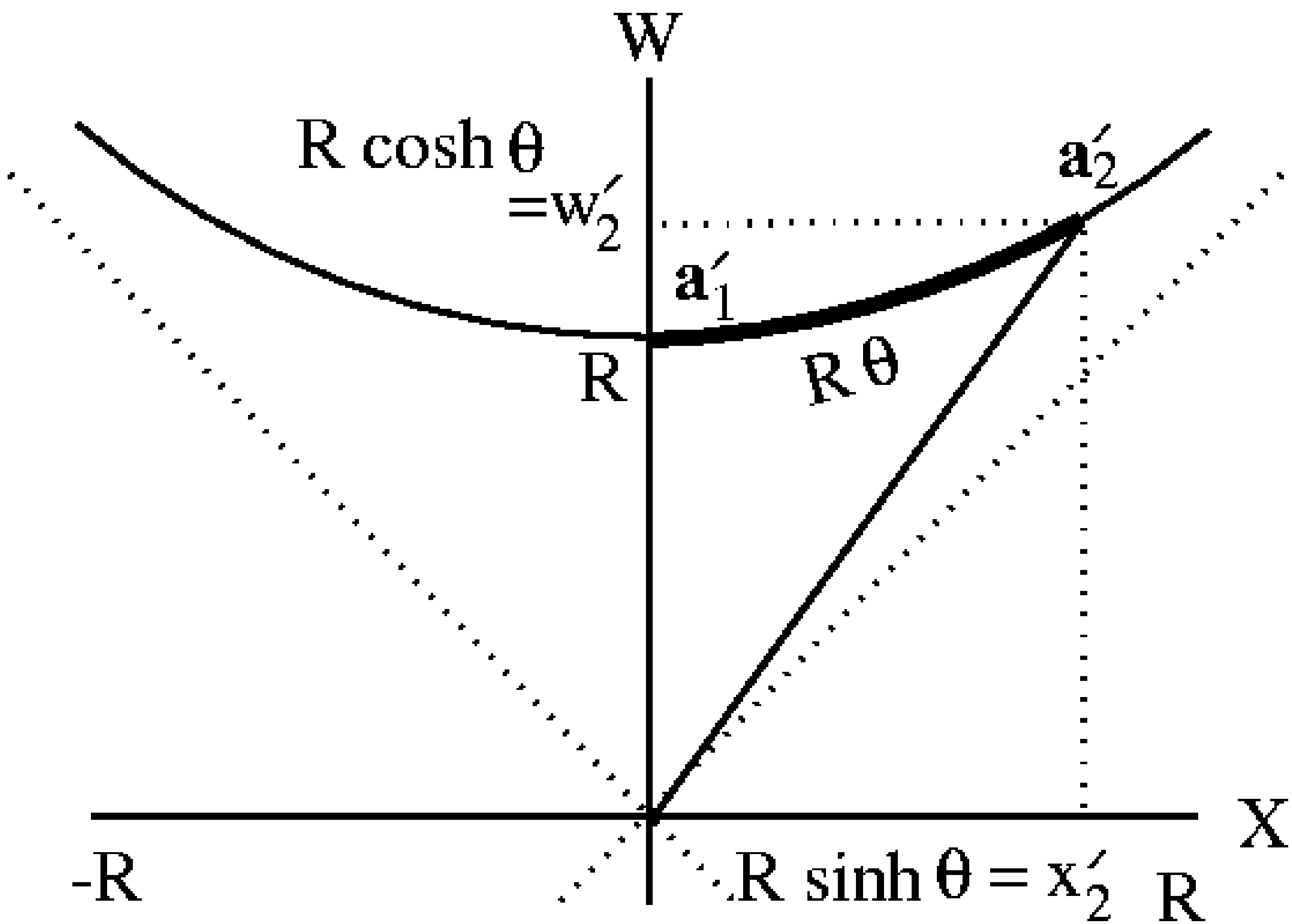

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distances on $\mathsf{S}^3 \subset \mathbb{R}^4$ or $\mathsf{H}^3 \subset \mathsf{M}^4$

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- metric on S^3 (or $\mathbb{R}^3$ or H^3):
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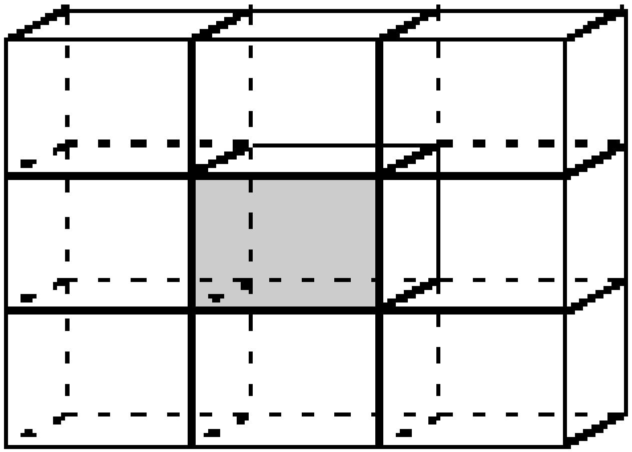
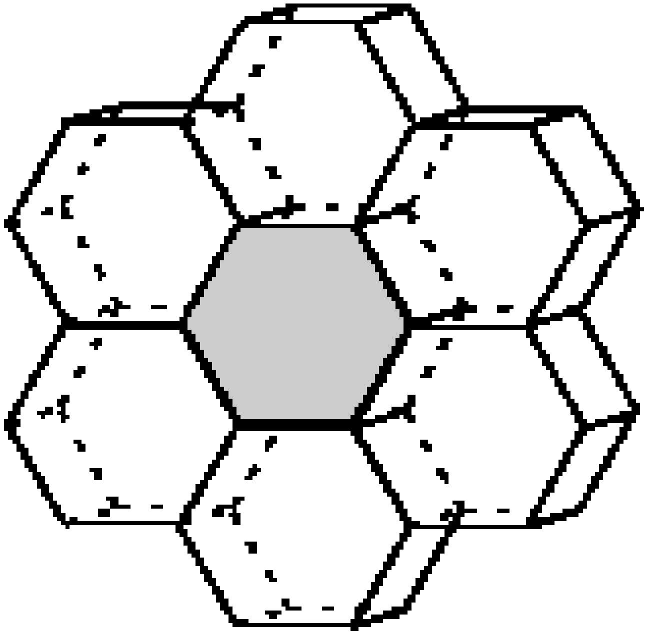
Cosmic topology: definitions





3D flat examples arXiv:astro-ph/9901364

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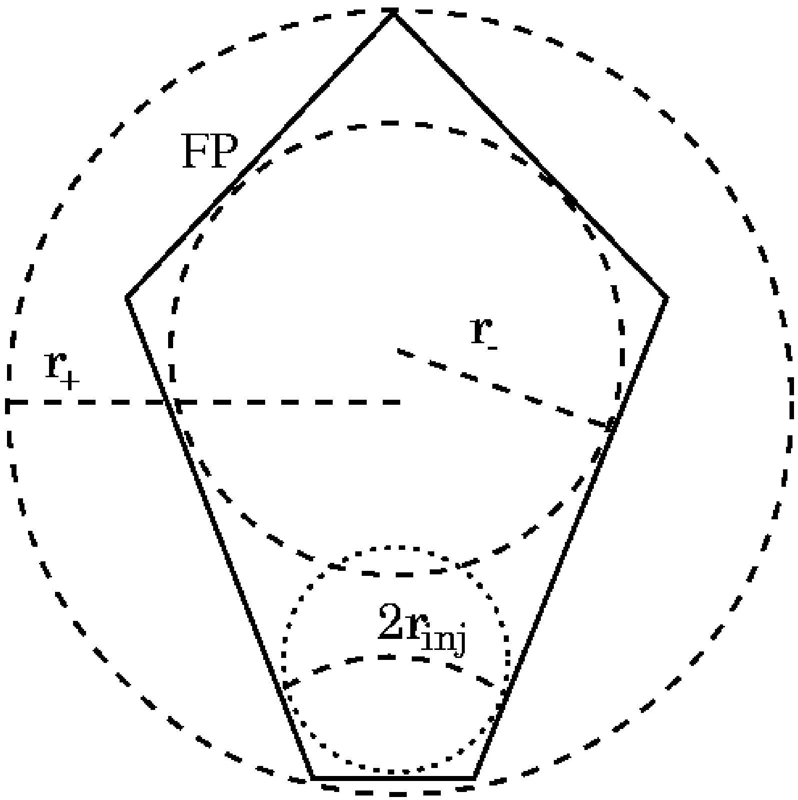

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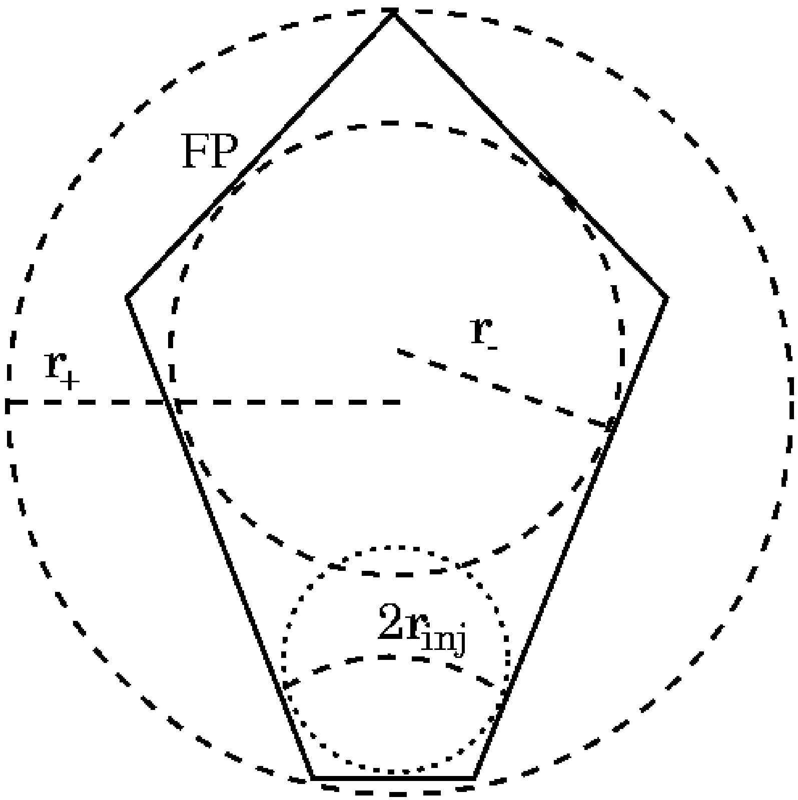

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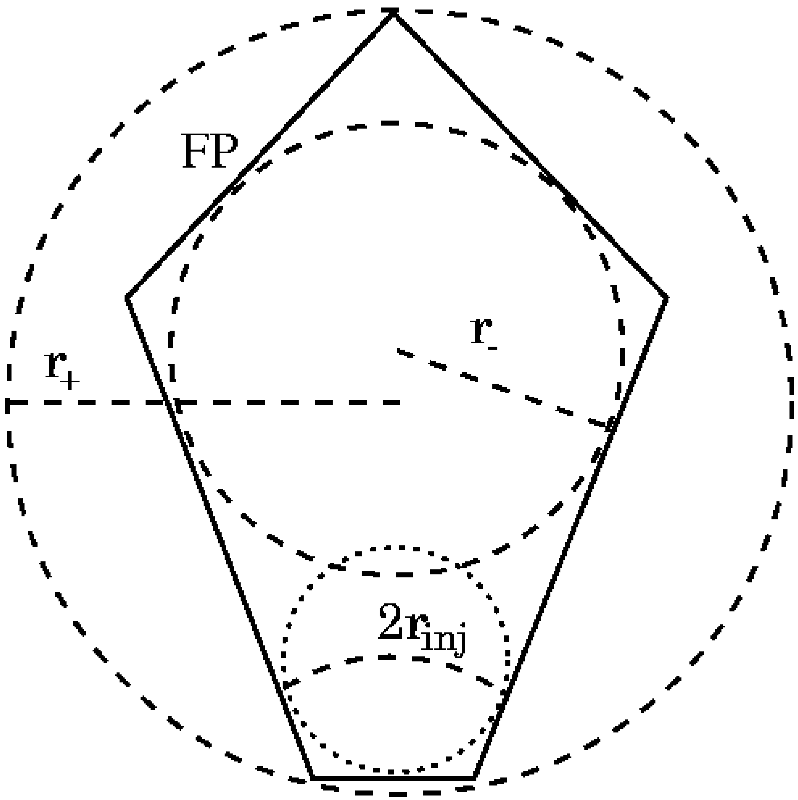

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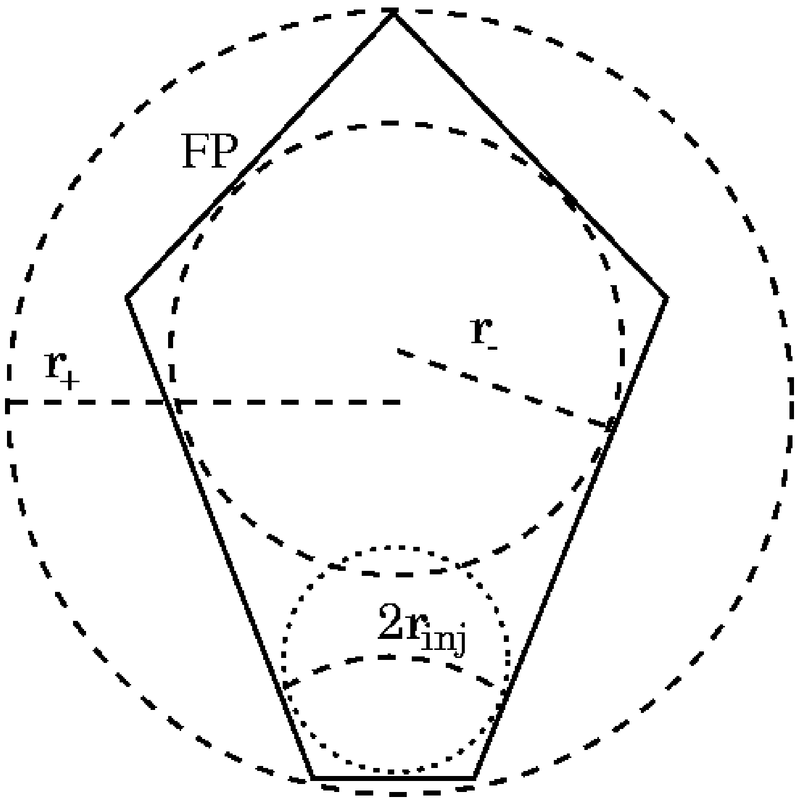

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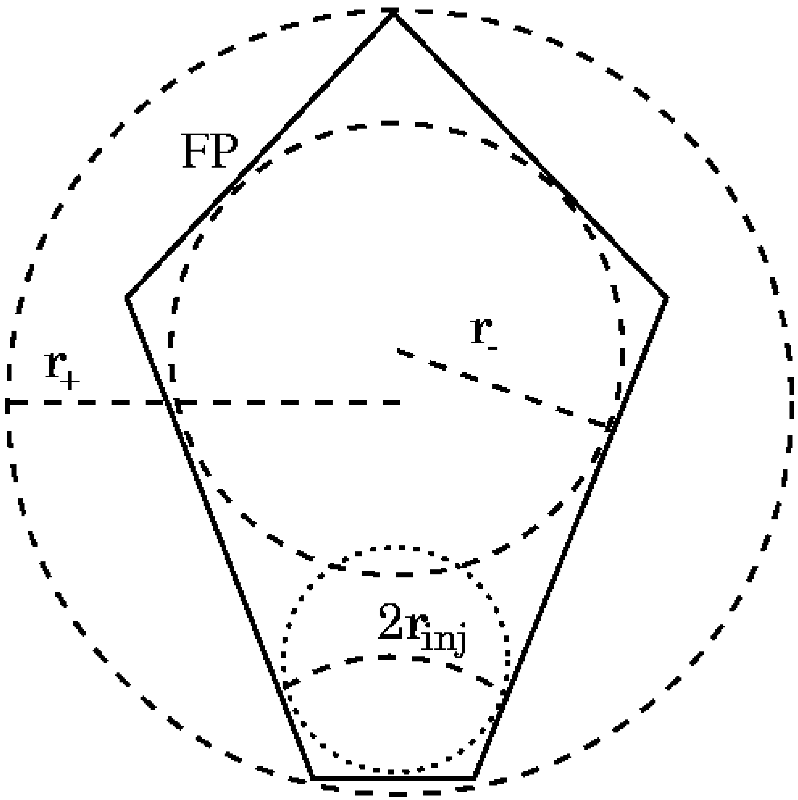

