Special and General Relativity

Boud Roukema

GR: intro

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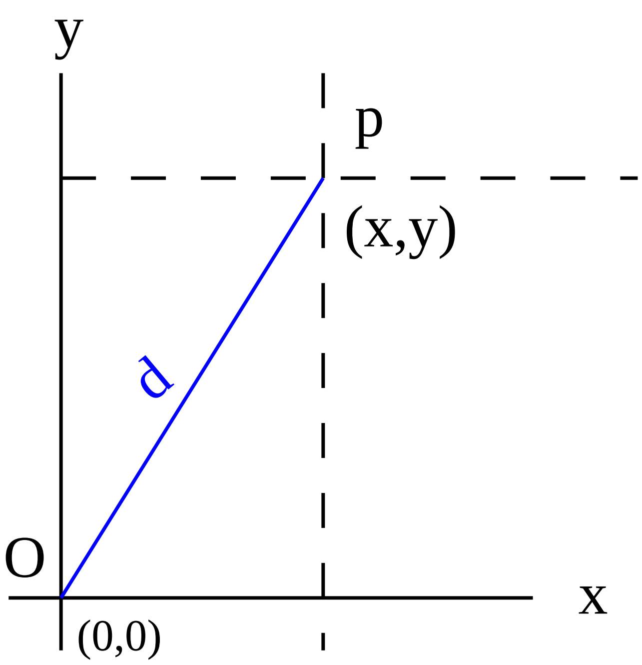
5. metric $\Leftarrow$ Einstein field equations

GR: coordinate transformations



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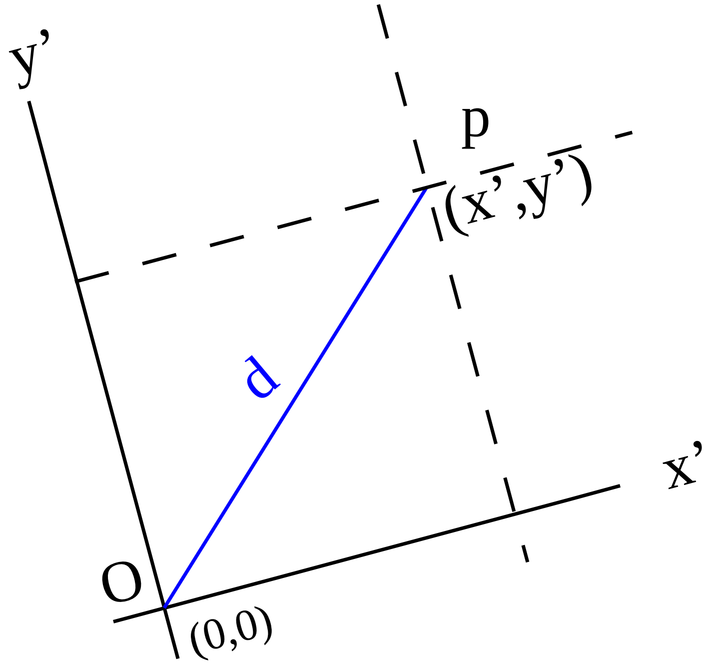

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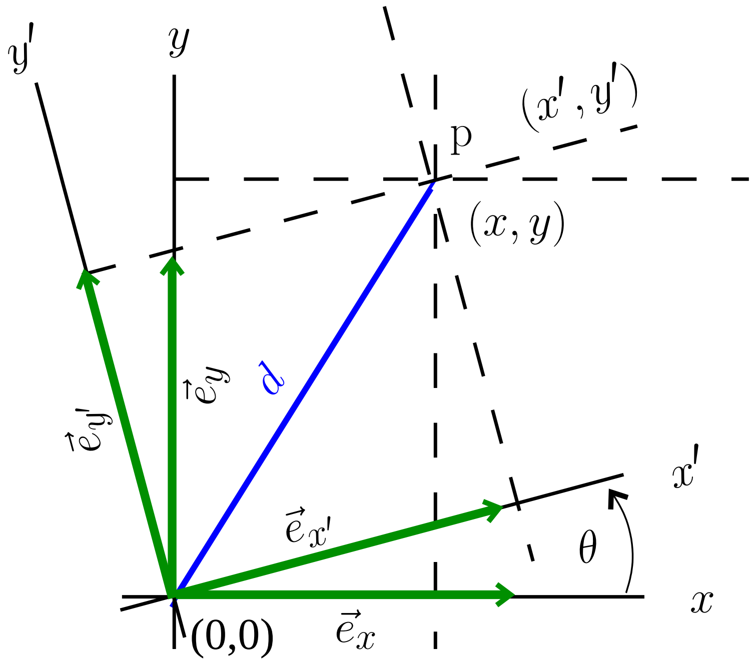

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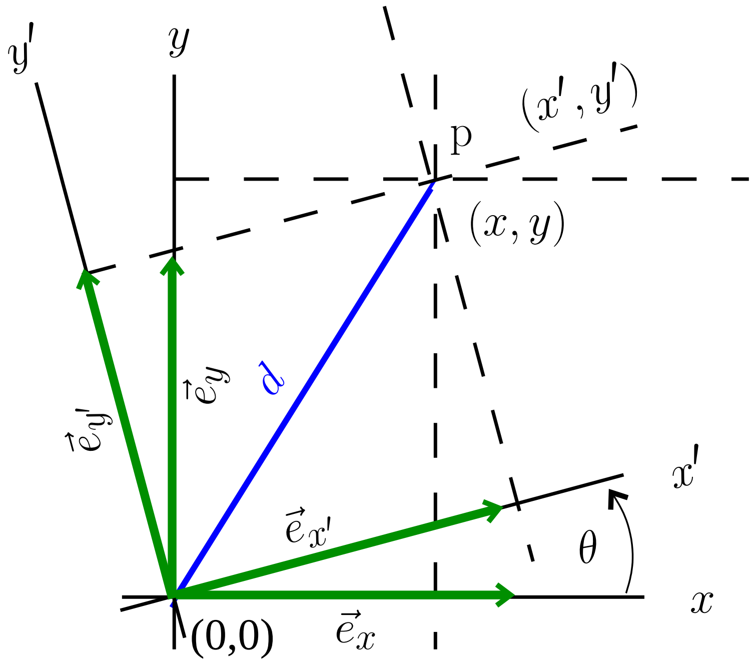