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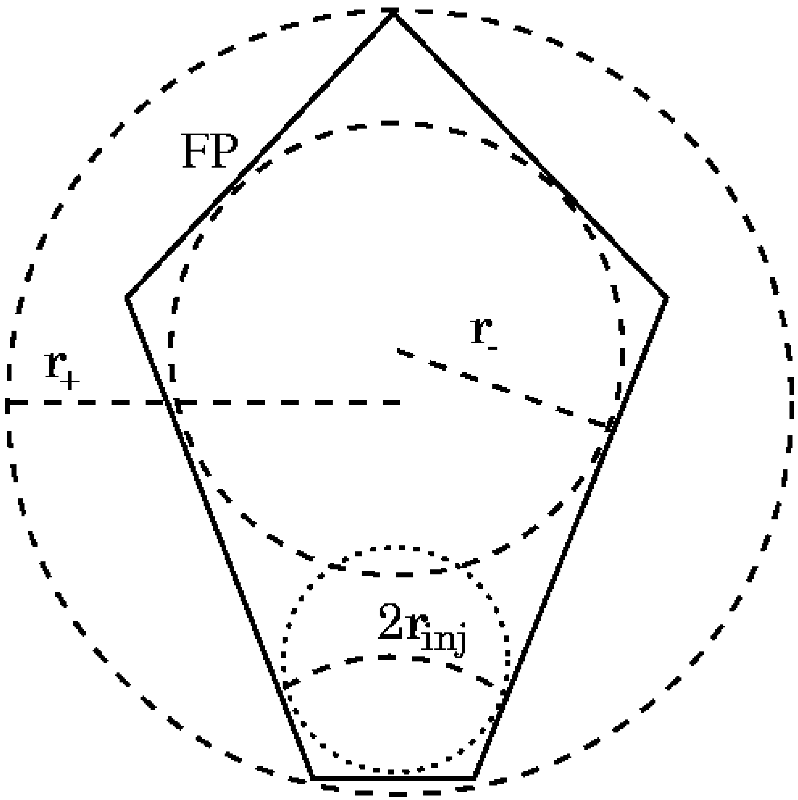

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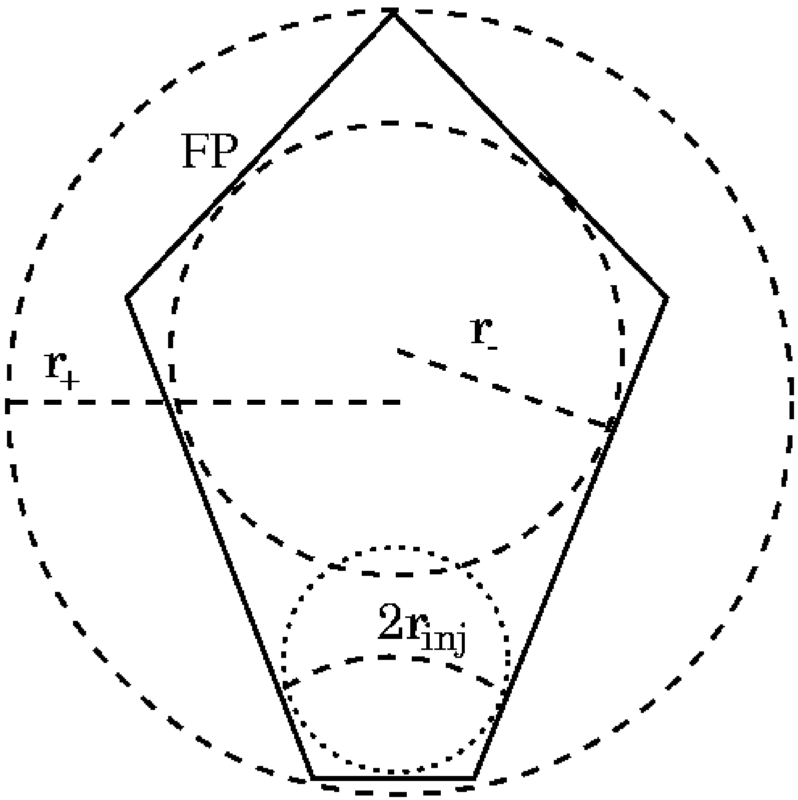

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    - w:Poincaré Conjecture *"Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere."* w:Grigori Perelman, arXiv:math/0211159 + arXiv:math/0303109 + arXiv:math/0307245

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  - countably infinite superset known
  - some finite, some infinite
  - active research area, e.g. arXiv:0705.4325 min. vol.

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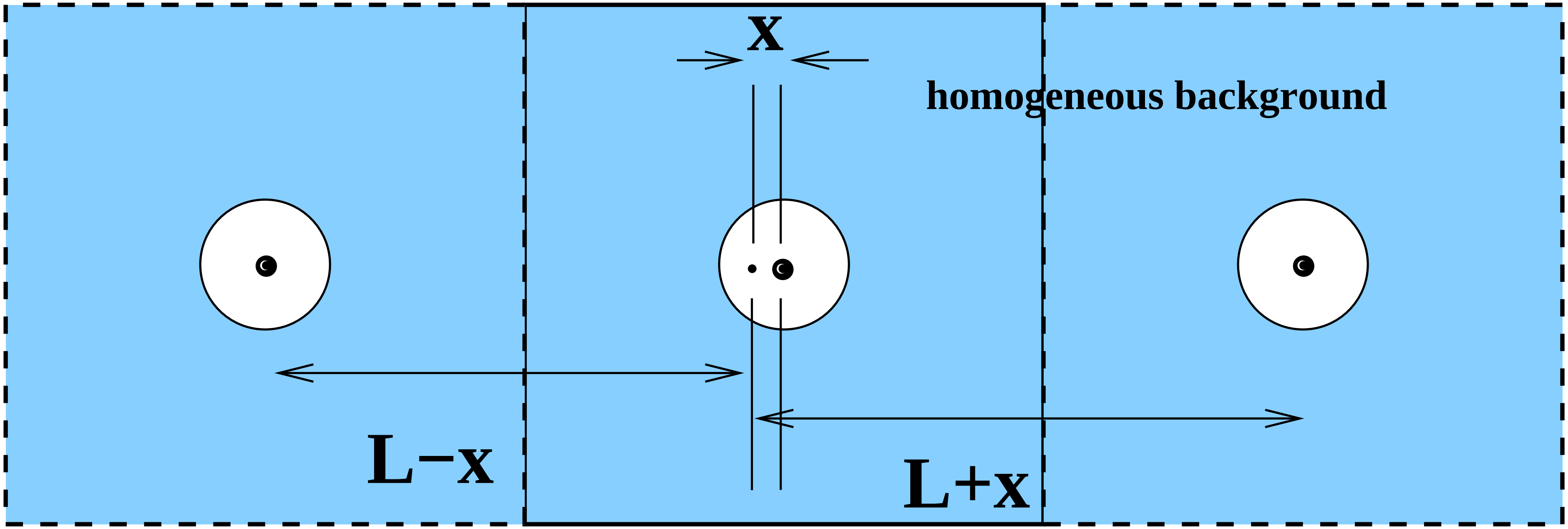

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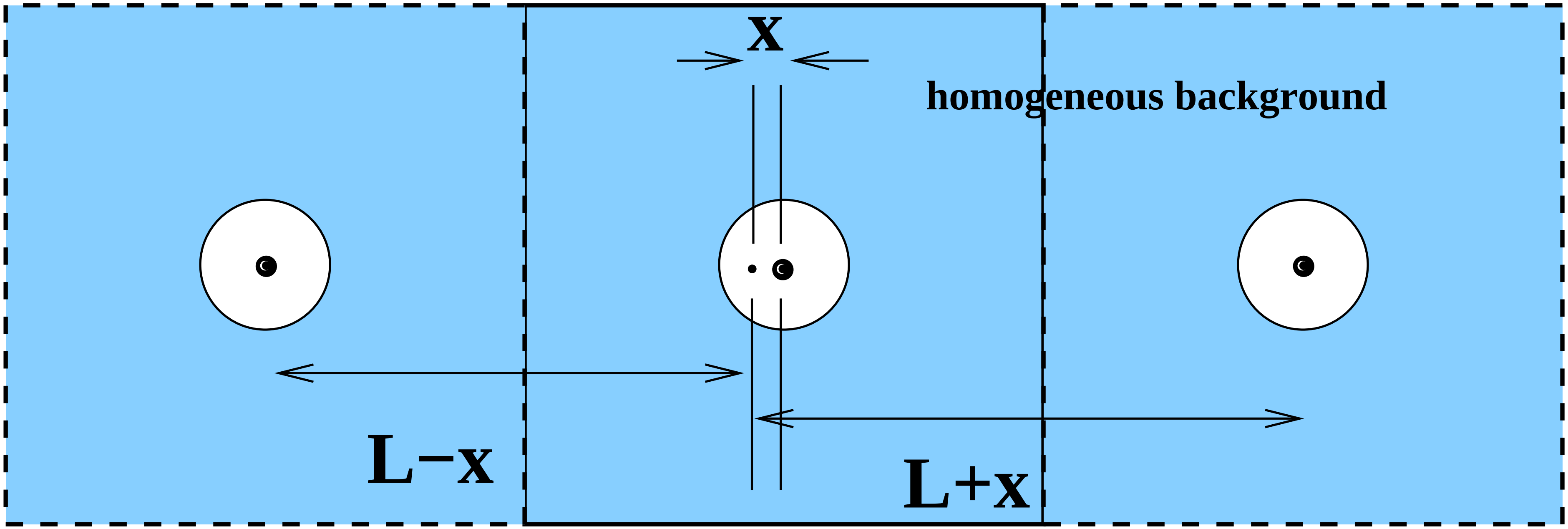

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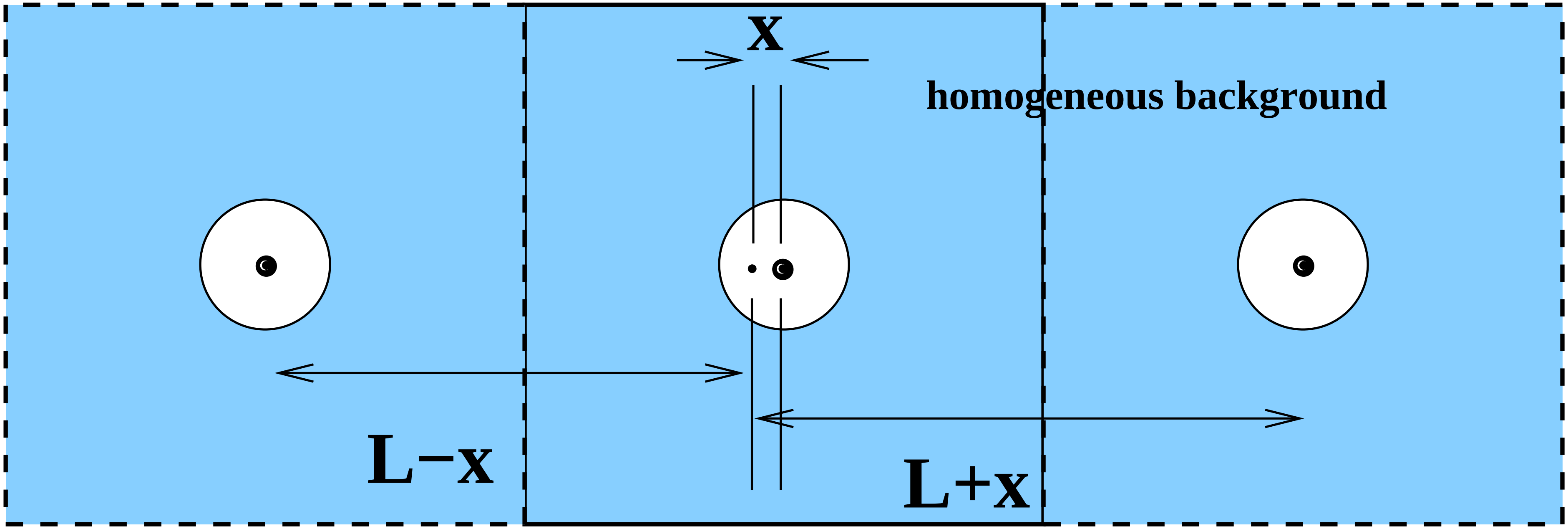

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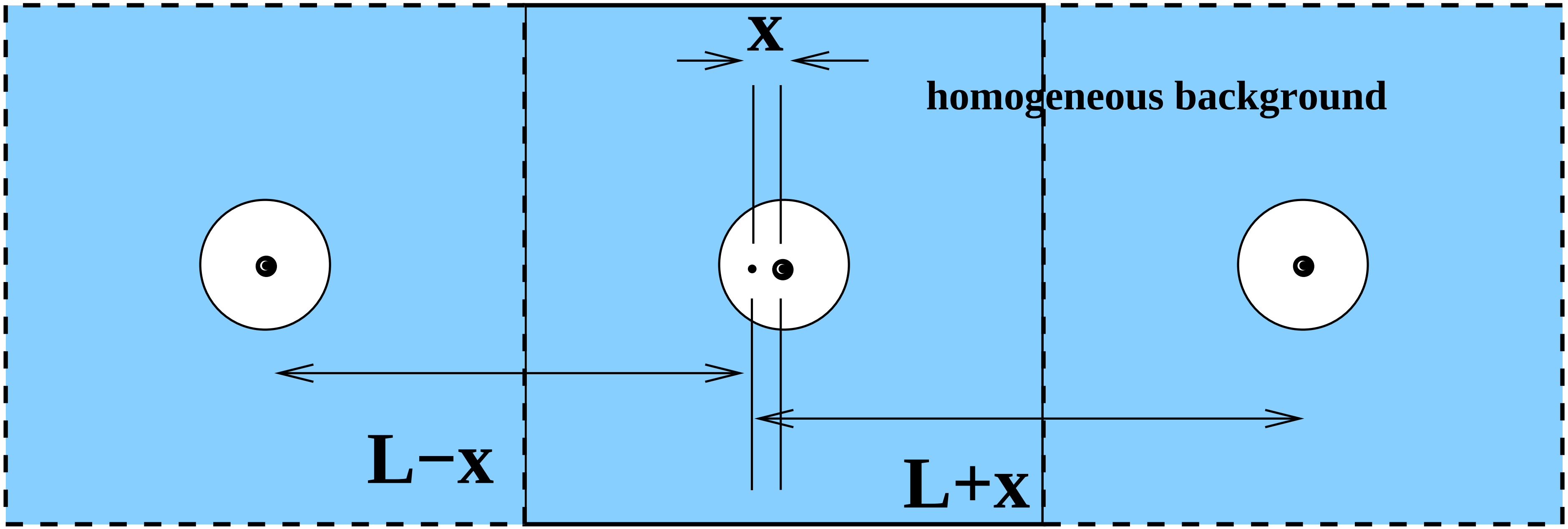

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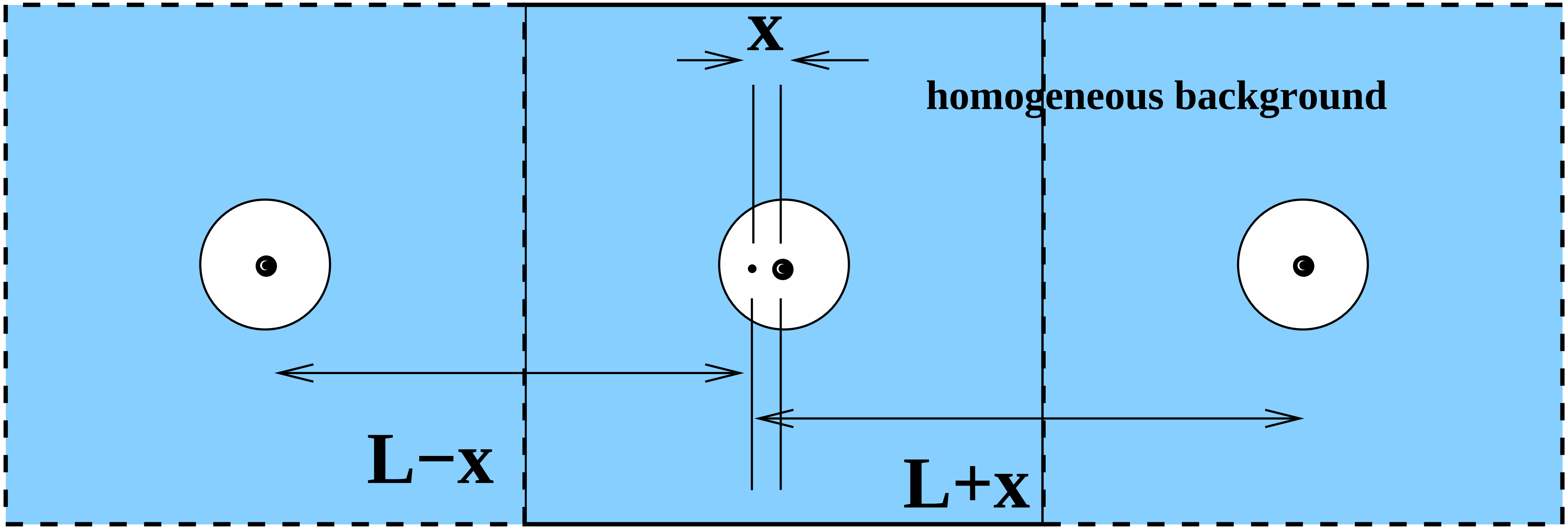

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topological acceleration—arXiv:astro-ph/0602159

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empirical strategies: arXiv:astro-ph/0010189

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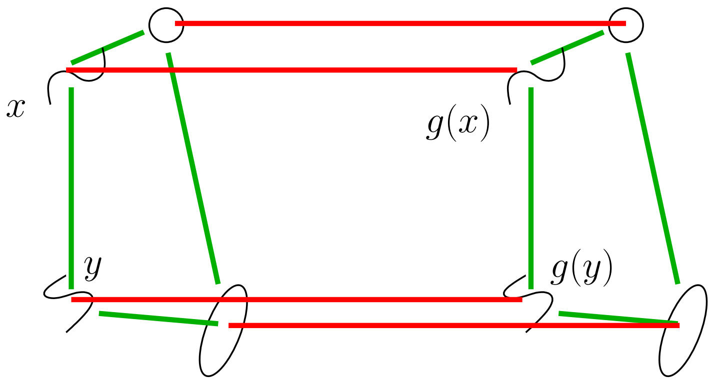
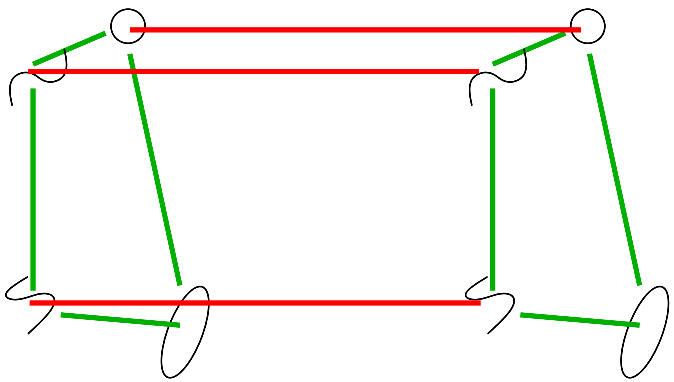
3D strategies—pair types





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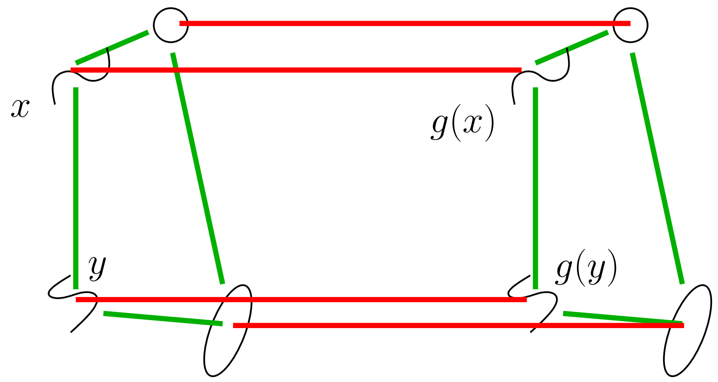

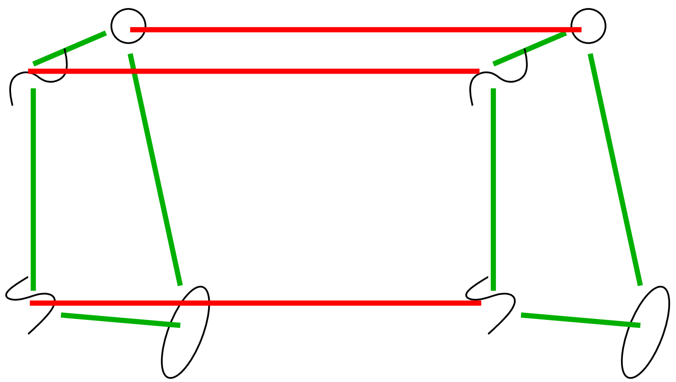

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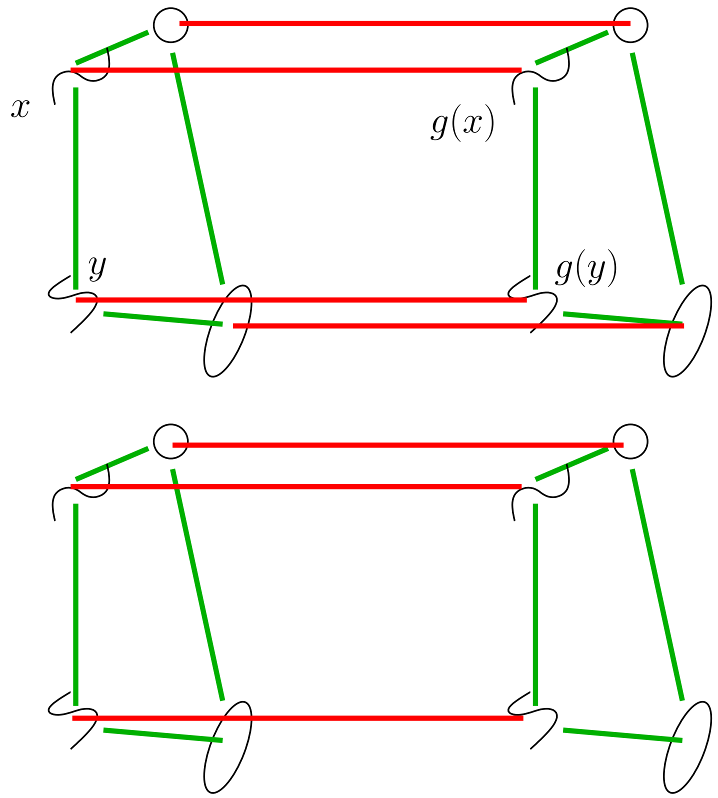

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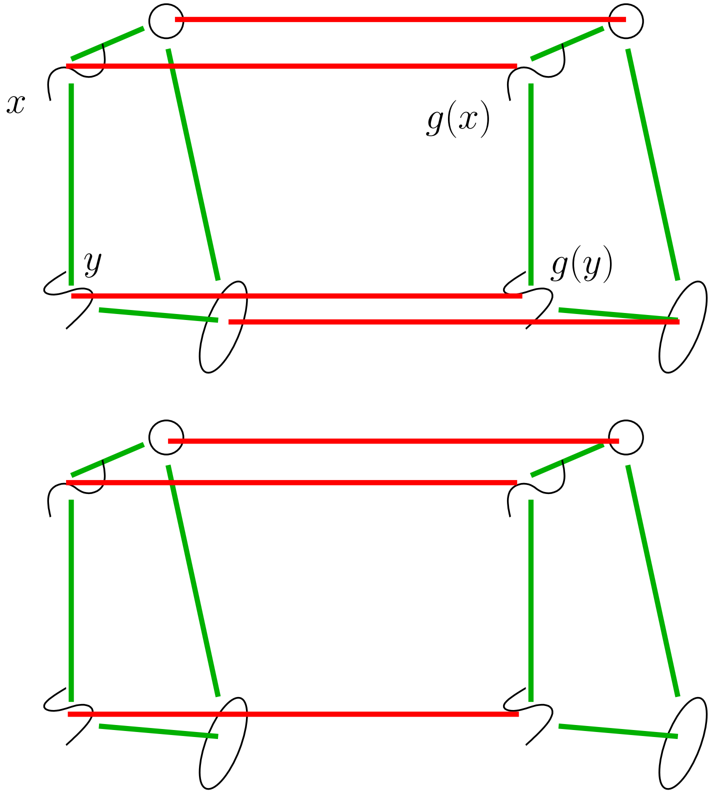

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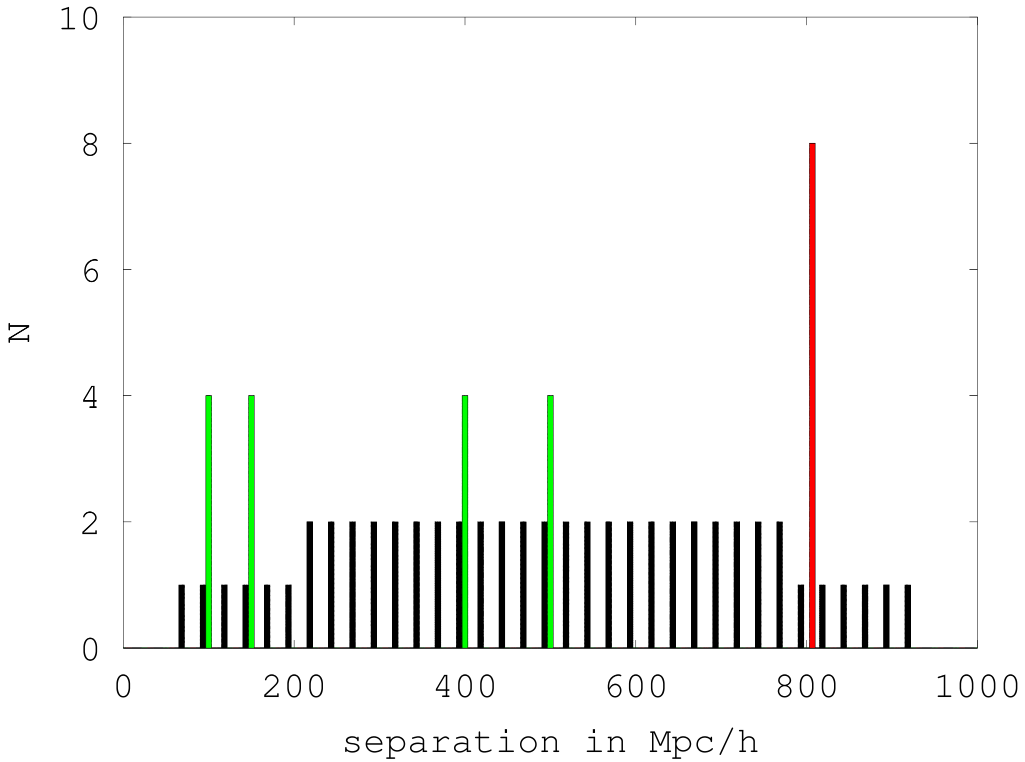

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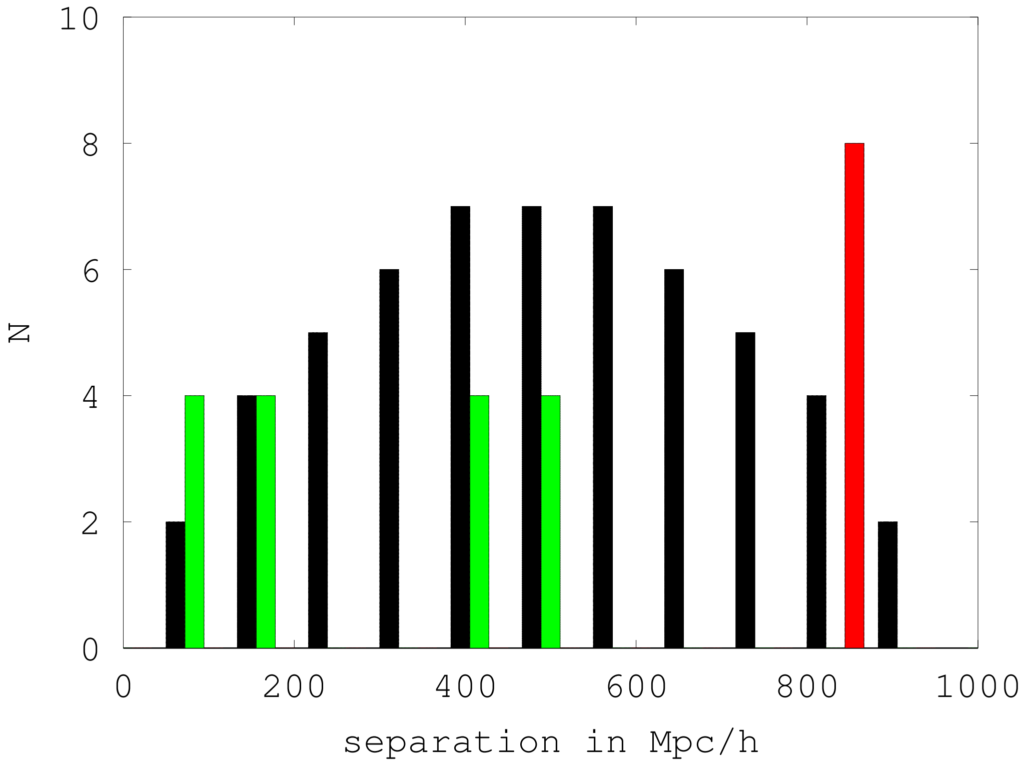

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- quadruples + successive filters + collect membership s of quadruples Fujii & Yoshii (2013)

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AGNs—successive filters

Marecki, Roukema, Bajtlik (2005) arXiv:astro-ph/0412181

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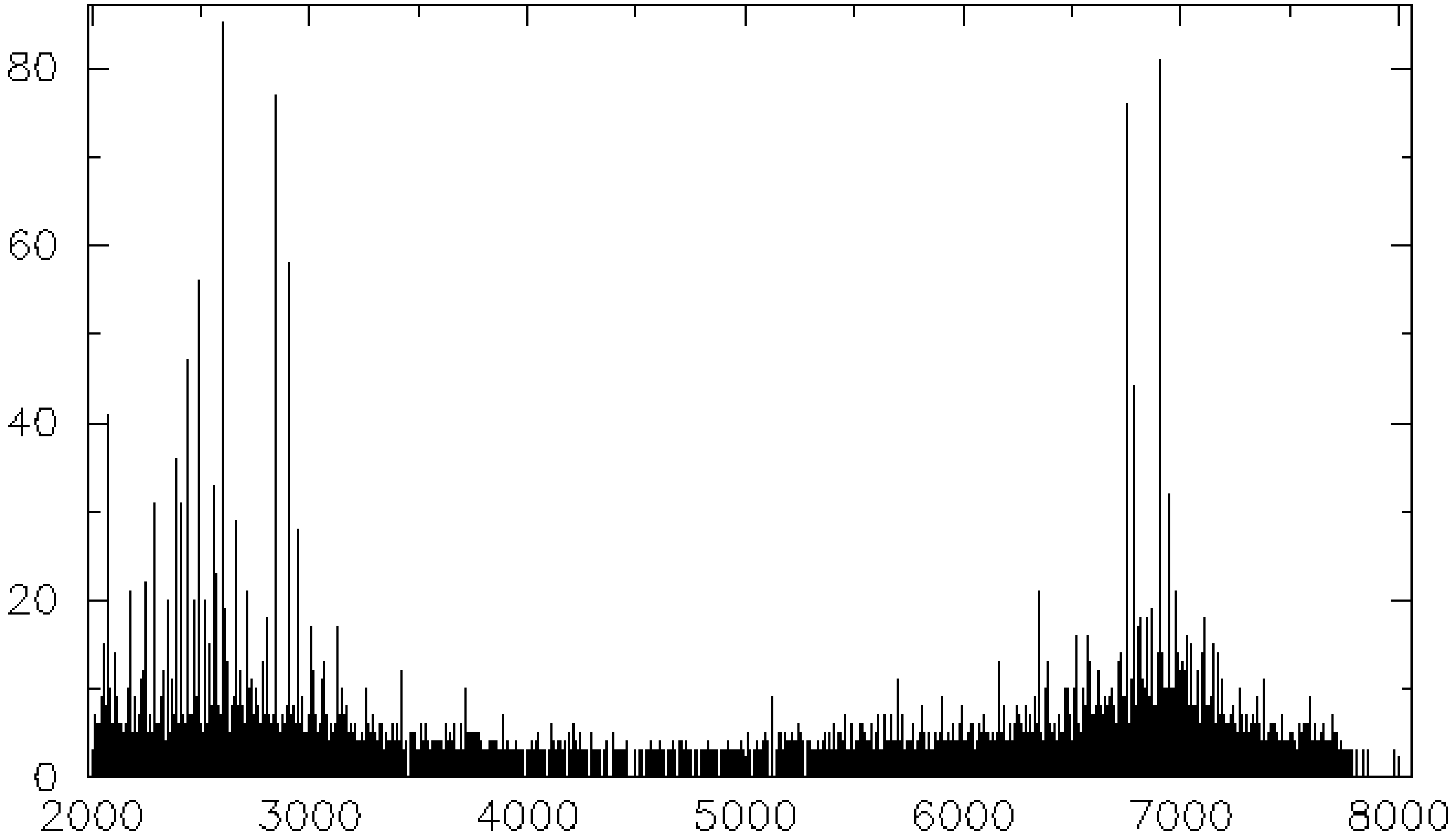
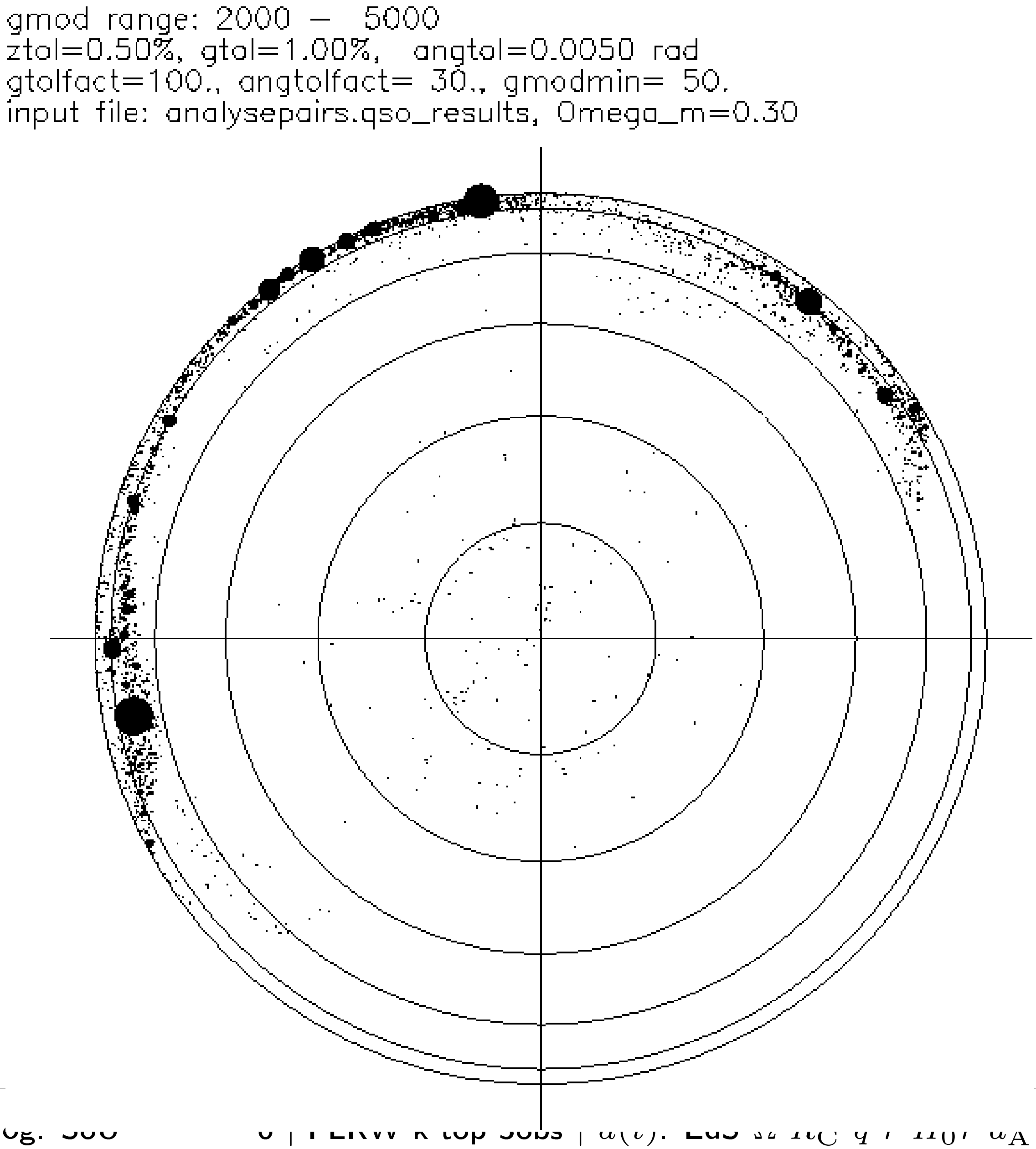