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

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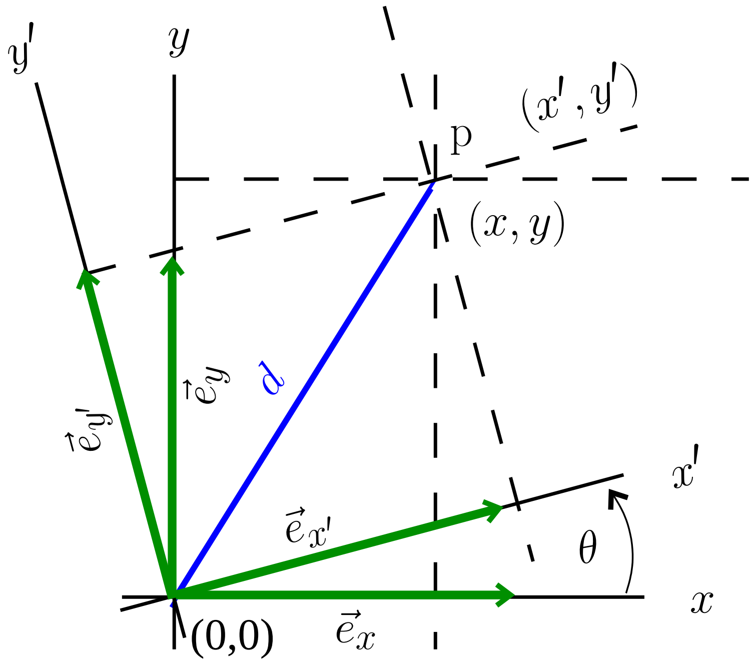

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

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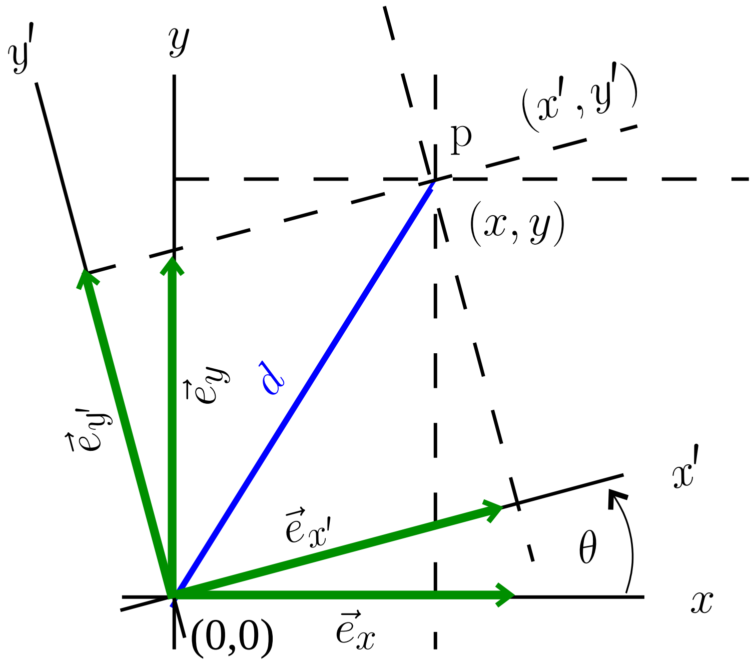

but $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} =$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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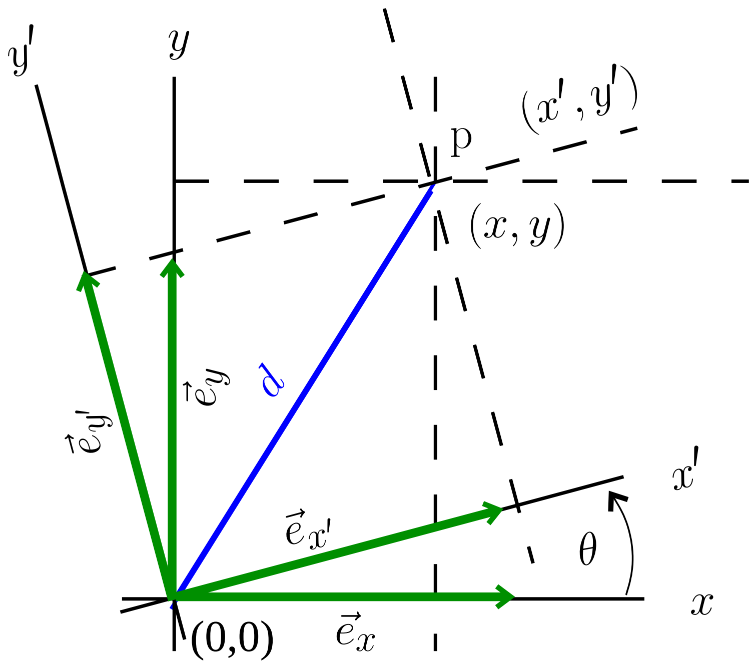

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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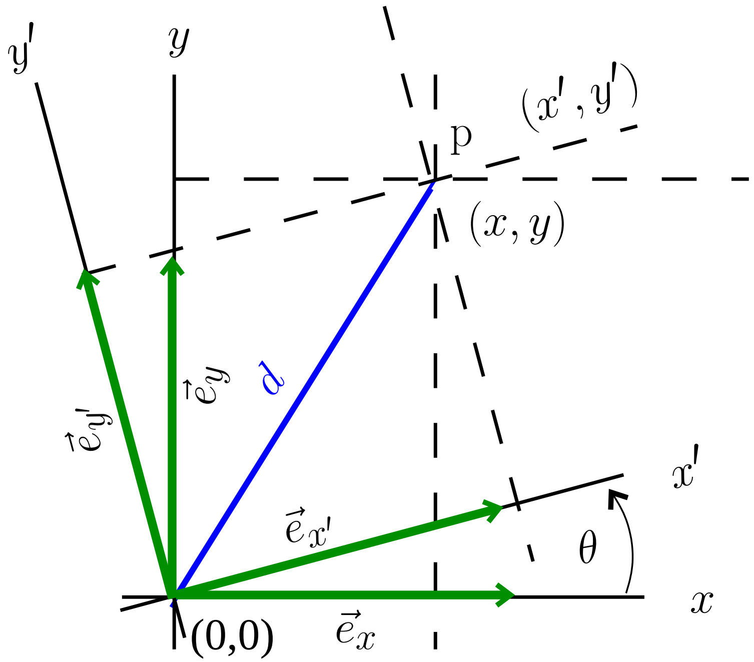

$$\vec{e}_{x'} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{e}_x + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{e}_y$$

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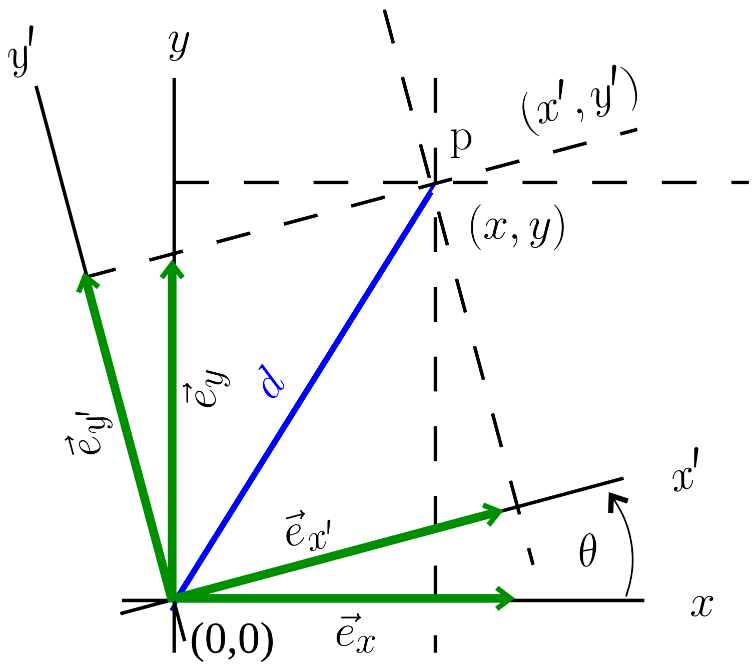

$$\vec{e}_{x'} = \Lambda^{x}_{x'}\,\vec{e}_x + \Lambda^{y}_{x'}\,\vec{e}_y$$

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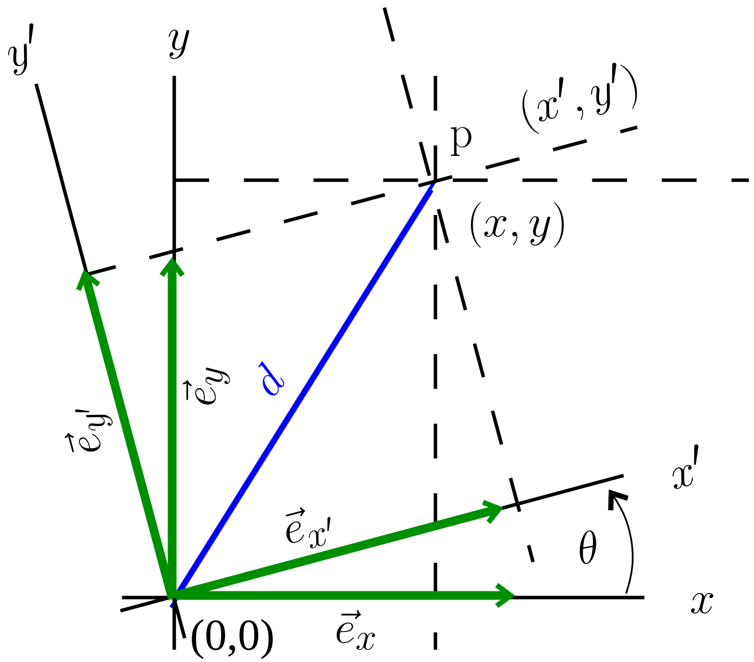

also:

$$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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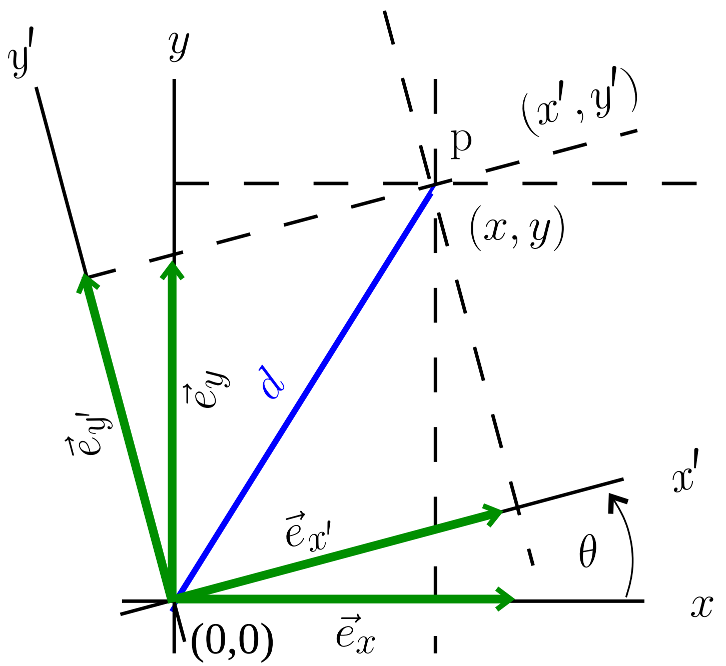

$$\vec{e}_{y'} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{e}_x + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{e}_y$$

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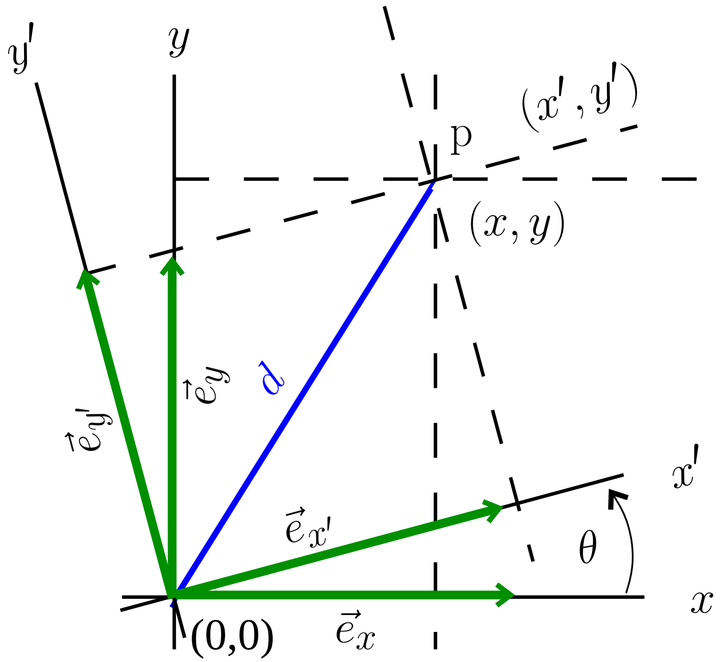

summary:

$$\vec{e}_{x'} = \Lambda^x_{x'}\,\vec{e}_x + \Lambda^y_{x'}\,\vec{e}_y,$$

$$\vec{e}_{y'} = \Lambda^x_{y'}\,\vec{e}_x + \Lambda^y_{y'}\,\vec{e}_y,$$

where $\Lambda^\alpha_{\beta'}$:= element

of inverse of $\Lambda^{\alpha'}_{\beta}$,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda^x_{x'}\,\vec{e}_x + \Lambda^y_{x'}\,\vec{e}_y, \qquad \vec{e}_{y'} = \Lambda^x_{y'}\,\vec{e}_x + \Lambda^y_{y'}\,\vec{e}_y,$$

$$\vec{p} \rightarrow_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

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$$\vec{p} = \sum_i p^i \vec{e}_i$$

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new basis vectors = sum of inverse $\Lambda \times$ old vectors

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new coords of vector $\vec{p}$ = $\Lambda \times$ old coords of same vector $\vec{p}$

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“contra” = inverse of change of basis vectors

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- $\vec{p}$ is invariant: no dependence on coords
- $\vec{p}$ is contravariant: p^i change inversely to $\vec{e}_i$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

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What is the relation between $(\phi_{,x'}, \phi_{,y'})$

and $(\phi_{,x}, \phi_{,y})$?