AGN Catalogues



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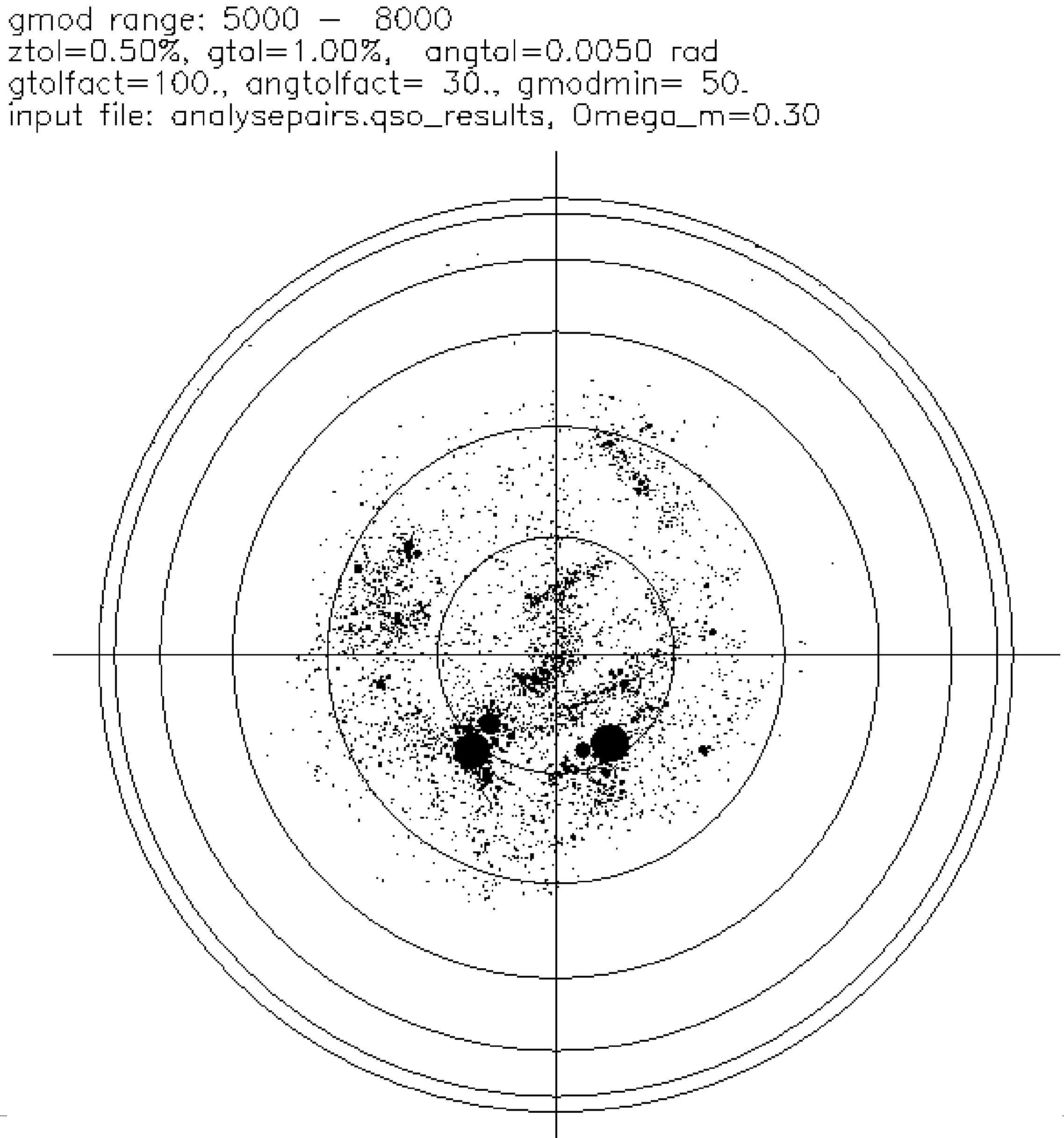

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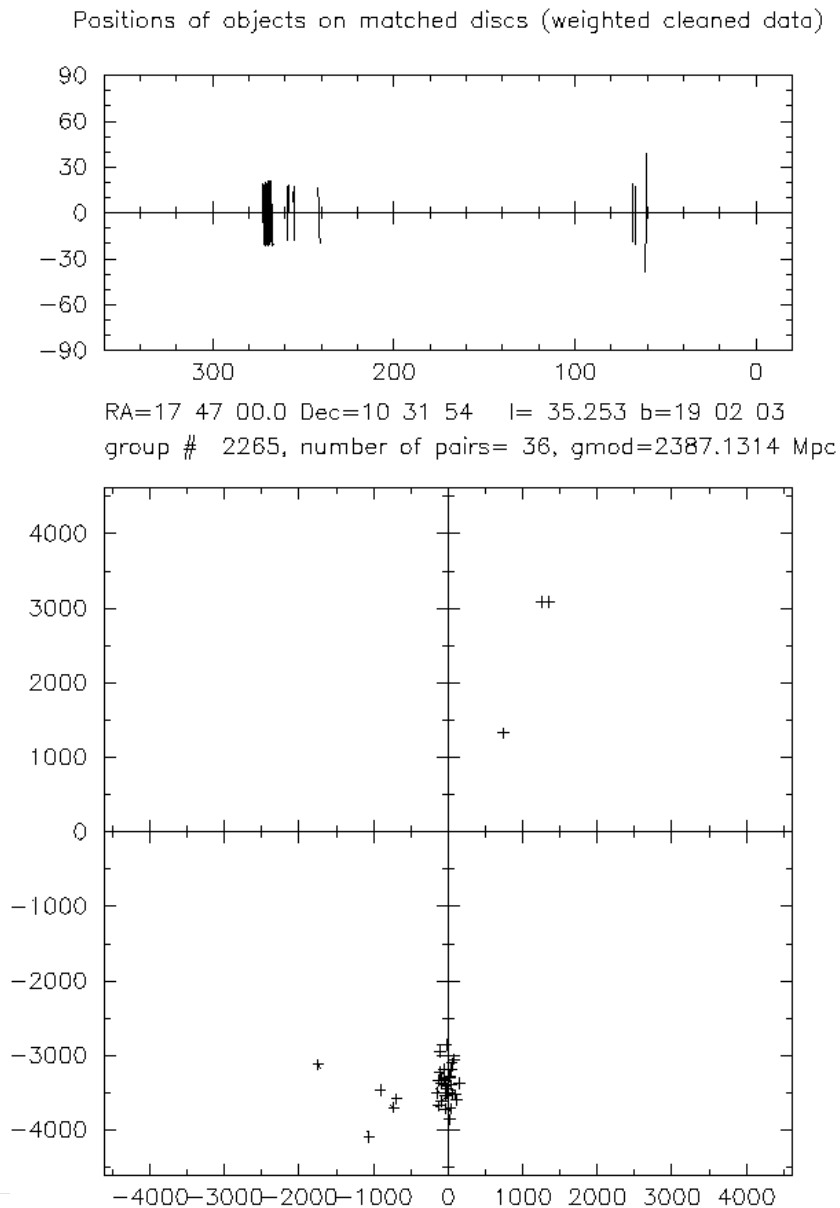

AGN Catalogues

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Fujii & Yoshii (2013) arXiv:1103.1466

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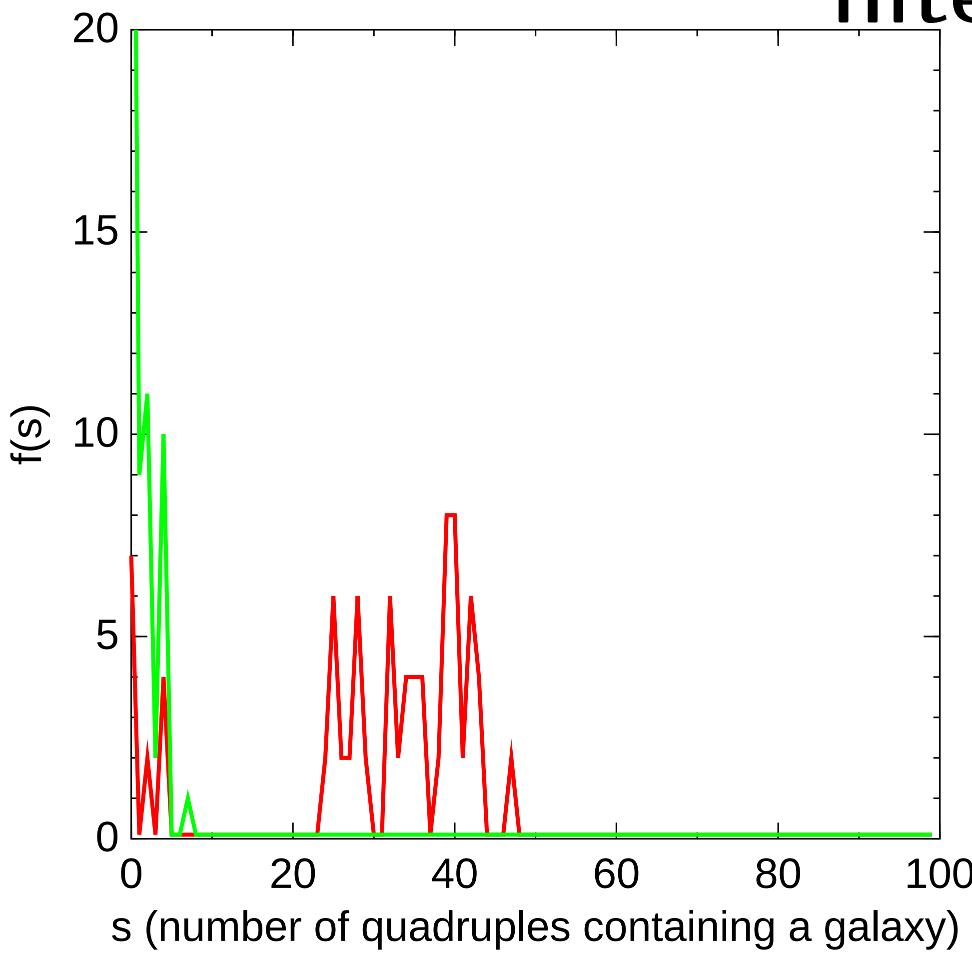
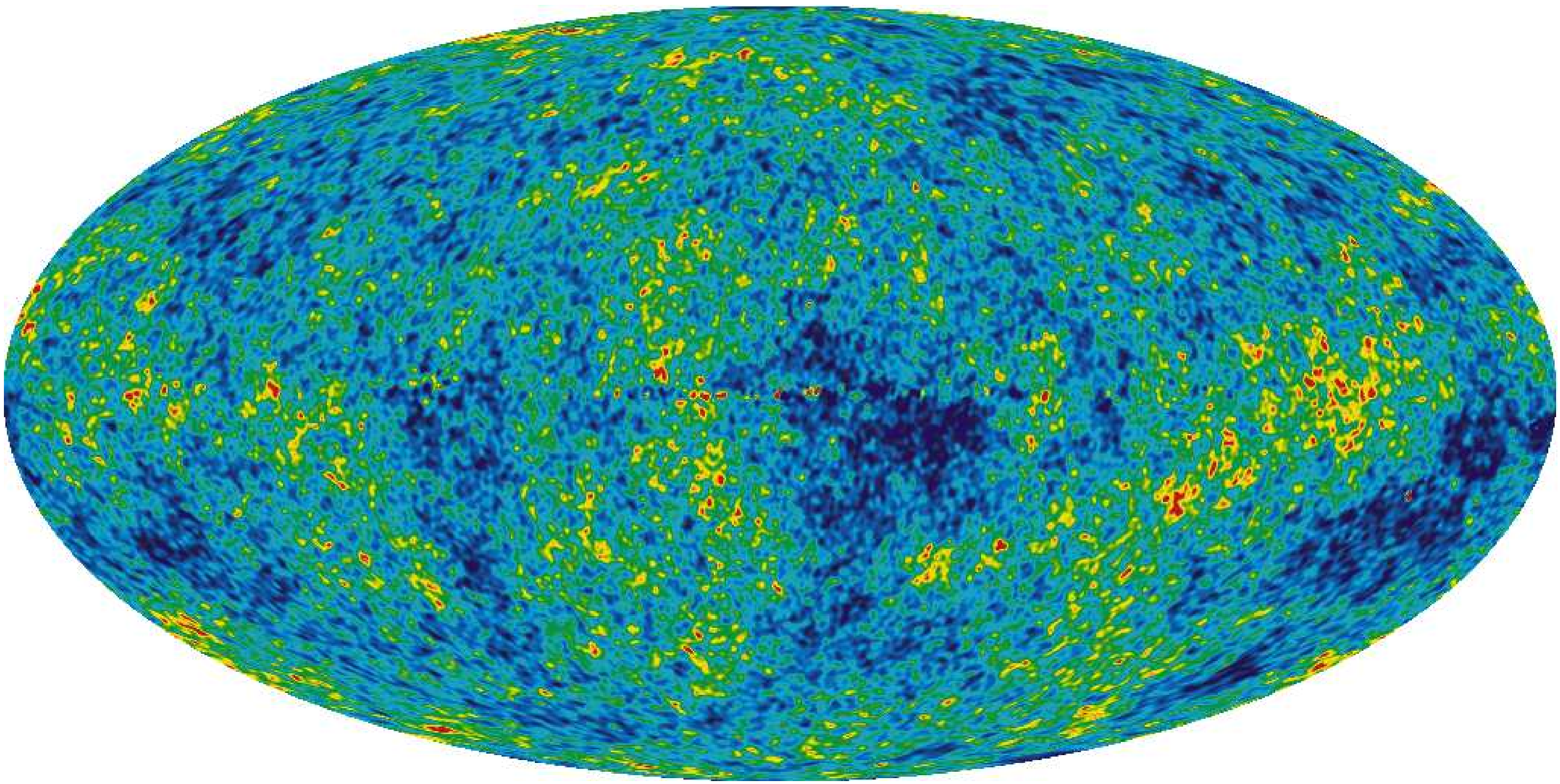


simulation of s histogram for Lyman break galaxies (LBGs) at $z \approx 6$

green: simply connected; red: T^3

ADS:2014MNRAS.437.1096R

2D methods: structure cutoff



WMAP 5yr ILC (internal linear combination)

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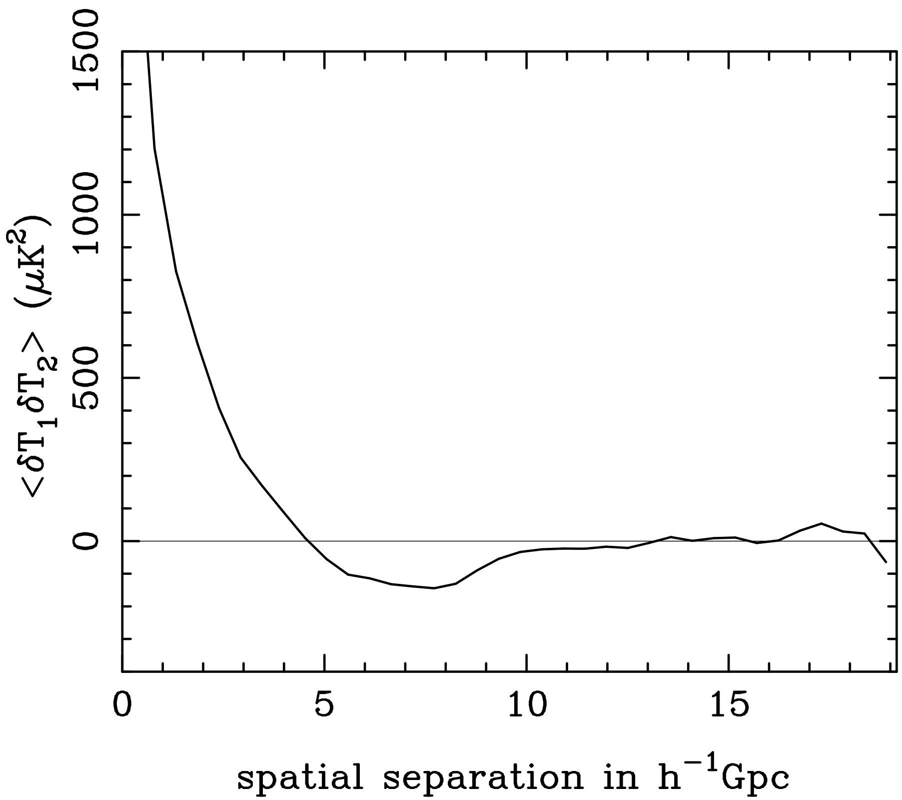