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ϕ depends either on x and y, or on x' and y'

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$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$ (example: rotation)

$x_{,x'} = \frac{\partial x}{\partial x'} = \cos\theta$

$x_{,y'} = \frac{\partial x}{\partial y'} = -\sin\theta \ldots$

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$$\begin{pmatrix} x \\ y \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \text{(general)}$$

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$$\tilde{\mathrm{d}}\phi = \left((\tilde{\mathrm{d}}\phi)_{x'}, (\tilde{\mathrm{d}}\phi)_{y'}\right) = \left((\tilde{\mathrm{d}}\phi)_{x}, (\tilde{\mathrm{d}}\phi)_{y}\right) \Lambda^{-1}$$

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$$(\tilde{\mathrm{d}}\phi)_{\mu'} = (\tilde{\mathrm{d}}\phi)_\nu \Lambda^\nu{}_{\mu'}$$

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basis vectors of different bases: $\vec{e}_{\mu'} = \Lambda^{\nu}{}_{\mu'} \vec{e}_{\nu}$

same vector: $(\vec{p})^{\mu'} = \Lambda^{\mu'}_{\nu} (\vec{p})^{\nu}$

same gradient (example 1-form): $(\tilde{\mathrm{d}}\phi)_{\mu'} = (\tilde{\mathrm{d}}\phi)_{\nu} \Lambda^{\nu}{}_{\mu'}$

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- vector $\vec{p}$ is invariant: no dependence on coords
- $\vec{p}$ is contravariant: components p^{ν} change inversely to

how $\vec{e}_{\mu}$ change; inverses: matrix $\{\Lambda^{\nu}{}_{\mu'}\}$ vs $\{\Lambda^{\beta'}_{\alpha}\}$

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- 1-form $\tilde{\mathrm{d}}\phi$ is invariant: no dependence on coords
- $\tilde{\mathrm{d}}\phi$ is covariant: components $(\tilde{\mathrm{d}}\phi)_{\mu}$ change like $\vec{e}_{\mu}$ (but left-multiply)

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w:Covariance and contravariance of vectors

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

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vector–1-form duality defined:

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vector–1-form duality defined:

$\langle \vec{p}, \tilde{q} \rangle = \sum_\mu p^\mu q_\mu$

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can be called $I \equiv \delta^\mu_\nu$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

$\langle \vec{p}, \tilde{q} \rangle = p^\mu q_\mu = \vec{p}(\tilde{q}) = \tilde{q}(\vec{p})$

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think: vector $\longrightarrow$ column vector
1-form $\longrightarrow$ row vector

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$\langle\ ,\ \rangle$ = (1,1)-tensor = “row-column” matrix $I = \delta^{\mu}_{\nu}$

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ordinary linear algebra: column vectors, row vectors, matrices

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(m, n)-tensor algebra: m column n row $m + n$-arrays

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e.g.: $(0, 2)$-tensor: metric $g_{\mu\nu}$

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using $\langle\ ,\ \rangle$, $(1, 0)$-tensor = vector = function of 1-forms

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(m,n)-tensor = function of m 1-forms and n vectors

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loosely speaking, the second $\otimes$ means “function of two vectors” (or 1-forms, or a vector and a 1-form) in *that particular left-to-right order*

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order of $V^* \otimes V^*$ = 2

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warning: the "rank" of tensors has two different meanings: w:Tensor_(intrinsic_definition)#Tensor_rank

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order of $V^* \otimes V^*$ = 2

dimension of $V^* \otimes V^*$ = 16 (for $V =$ spacetime)

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also written: $\vec{A} \cdot \vec{B}$ “dot product”

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in general, for a 2-form T, $T(\vec{A}, \vec{B}) \neq T(\vec{B}, \vec{A})$

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$$\mathbf{g} = g_{xy}\tilde{e}^x \otimes \tilde{e}^y$$

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$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

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we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

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$\Rightarrow \mathbf{g} = g_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr} \tilde{e}^r \otimes \tilde{e}^r + g_{r\theta} \tilde{e}^r \otimes \tilde{e}^\theta + g_{\theta r} \tilde{e}^\theta \otimes \tilde{e}^r + g_{\theta\theta} \tilde{e}^\theta \otimes \tilde{e}^\theta$

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check: $\mathbf{g}(\vec{e}_r, \vec{e}_r) = g_{rr}$?

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$\mathbf{g}(\vec{e}_r, \vec{e}_r) = (g_{rr} \tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta} \tilde{e}^\theta \otimes \tilde{e}^\theta)(\vec{e}_r, \vec{e}_r)$

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$\mathbf{g}(\vec{e}_r, \vec{e}_r) = g_{rr}\, \tilde{e}^r(\vec{e}_r)\, \tilde{e}^r(\vec{e}_r) + g_{\theta\theta}\, \tilde{e}^\theta(\vec{e}_r)\, \tilde{e}^\theta(\vec{e}_r)$

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e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

$$\mathbf{g}(\vec{e}_r, \vec{e}_r) = g_{rr}\left\langle \tilde{e}^r, \vec{e}_r \right\rangle \left\langle \tilde{e}^r, \vec{e}_r \right\rangle + g_{\theta\theta}\left\langle \tilde{e}^\theta, \vec{e}_r \right\rangle \left\langle \tilde{e}^\theta, \vec{e}_r \right\rangle$$

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e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

$\mathbf{g}(\vec{e}_r, \vec{e}_r) = g_{rr} \times 1 \times 1 + g_{\theta\theta} \times 0 \times 0$ by duality through scalar product $\langle\ ,\ \rangle$

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e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

$\mathbf{g}(\vec{e}_r, \vec{e}_r) = g_{rr}$ self-consistent definition

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lower an index: $g_{\mu\nu}A^\mu = A_\nu$

raise an index: $g^{\mu\nu}B_\nu = B^\mu$

GR: what is a coordinate?

a coordinate, e.g. x^0 or x^1 is a scalar field on the 4-manifold

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a coordinate system x^μ = set of four scalar fields on the 4-manifold

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(Bertschinger writes $x^\mu_{\mathbf{X}}$ to show dependence on position $\mathbf{X}$ in manifold $\neq$ vector space)

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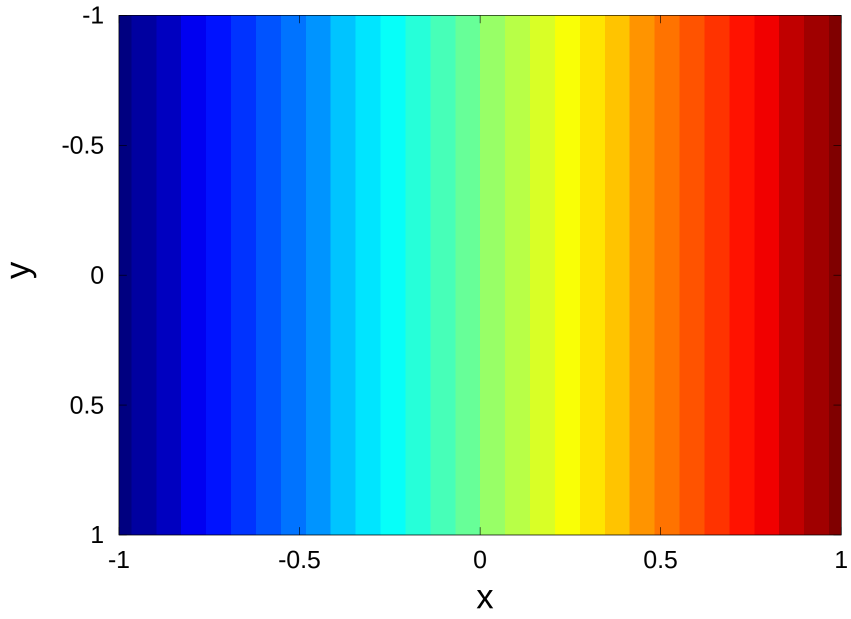

e.g. on $\mathbb{R}^2$

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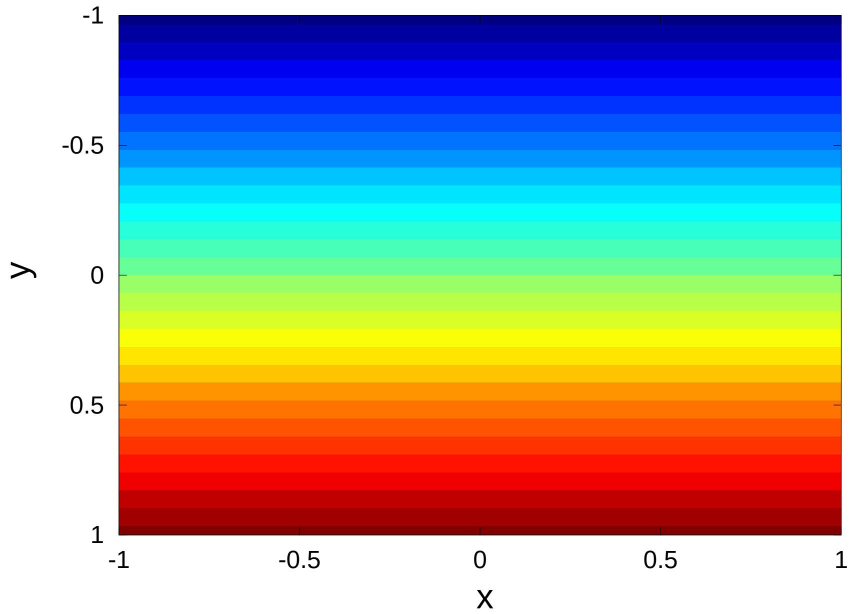

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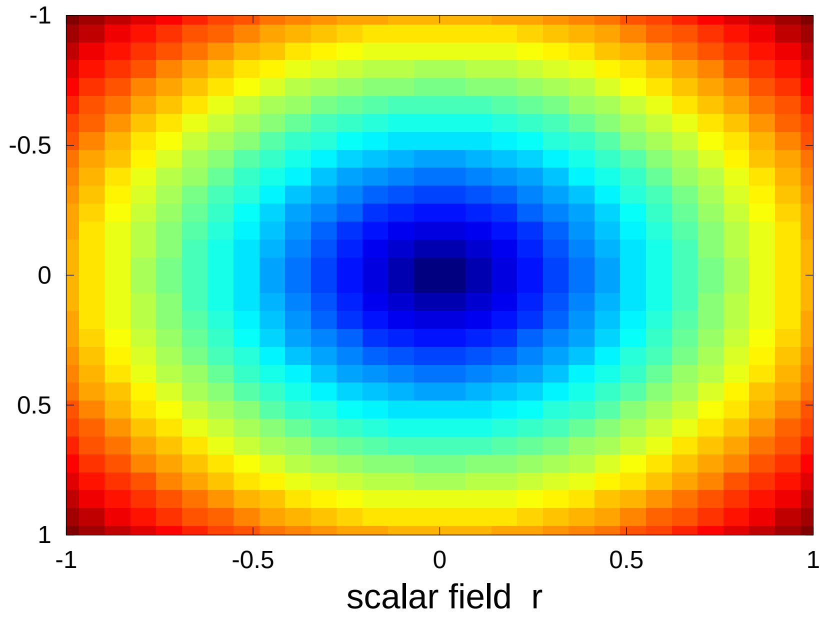

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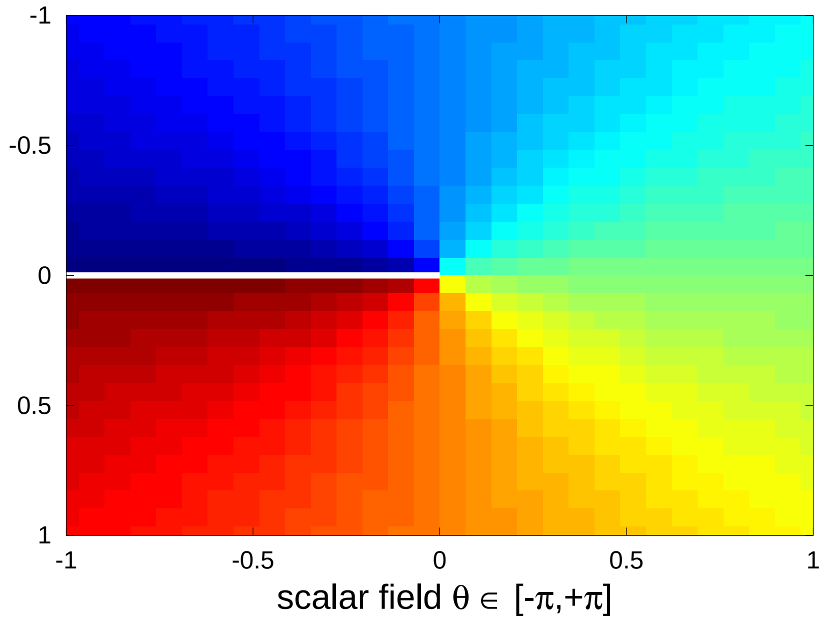