3D: structures bigger than FD cannot exist

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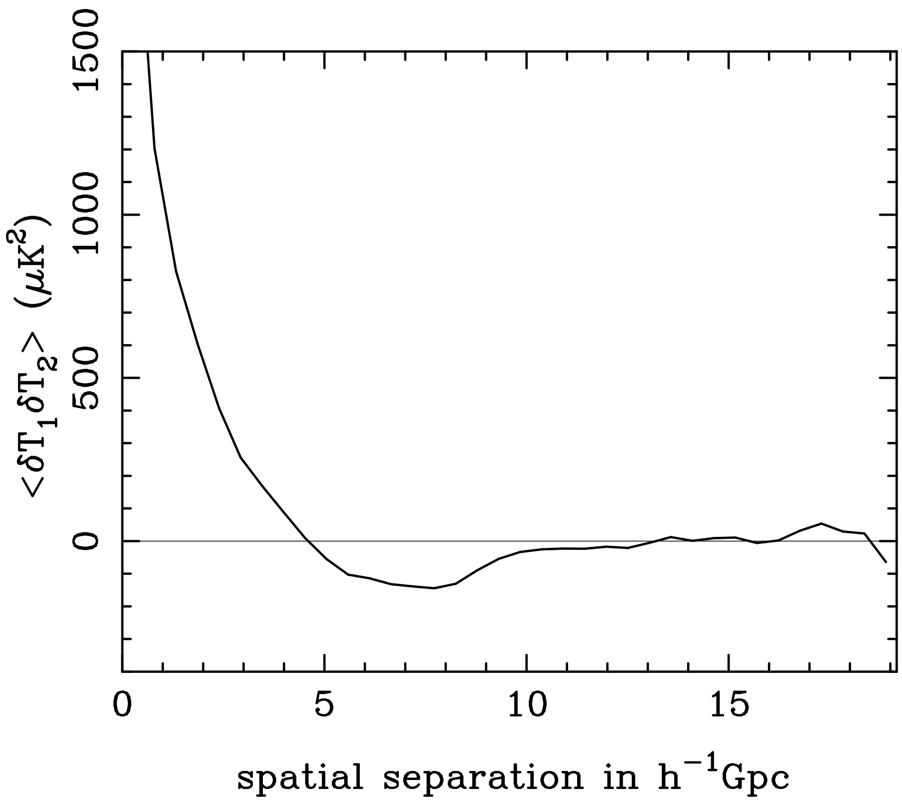

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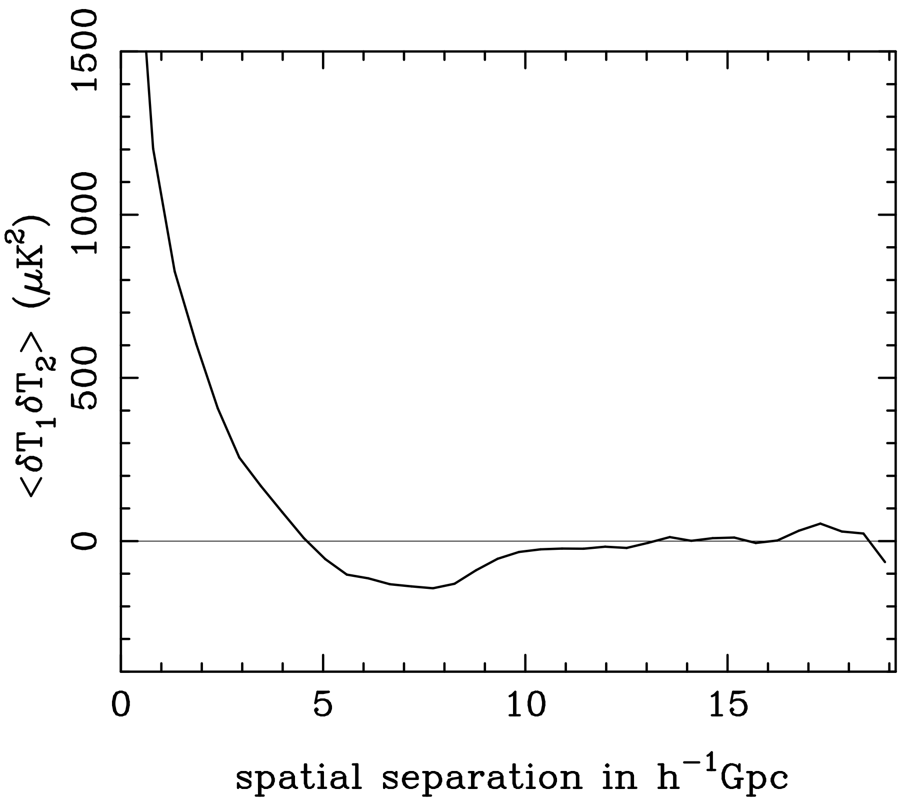

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Starobinsky (1993); Stevens et al. (1993)

The Identified Circles Principle

- discovery of principle: Cornish, Spergel & Starkman (1996)

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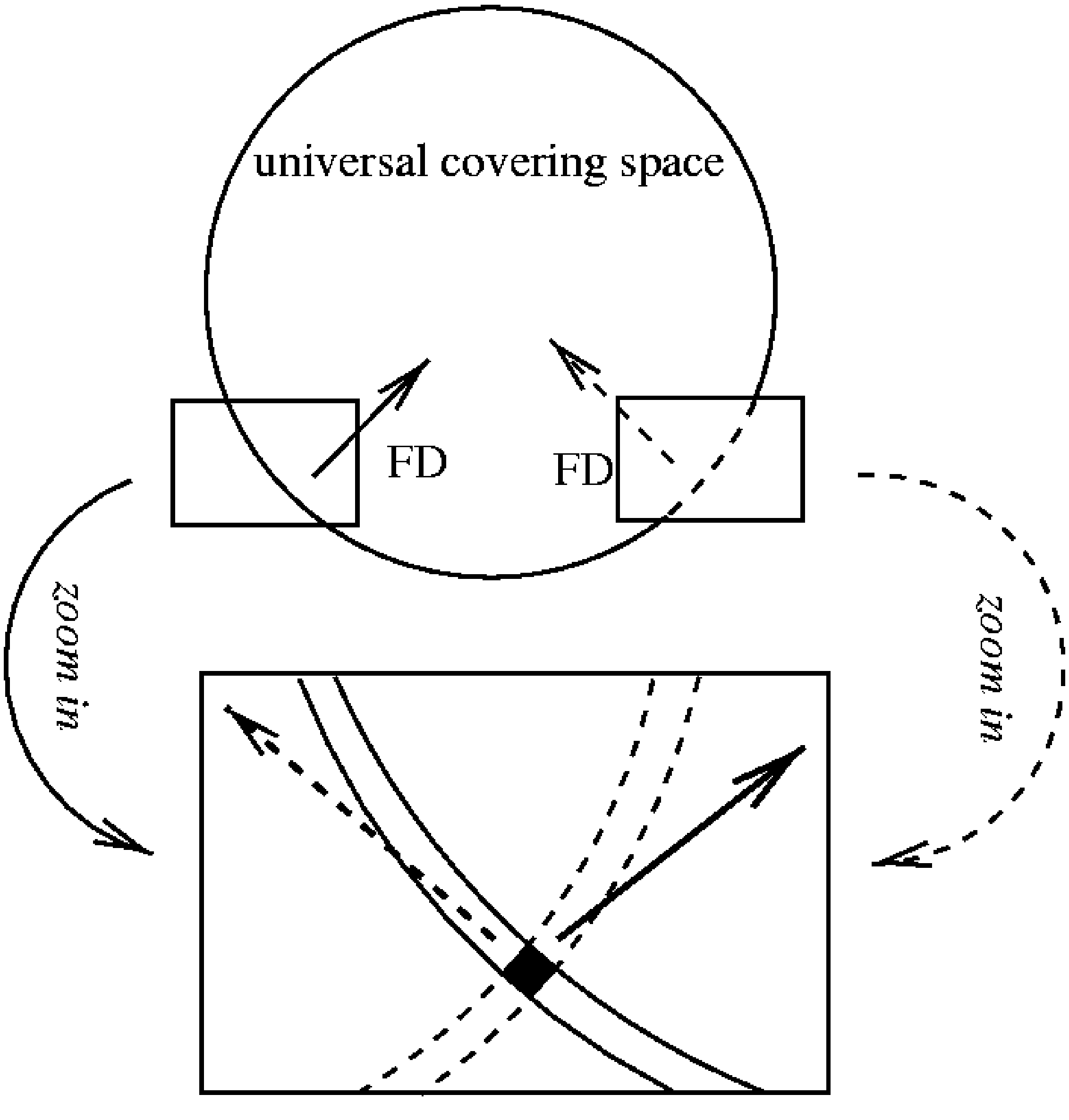

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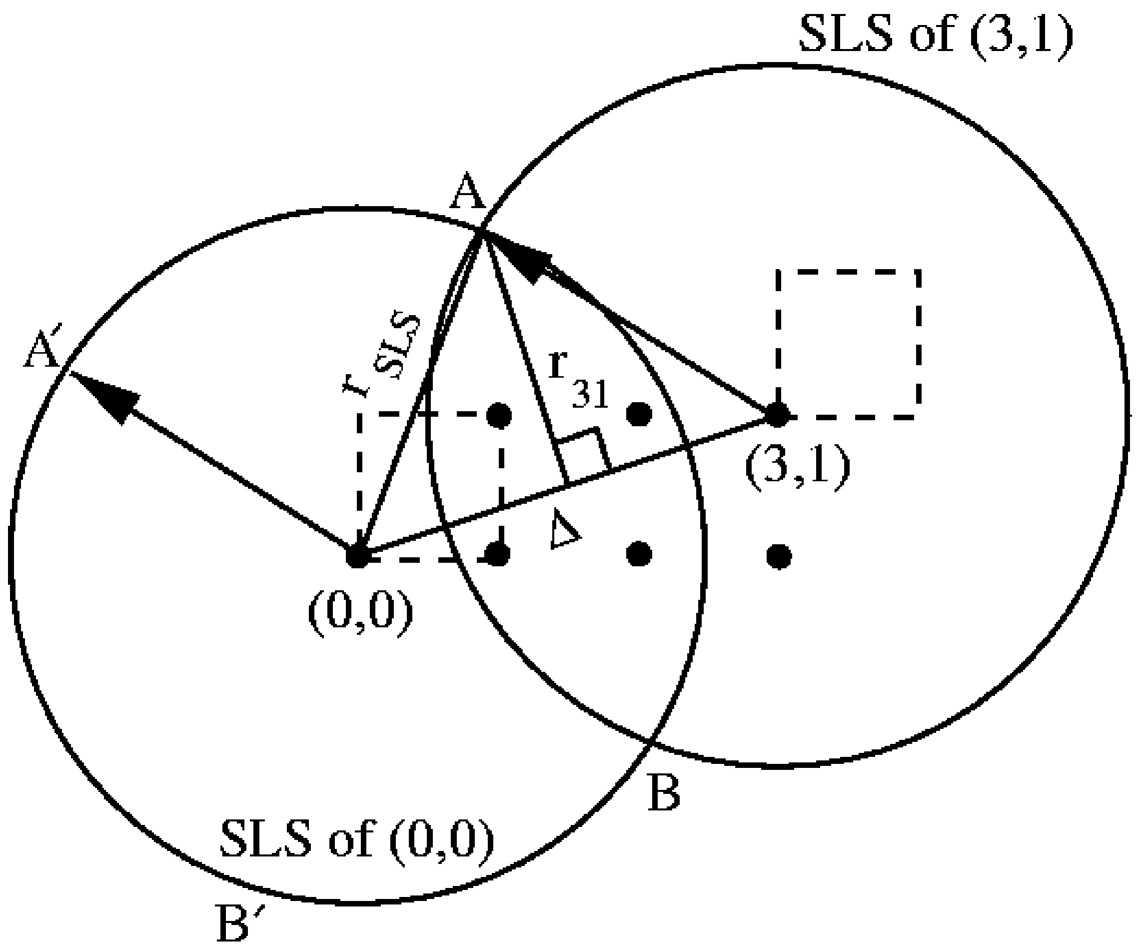
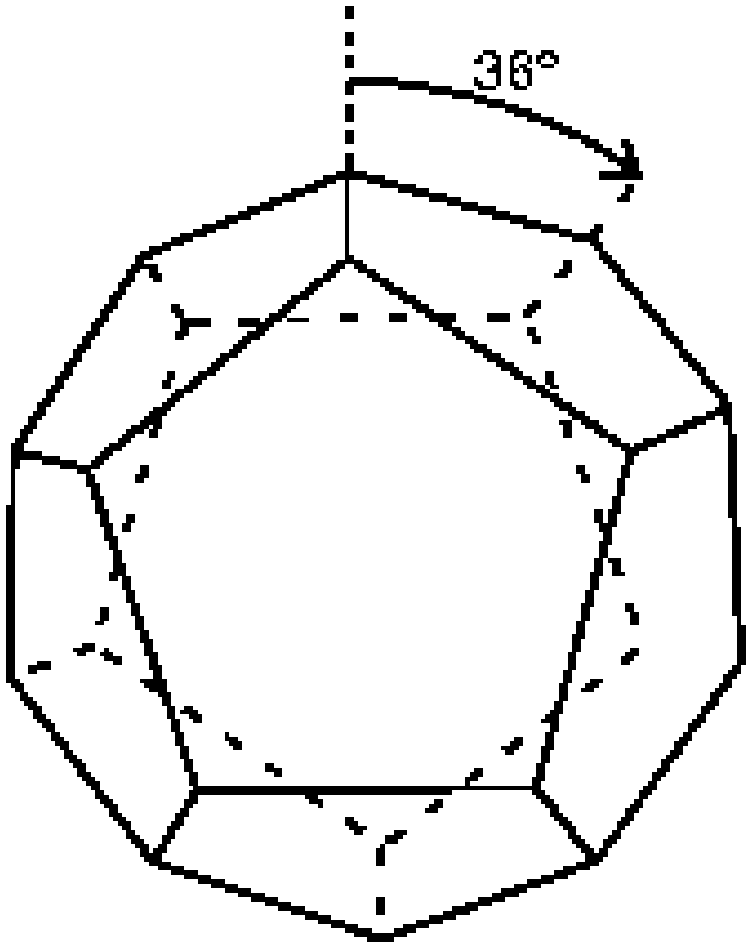
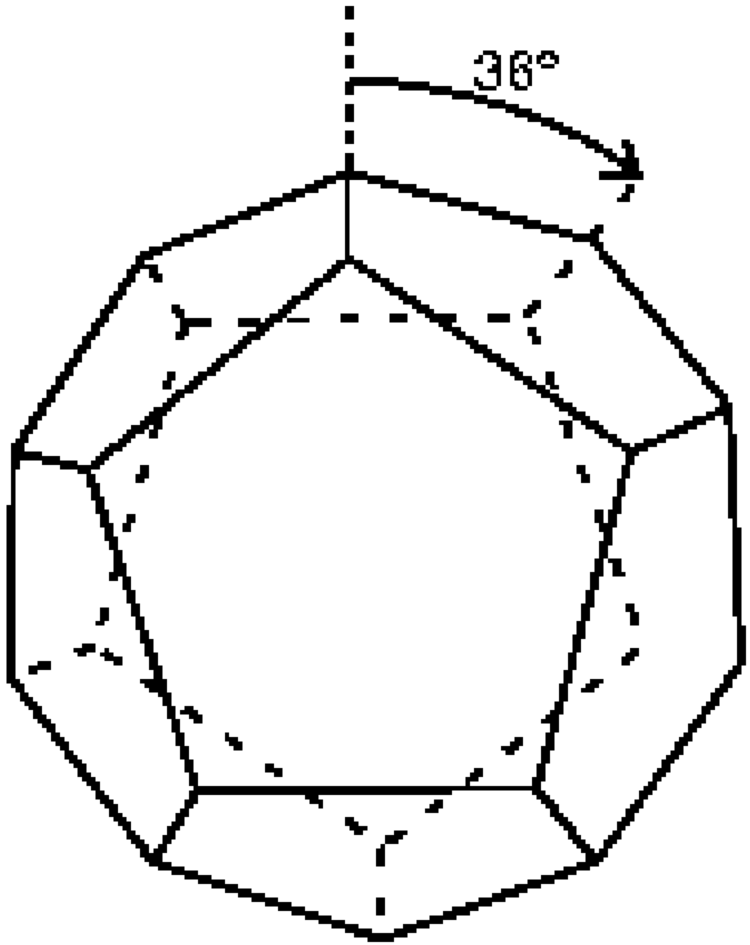