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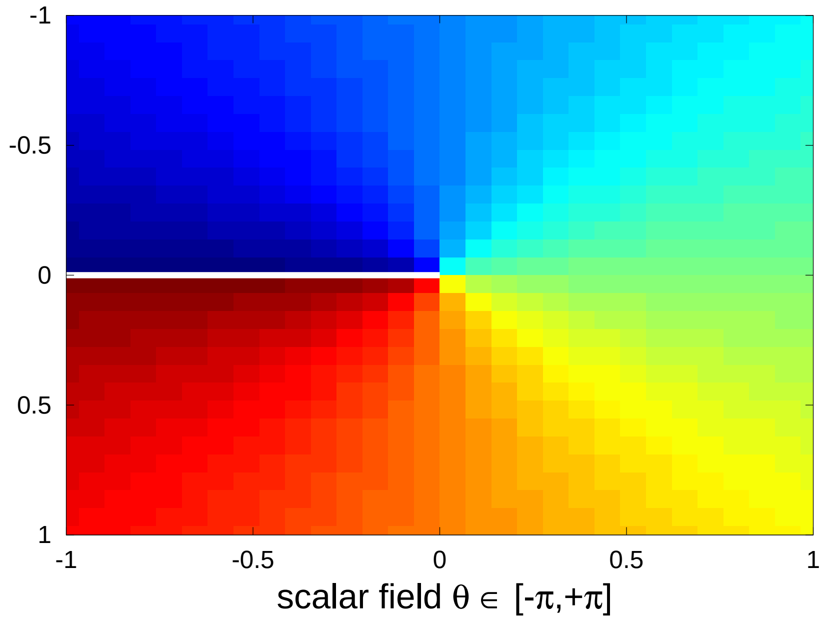

e.g. on $\mathbb{R}^2$

coordinate singularity $\neq$ singularity in manifold

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(Bertschinger writes $\widetilde{\nabla}$ for the gradient $\tilde{\mathrm{d}}$)

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GR: metric using $\mathrm{d}\vec{x}$

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$\mathrm{d}s^2 := |\mathrm{d}\vec{x}|^2 = \mathbf{g}(\mathrm{d}\vec{x}, \mathrm{d}\vec{x}) = \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x}$ coordinate-free

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$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu x^\nu$ if x^μ are a coordinate basis

GR: e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$

$g_{r\theta}$ and g_{xy}

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$$

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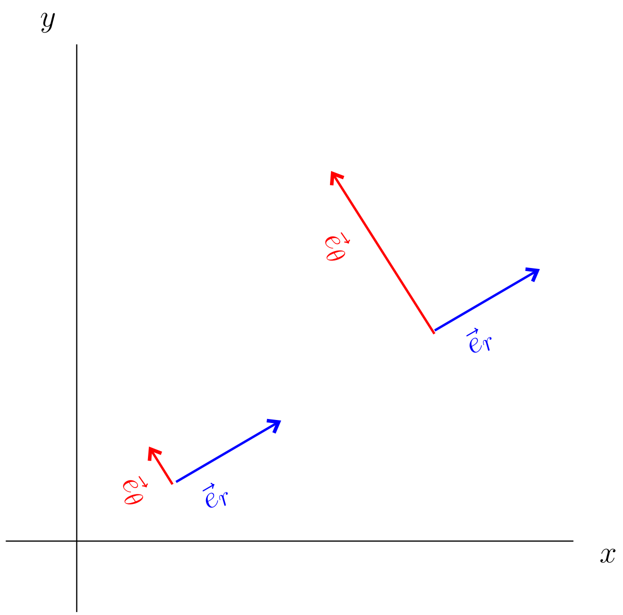

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$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \Rightarrow g^{xx} = 1 = g^{yy}, g^{xy} = 0 = g^{yx}$

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but $g^{rr} = 1 \neq g^{\theta\theta} = r^{-2}, g^{r\theta} = 0 = g^{\theta r}$

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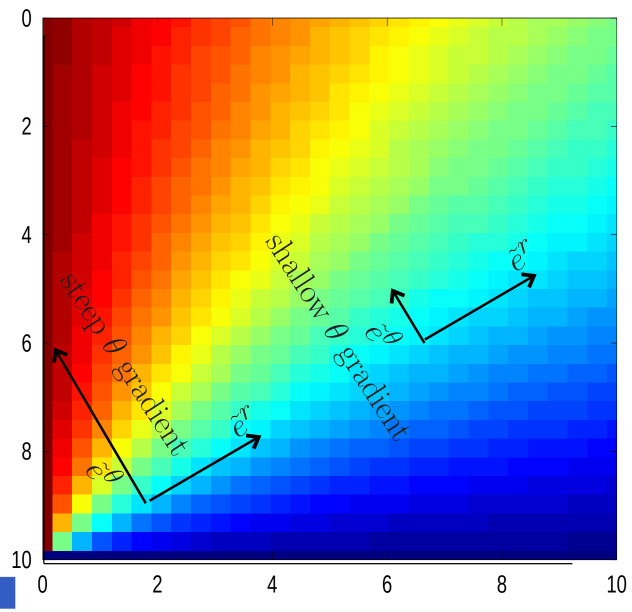

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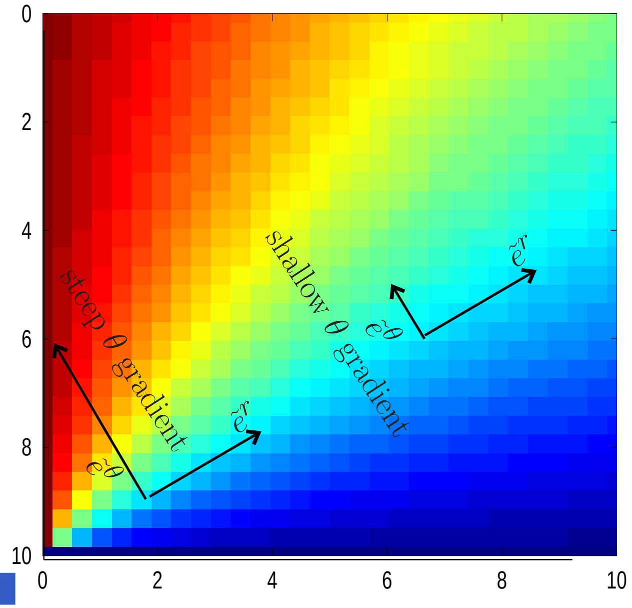
$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \Rightarrow g^{xx} = 1 = g^{yy}, g^{xy} = 0 = g^{yx}$

but $g^{rr} = 1 \neq g^{\theta\theta} = r^{-2}, g^{r\theta} = 0 = g^{\theta r}$



so $\tilde{e}^r \cdot \tilde{e}^r = 1$, $\tilde{e}^\theta \cdot \tilde{e}^\theta = r^{-2} \neq 1$

GR: gradient of a vector: $\nabla\vec{A}$

gradient of scalar field: $\tilde{\mathrm{d}}\phi \equiv \widetilde{\nabla}\phi$

GR: gradient of a vector: $\nabla\vec{A}$

what is gradient of vector field $\widetilde{\nabla}\vec{A}$?

GR: gradient of a vector: $\nabla\vec{A}$

$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$$

GR: gradient of a vector: $\nabla\vec{A}$

$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu\vec{e}_\nu)$$

$$= \tilde{e}^\mu\partial_\mu(A^\nu\vec{e}_\nu)$$

GR: gradient of a vector: $\nabla\vec{A}$

$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$$

$$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$$

$$= \tilde{e}^\mu \otimes [(\partial_\mu A^\nu)\vec{e}_\nu + A^\nu \partial_\mu \vec{e}_\nu]$$ by product rule and linearity

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$$= \tilde{e}^\mu\partial_\mu(A^\nu\vec{e}_\nu)$$

$$= \partial_\mu A^\nu\tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu\tilde{e}^\mu \otimes \partial_\mu\vec{e}_\nu$$

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$$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \partial_\mu \vec{e}_\nu$$

give a name to the second part: it must be a linear combination of basis vectors $\vec{e}_\lambda$

GR: gradient of a vector: $\nabla \vec{A}$

$\widetilde{\nabla} \vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$

$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$

$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \partial_\mu \vec{e}_\nu$

define $\Gamma^\lambda{}_{\nu\mu} \vec{e}_\lambda := \partial_\mu \vec{e}_\nu$ Christoffel symbols of second kind (symmetric defn)

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so $\widetilde{\nabla} \vec{A} = \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \Gamma^\lambda{}_{\nu\mu} \vec{e}_\lambda$

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$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \Gamma^\lambda{}_{\nu\mu} \tilde{e}^\mu \otimes \vec{e}_\lambda$ since any $\Gamma^\lambda{}_{\nu\mu}$ is a scalar

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since name of summation index is arbitrary, e.g.
$\sum_\lambda x^{-2\lambda} = \sum_\mu x^{-2\mu} = \sum_\nu x^{-2\nu}$

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$= (\partial_\mu A^\nu + A^\lambda {\Gamma^\nu}_{\lambda\mu}) \tilde{e}^\mu \otimes \vec{e}_\nu$

$$\boxed{\nabla_\mu A^\nu := {A^\nu}_{;\mu} := \partial_\mu A^\nu + A^\lambda {\Gamma^\nu}_{\lambda\mu}}$$

w:covariant derivative of vector

GR: gradient of a vector: $\nabla\vec{A}$

$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$

$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$

$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \partial_\mu \vec{e}_\nu$

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$= (\partial_\mu A^\nu + A^\lambda \Gamma^\nu{}_{\lambda\mu}) \tilde{e}^\mu \otimes \vec{e}_\nu$

$$\boxed{\nabla_\mu A^\nu := A^\nu{}_{;\mu} := A^\nu{}_{,\mu} + A^\lambda \Gamma^\nu{}_{\lambda\mu}}$$

w:covariant derivative of vector

GR: $\tilde{\nabla}$ and $\Gamma^{\nu}_{\lambda\mu}$

mathematically deeper: $\tilde{\nabla}$, usually written just as ∇, is the w:Levi-Civita connection

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- $\widetilde{\nabla}$ applied to a (m,n)-tensor field on a manifold gives an $(m,n+1)$-tensor field

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so far we showed how $\tilde{\nabla}$ applied to a $(0, 0)$-tensor field = scalar field ϕ gives a $(0, 1)$-tensor field = one-form field = $(\tilde{\mathrm{d}}\phi)_{\mu}\vec{e}^{\mu}$

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and $\widetilde{\nabla}$ on a $(1,0)$-tensor field = vector field $\vec{A}$ gives a $(1,1)$-tensor with components $\nabla_{\mu}A^{\nu} = \partial_{\mu}A^{\nu} + A^{\lambda}\Gamma^{\nu}{}_{\lambda\mu}$

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- tensors: $\widetilde{\nabla}\phi = \widetilde{\nabla}_\mu \phi \vec{e}^\mu$,

GR: $\widetilde{\nabla}$ and $\Gamma^{\nu}{}_{\lambda\mu}$

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- not components of tensor: $\Gamma^{\nu}{}_{\lambda\mu}$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

how does a one-form change with position? $\widetilde{\nabla}\tilde{A} =$?

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

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relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

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$\partial_\mu \delta^\nu{}_\lambda = 0$ (obviously)

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$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$

can we use the product rule with this scalar product?

$\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = ?$

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can we use the product rule with this scalar product?

$\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \partial_\mu \left(A_\nu B^\nu \right)$ in some coordinate basis

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evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

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can we use the product rule with this scalar product?

$\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \partial_\mu \left(A_\nu B^\nu \right)$

$= (\partial_\mu A_\nu) B^\nu + A_\nu (\partial_\mu B^\nu)$ by product rule on functions

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$$= (\partial_\mu A_\nu) B^\nu + A_\nu (\partial_\mu B^\nu)$$

$= \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$ since

$$\partial_\mu \tilde{A} = (\partial_\mu A_0, \partial_\mu A_1, \partial_\mu A_2, \partial_\mu A_3)$$

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$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$

product rule holds: $\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$

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so $0 = \langle \partial_\mu \tilde{e}^\nu, \vec{e}_\lambda \rangle + \langle \tilde{e}^\nu, \partial_\mu \vec{e}_\lambda \rangle$

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so $0 = \langle \partial_\mu \tilde{e}^\nu, \vec{e}_\lambda \rangle + \langle \tilde{e}^\nu, \partial_\mu \vec{e}_\lambda \rangle$

$= \left\langle F^\nu{}_{\kappa\mu} \tilde{e}^\kappa, \vec{e}_\lambda \right\rangle + \left\langle \tilde{e}^\nu, \Gamma^\kappa{}_{\lambda\mu} \vec{e}_\kappa \right\rangle$