The Poincaré Dodecahedral 3-Manifold



- FD = positively curved dodecahedron

The Poincaré Dodecahedral 3-Manifold



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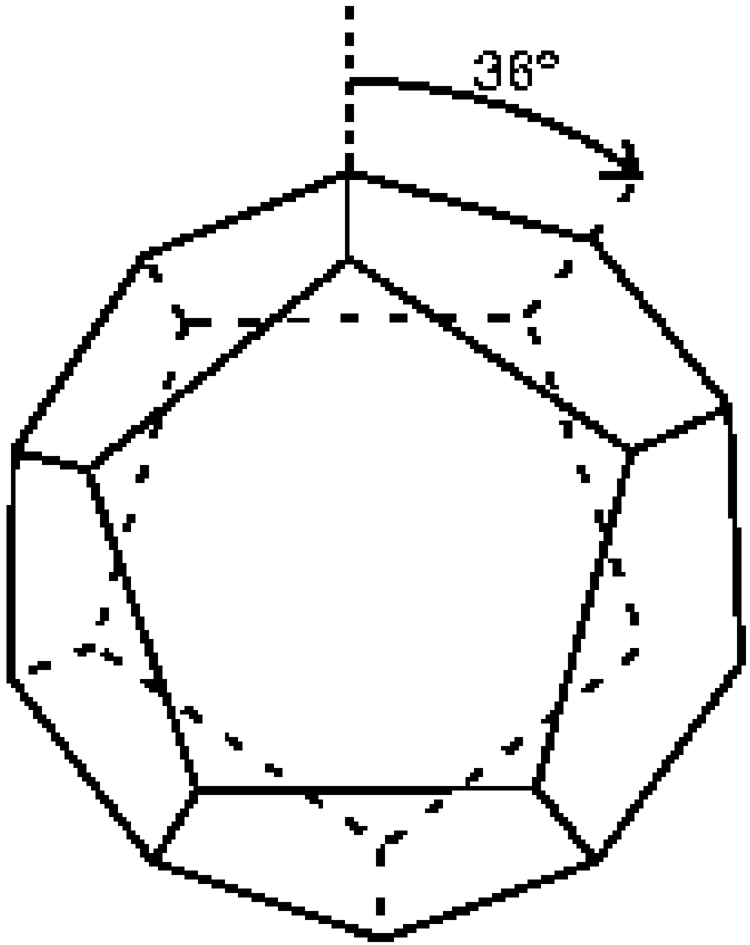

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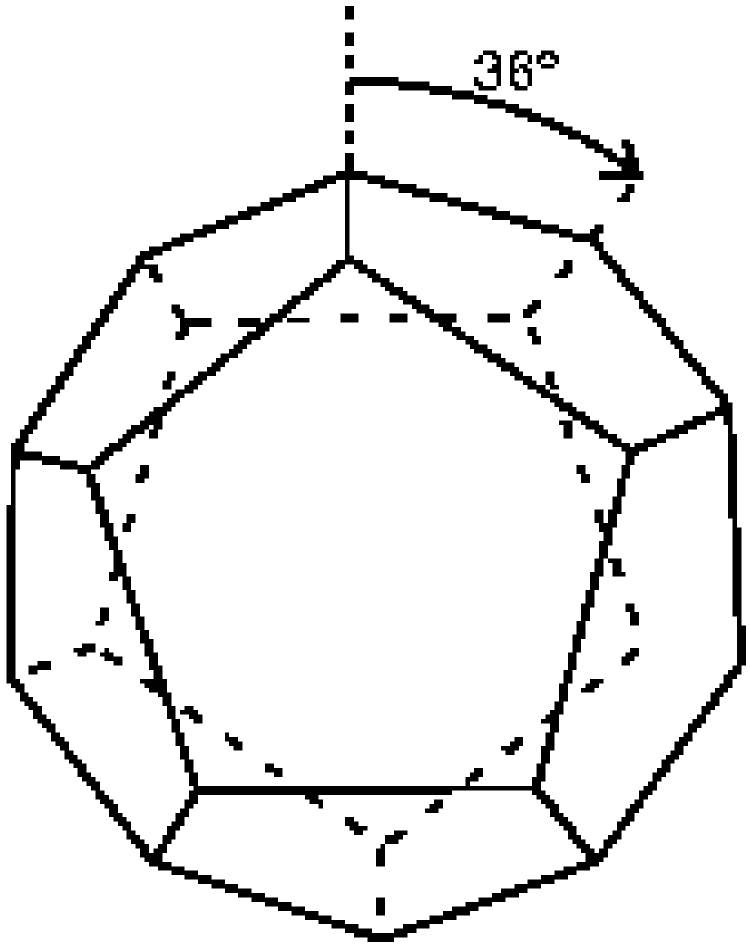

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- 120 copies of FD tile (exactly fill) S^3
- Luminet et al. (2003): S^3/I^* favoured by WMAP statistics

Optimal cross-correlation method

- extension to identified circles principle:

Optimal cross-correlation method

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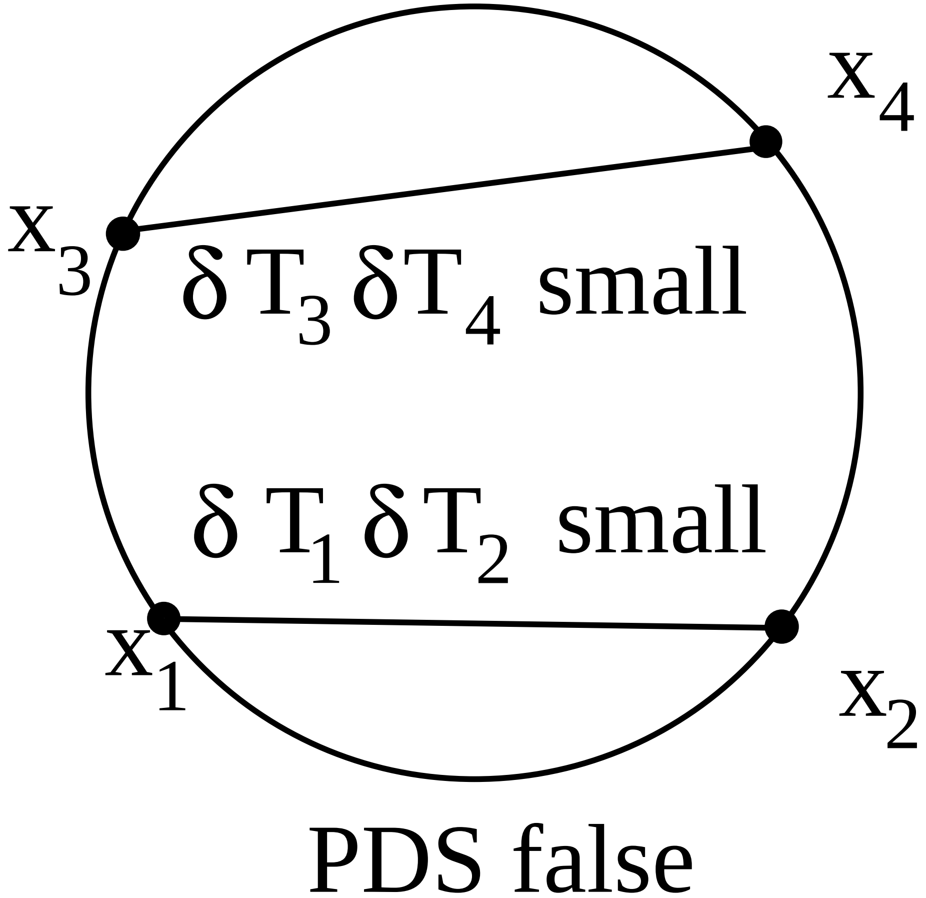

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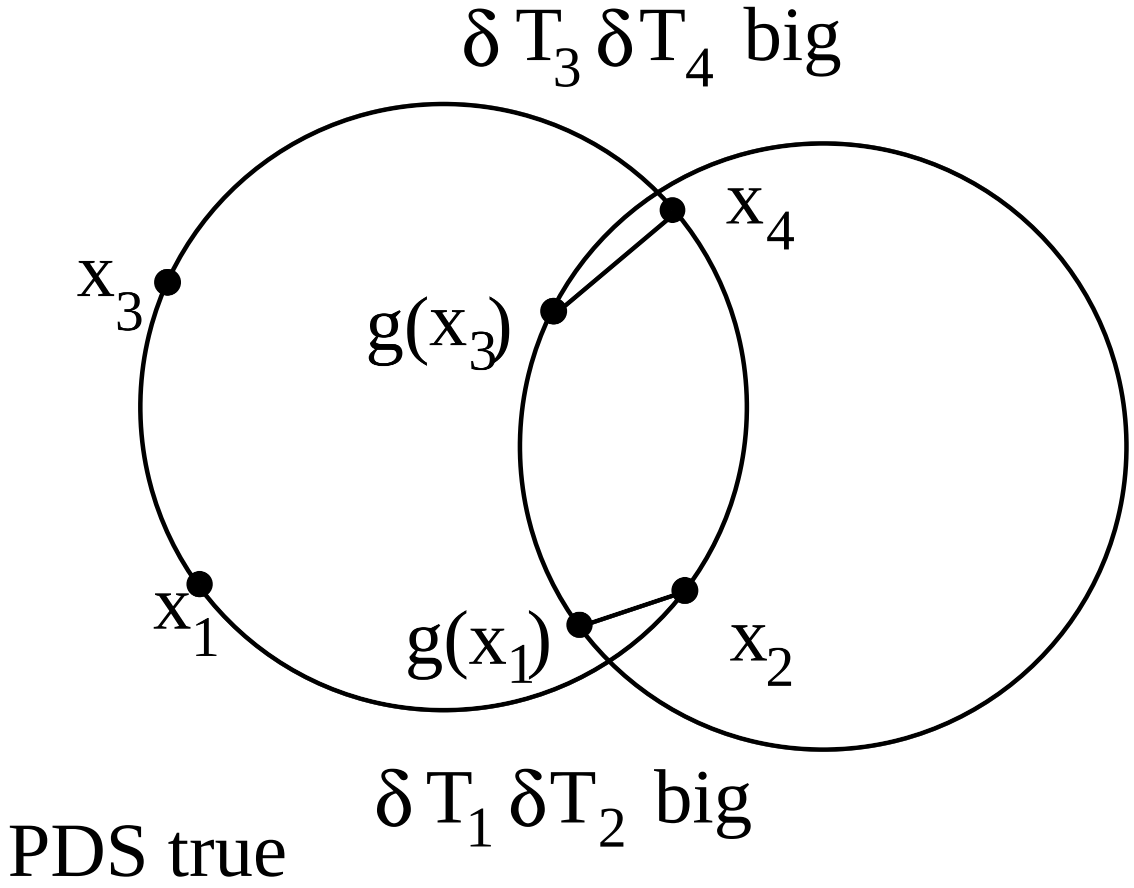

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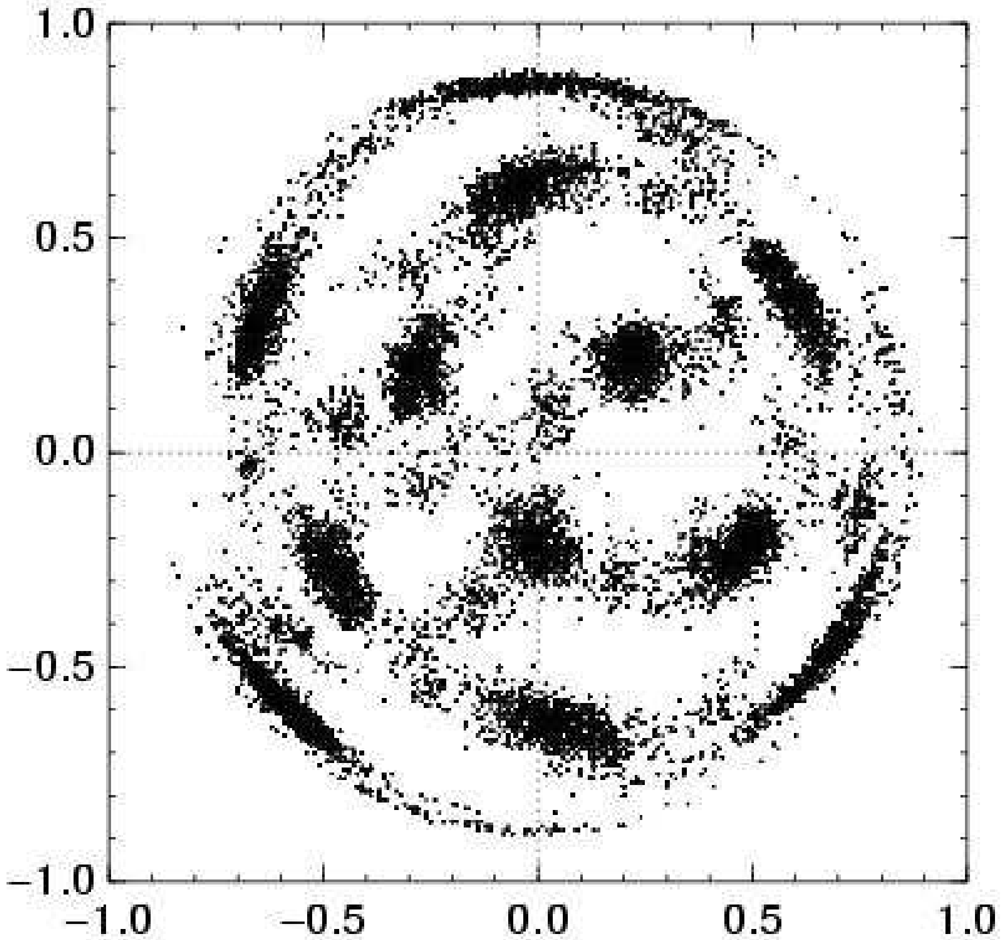
- allow arbitrary ϕ so that accidental correlations are likely to give an invalid value $\phi \neq \pm 36^\circ$

WMAP + Poincaré S^3/I^*



ILC + kp2

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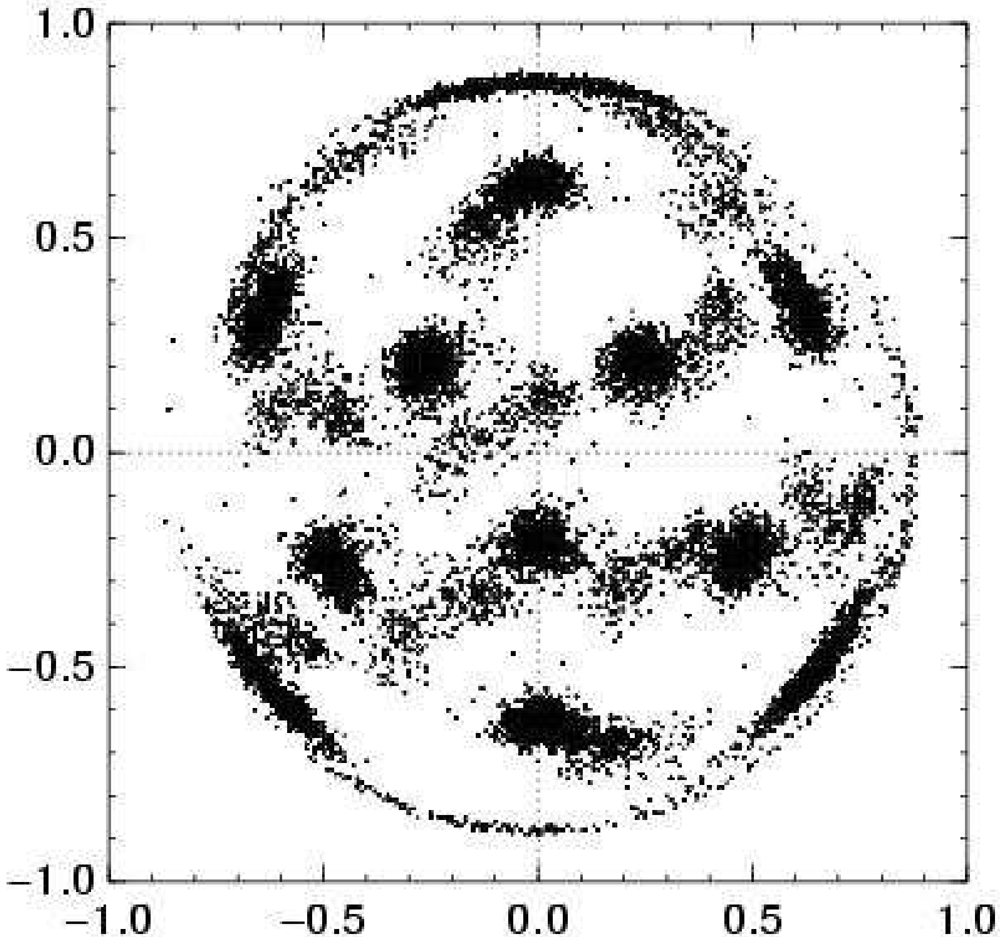

ILC + nomask

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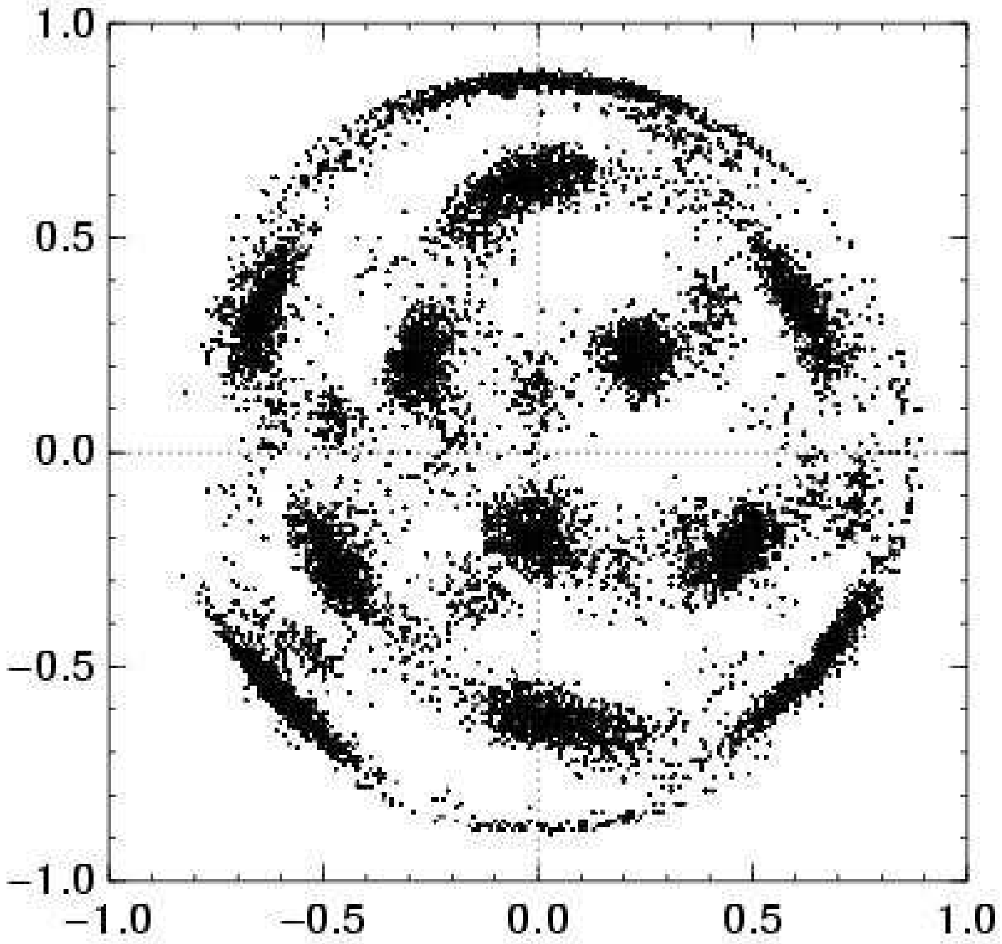

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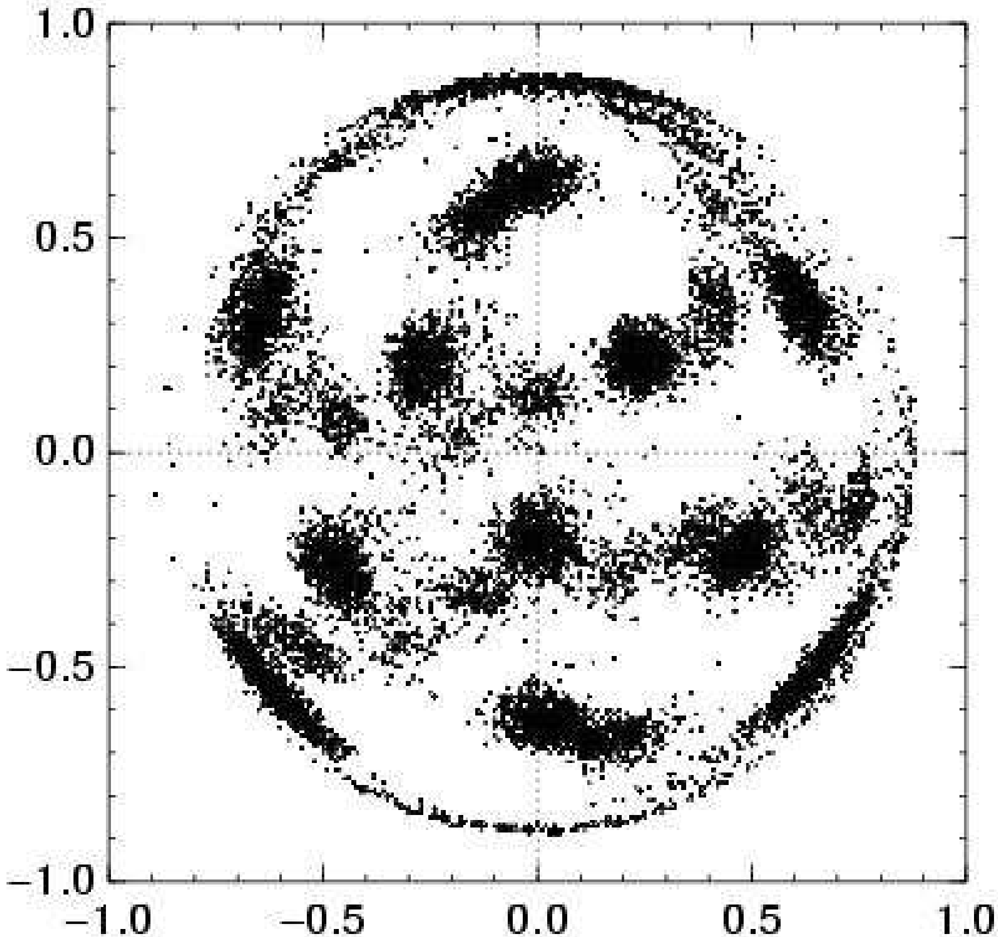

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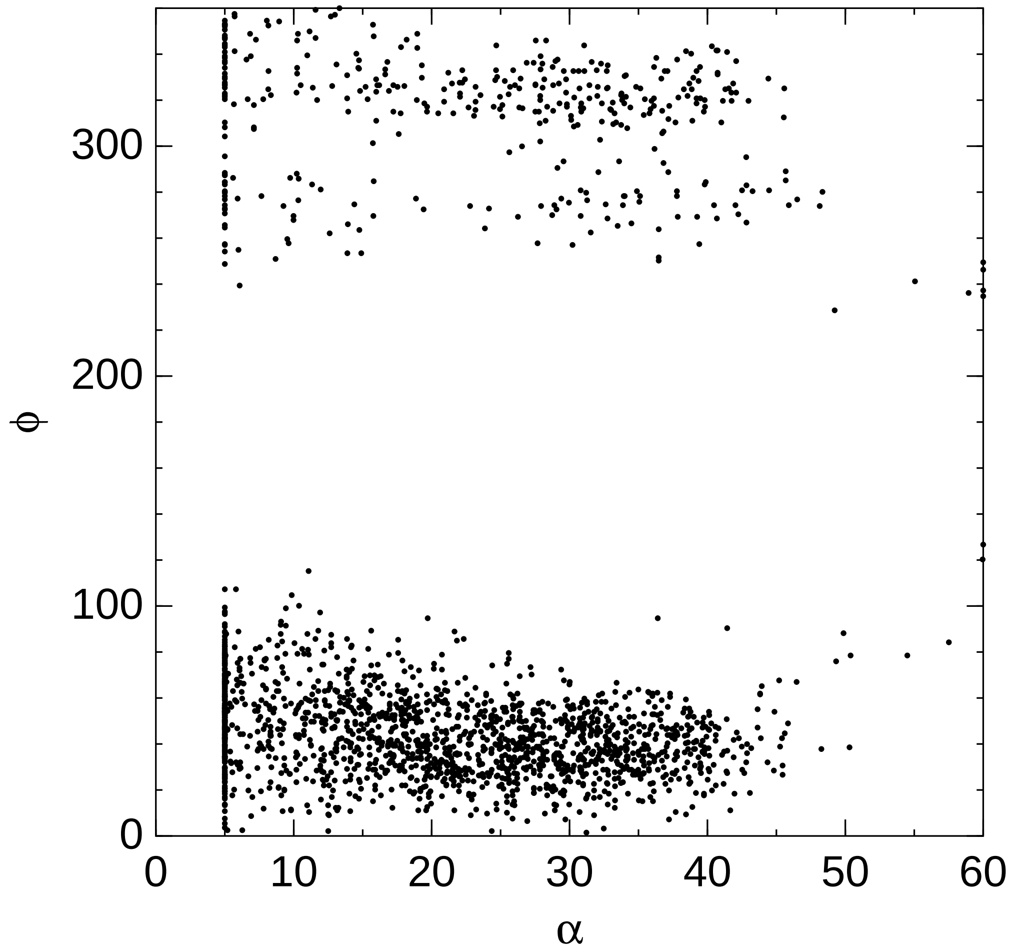
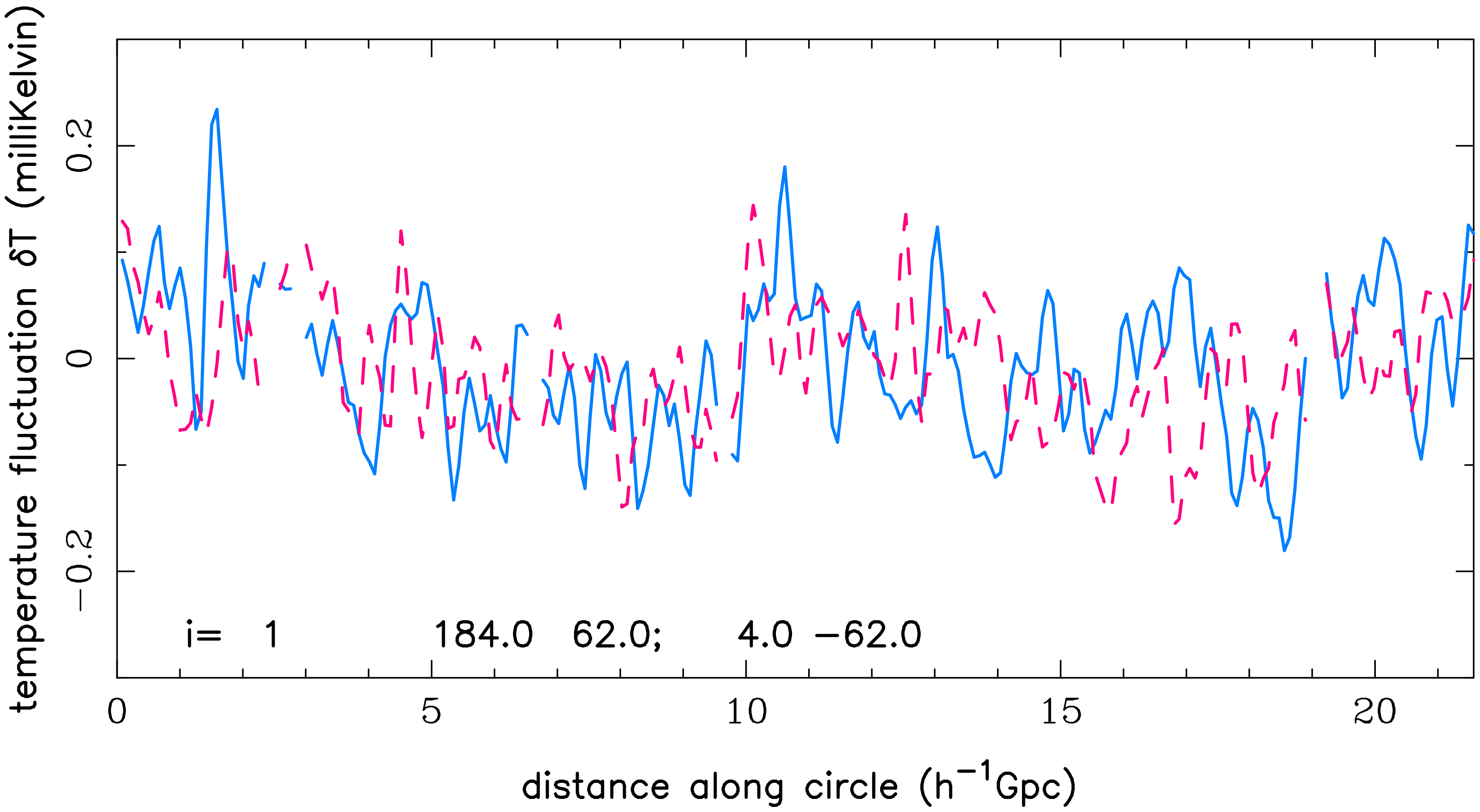


ϕ vs matched circle size α

WMAP + Poincaré S^3/I^*

$P_{\min}$	n	α°	$\sigma^\circ_{\langle\alpha\rangle}$	ϕ°	$\sigma^\circ_{\langle\phi\rangle}$
0.4	12589.0	20.6	0.6	**39.0**	**2.4**
0.5	6537.5	20.8	0.7	**38.7**	**2.2**
0.6	2961.0	22.1	0.5	**37.4**	**2.1**

RBSG08 matched circles



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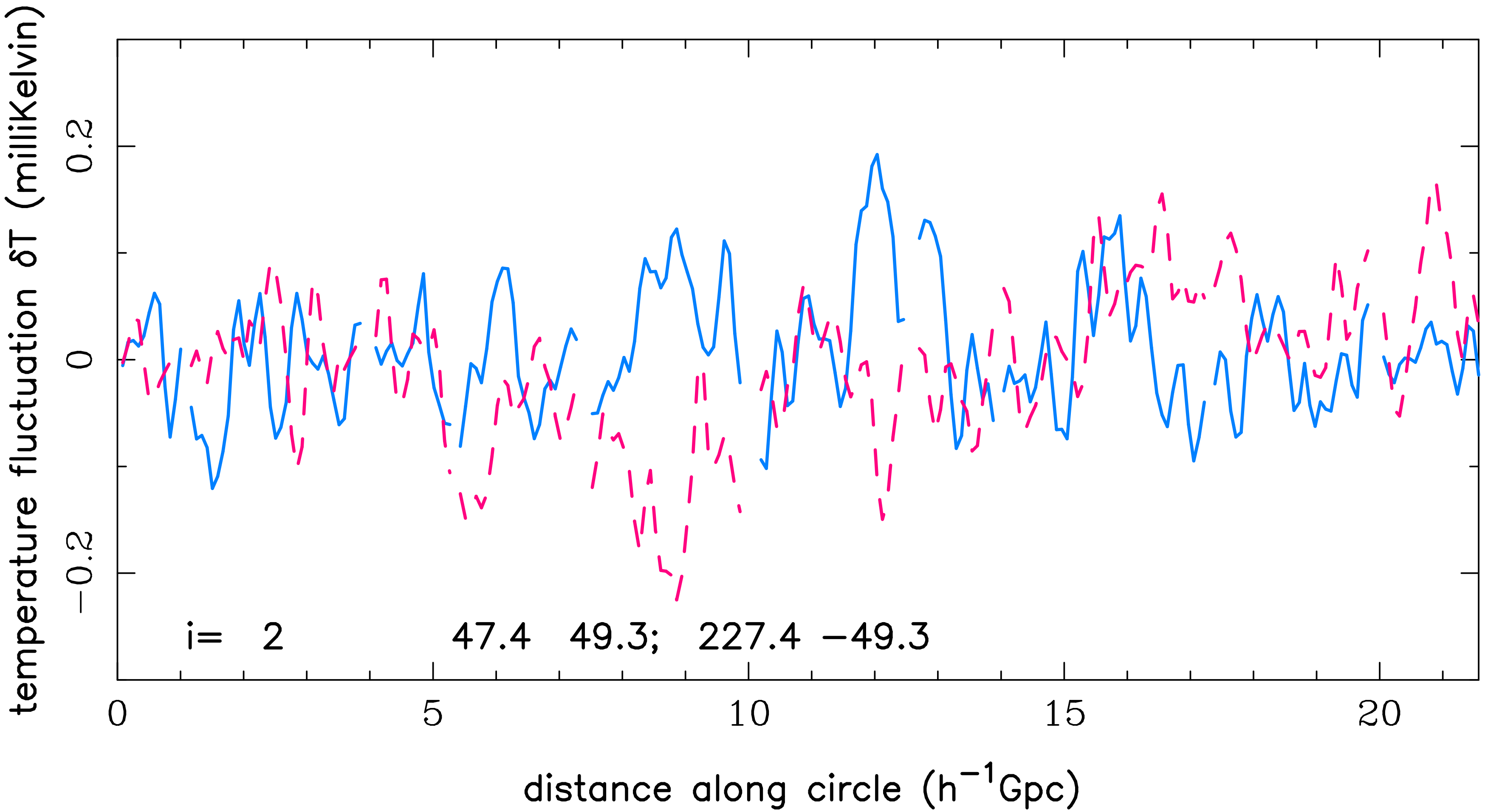