GR: gradient of one-form $\widetilde{\nabla} \tilde{A}$

evaluating $\widetilde{\nabla} \tilde{A}$ as we did $\widetilde{\nabla} \vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu} \tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

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so $0 = \langle \partial_\mu \tilde{e}^\nu, \vec{e}_\lambda \rangle + \langle \tilde{e}^\nu, \partial_\mu \vec{e}_\lambda \rangle$

$$= F^\nu{}_{\kappa\mu} \langle \tilde{e}^\kappa, \vec{e}_\lambda \rangle + \Gamma^\kappa{}_{\lambda\mu} \langle \tilde{e}^\nu, \vec{e}_\kappa \rangle$$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

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how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

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GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

similarly, we can write the $(0,3)$-tensor

$$\widetilde{\nabla}\mathbf{g} = (\nabla_\lambda g_{\mu\nu})\tilde{e}^\lambda \otimes \tilde{e}^\mu \otimes \tilde{e}^\nu$$

giving $\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$

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definition:

locally flat coordinate system if $\vec{e}_\mu \cdot \vec{e}_\nu = \pm\delta^\mu_\nu$ in limit near a point $\forall \mu, \nu \in \{0, 1, 2, 3\}$

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smooth pseudo-4-manifold M: $\forall$ point $\mathbf{x} \in M$, $\exists$ a locally flat coordinate system $\{\vec{e}^{\bar{\mu}}, \bar{\mu} = 0, 1, 2, 3\}$ at $\mathbf{x}$, i.e. where $g_{\bar{\mu}\bar{\nu}} = \vec{e}_{\bar{\mu}} \cdot \vec{e}_{\bar{\nu}} = \pm\delta^{\bar{\mu}}_{\bar{\nu}}$ in limit of neighbourhood of $\mathbf{x}$

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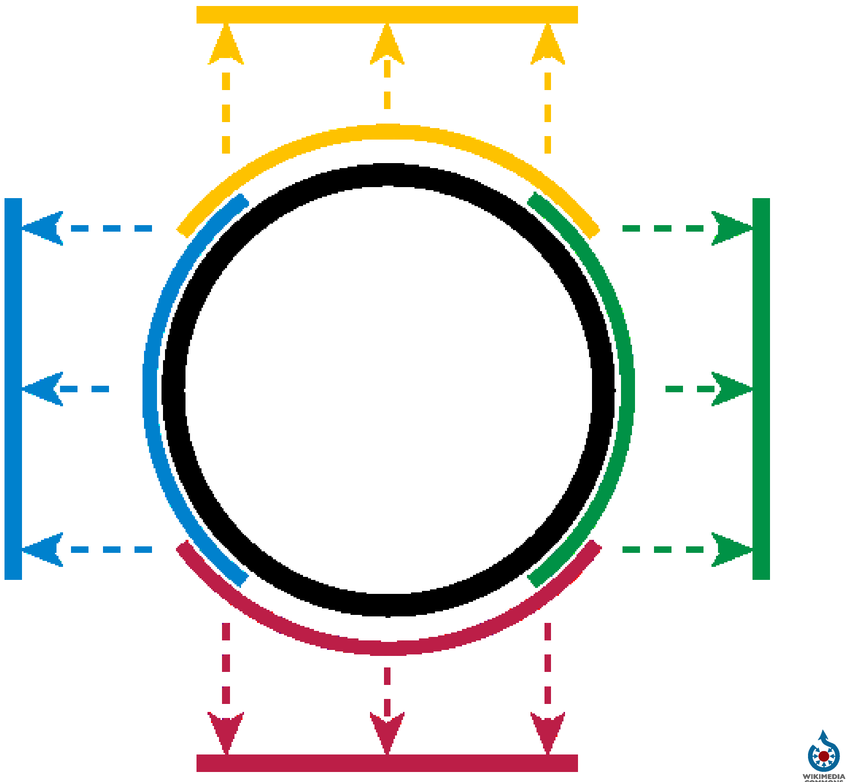

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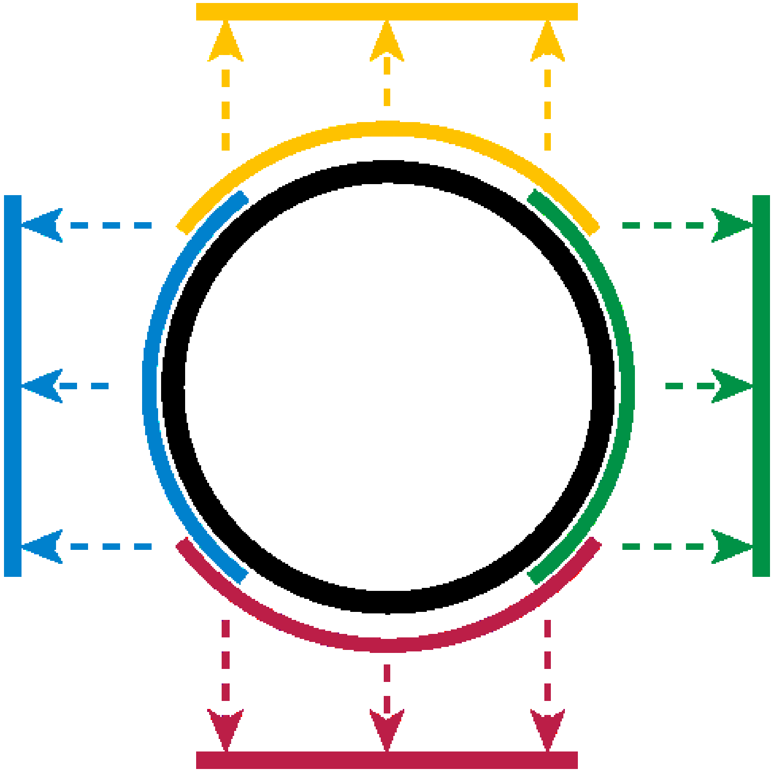

w:Manifold

- chart = function from part of pseudo-4-manifold M to part of M^4 (Minkowski)
- atlas = set of overlapping charts that cover M

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for relativity, $\exists \vec{e}^{\bar{\mu}}$ with $g_{\bar{i}\,\bar{j}} = \delta_{\bar{i}\,\bar{j}} = g^{\bar{i}\,\bar{j}}$ for a 3-spatial slice at $\mathbf{x}$ or $g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(-1,1,1,1) = g^{\bar{\mu}\bar{\nu}}$ for spacetime

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use of the above and changing certain index labels gives

$\boxed{\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) \text{ in a coordinate basis}}$

GR:

w:Einstein field equations

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[w:Einstein field equations]

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can be thought of as a *consequence* of the model

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GR: an approximation method: ADM

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GR: a numerical method: cactus

+ Cactus - http://cactuscode.org

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