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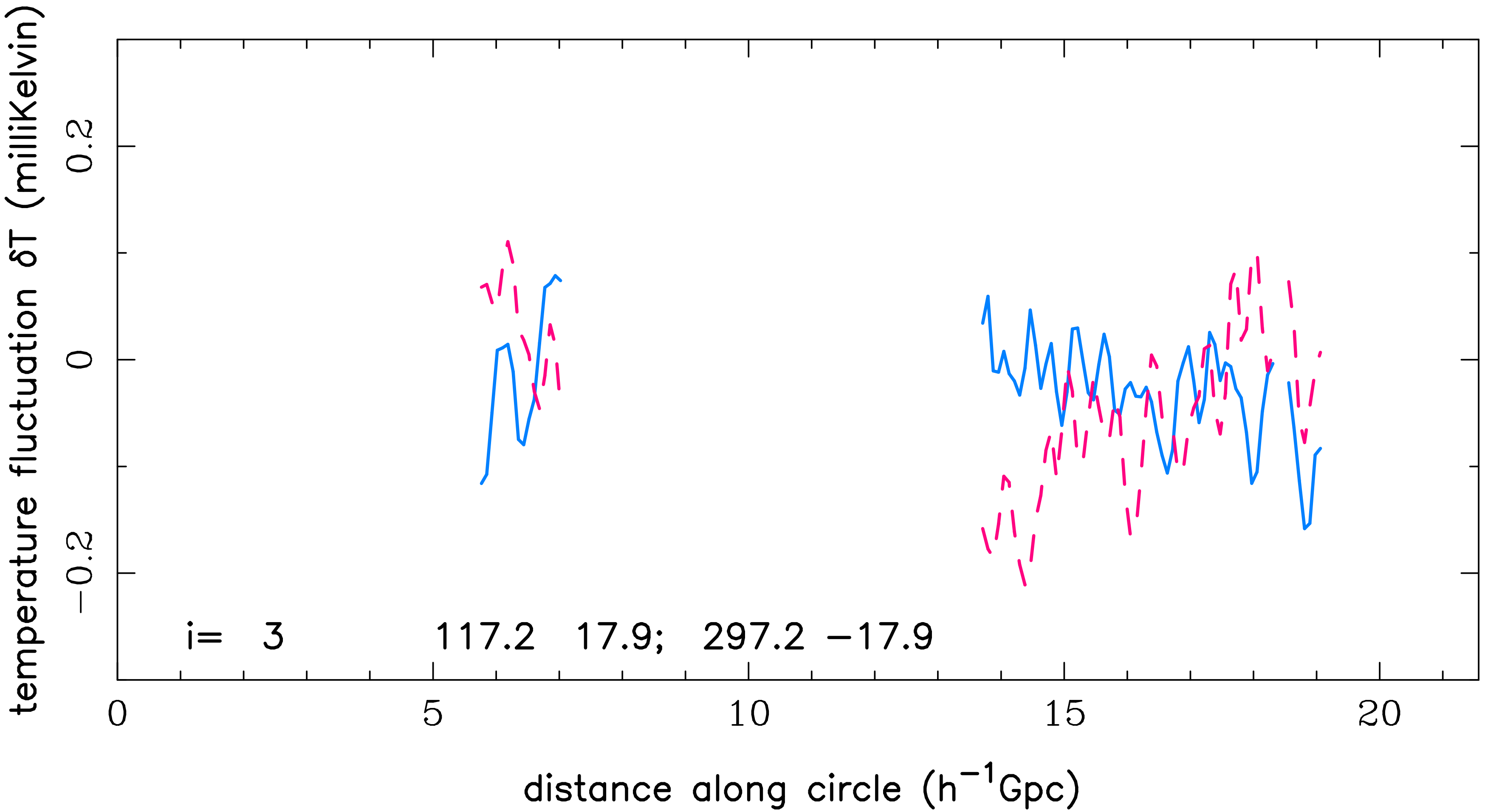

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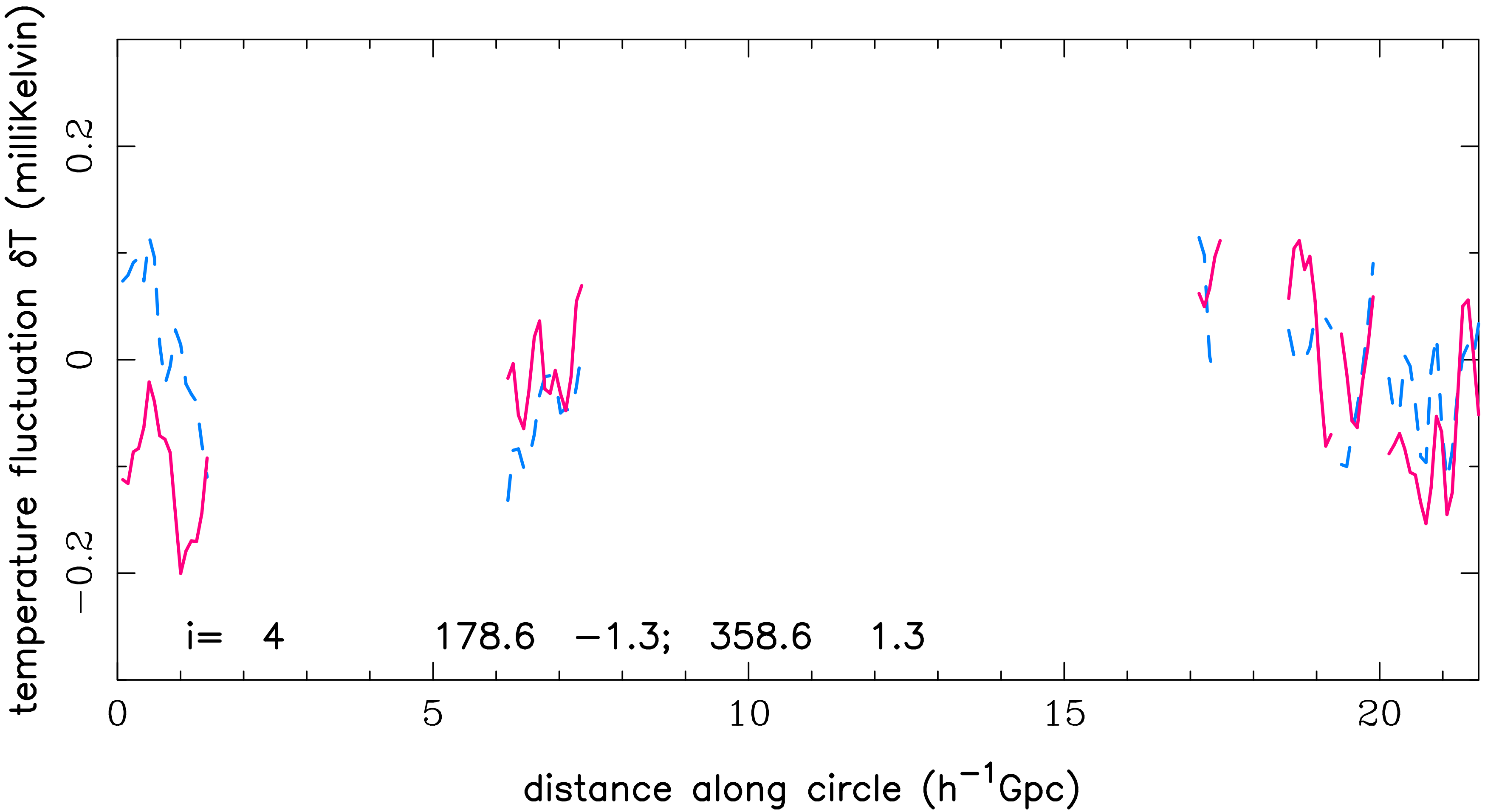

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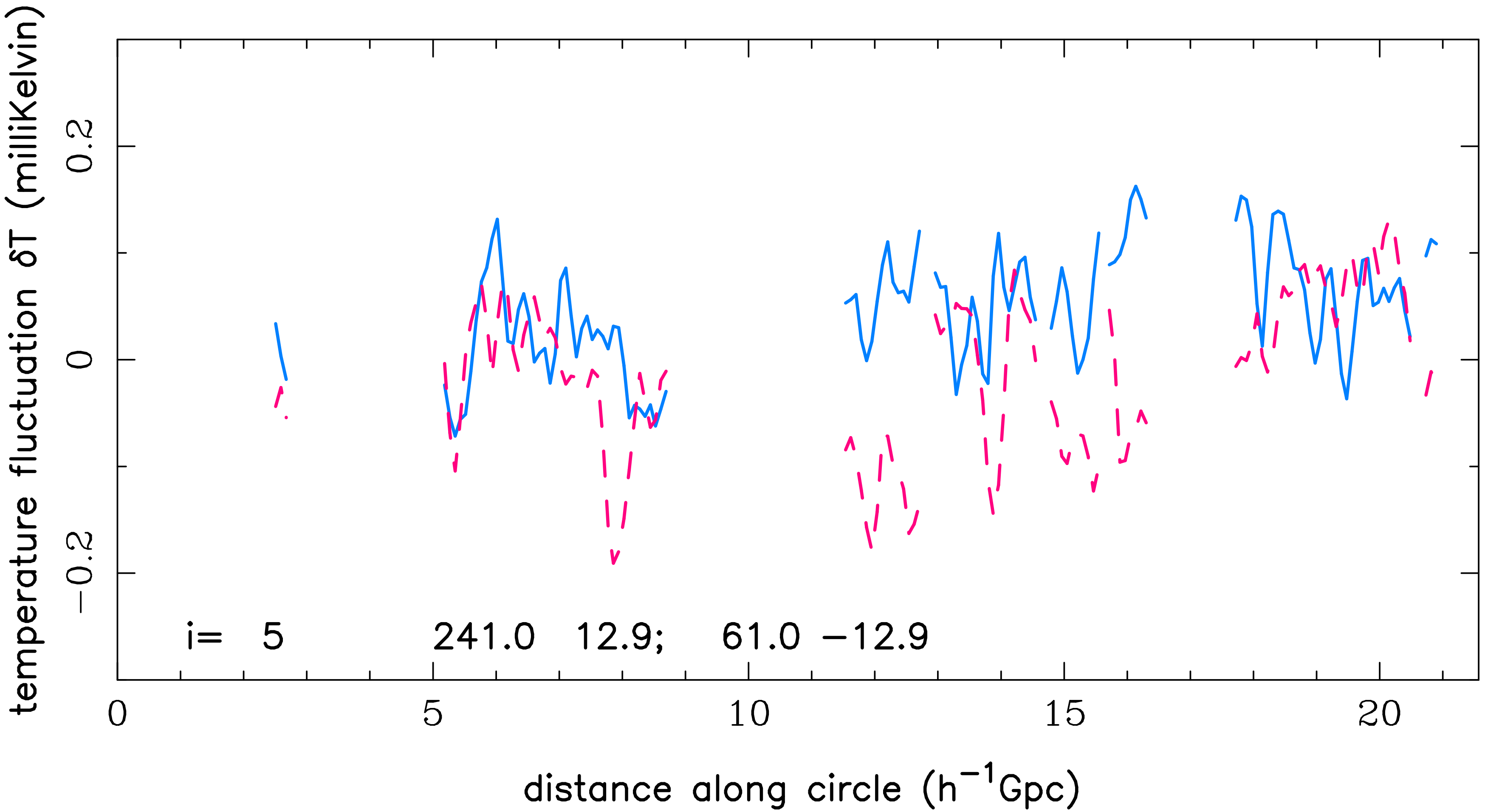

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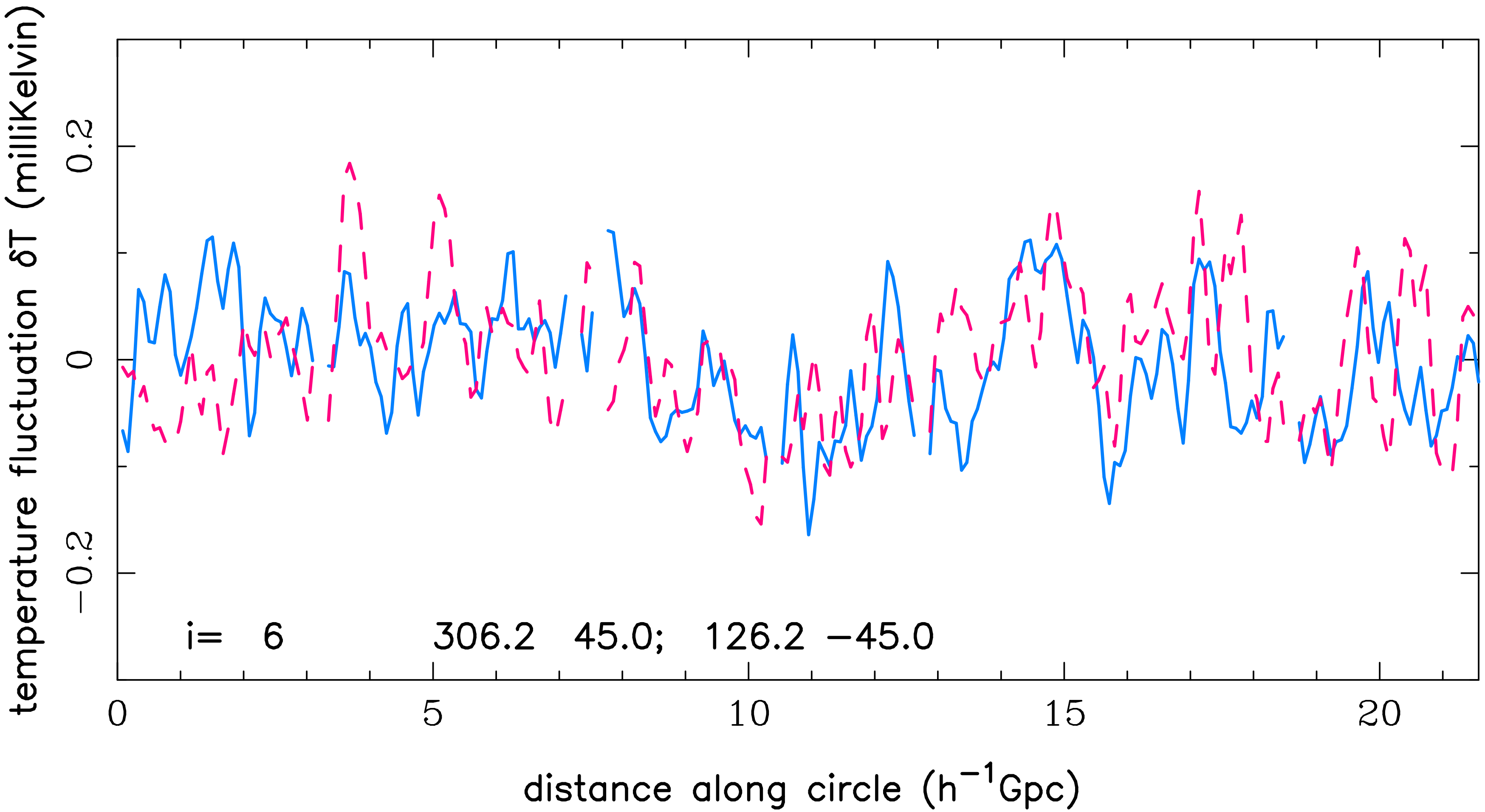

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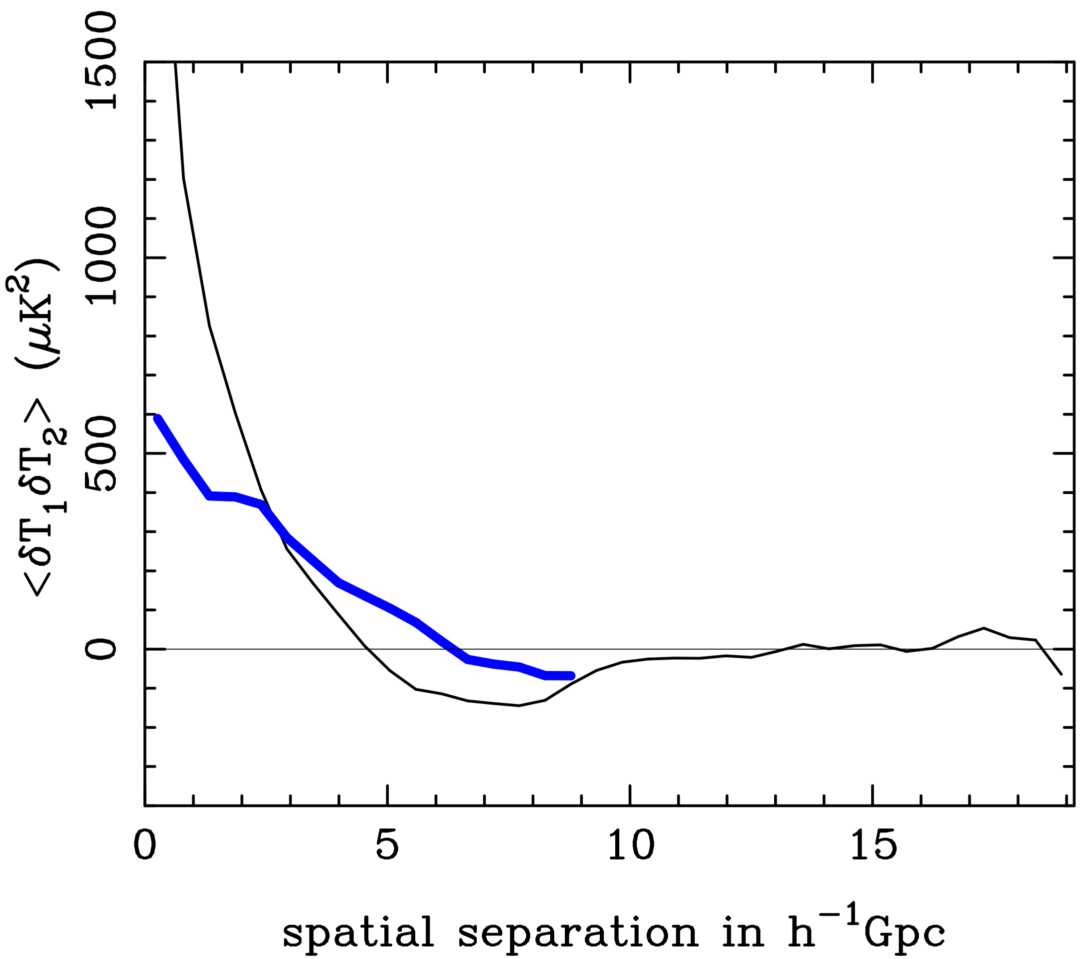

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- Planck (2013): (i) perturbation statististics assumption method; + (ii) identified circles: small correlation signal from S^3/I^* and other well-proportioned spaces, but consistent with noise

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