Special and General Relativity

Boud Roukema

GR: intro

1. spacetime = 4D (curved) pseudo-Riemannian manifold M with metric $\mathbf{g}$

GR: intro

1. spacetime = 4D (curved) pseudo-Riemannian manifold M with metric $\mathbf{g}$

2. $\forall$ spacetime point $\mathbf{x}$ $\exists$ 4D Minkowski tangent space $T_{\mathbf{x}}M$ at $\mathbf{x}$

GR: intro

1. spacetime = 4D (curved) pseudo-Riemannian manifold M with metric $\mathbf{g}$

2. $\forall$ spacetime point $\mathbf{x}$ $\exists$ 4D Minkowski tangent space $T_{\mathbf{x}}M$ at $\mathbf{x}$

= vector space (e.g. 4-momentum vectors)

GR: intro

1. spacetime = 4D (curved) pseudo-Riemannian manifold M with metric $\mathbf{g}$
2. $\forall$ spacetime point $\mathbf{x}$ $\exists$ 4D Minkowski tangent space $T_{\mathbf{x}}M$ at $\mathbf{x}$

= vector space, $\mathbf{g} \Rightarrow$ lengths of vectors in $T_{\mathbf{x}}M$

GR: intro

1. spacetime = 4D (curved) pseudo-Riemannian manifold M with metric $\mathbf{g}$

2. $\forall$ spacetime point $\mathbf{x}$ $\exists$ 4D Minkowski tangent space $T_{\mathbf{x}}M$ at $\mathbf{x}$

= vector space, $\mathbf{g} \Rightarrow$ lengths of vectors in $T_{\mathbf{x}}M$

3. also, $\forall \mathbf{x} \in M$, $\exists$ 4D Mink. cotangent space $T^*_{\mathbf{x}}M$

GR: intro

1. spacetime = 4D (curved) pseudo-Riemannian manifold M with metric $\mathbf{g}$

2. $\forall$ spacetime point $\mathbf{x}$ $\exists$ 4D Minkowski tangent space $T_{\mathbf{x}}M$ at $\mathbf{x}$

= vector space, $\mathbf{g} \Rightarrow$ lengths of vectors in $T_{\mathbf{x}}M$

3. also, $\forall \mathbf{x} \in M$, $\exists$ 4D Mink. cotangent space $T^*_{\mathbf{x}}M$

= dual vector space (think: contour map, gradients)

GR: intro

1. spacetime = 4D (curved) pseudo-Riemannian manifold M with metric $\mathbf{g}$

2. $\forall$ spacetime point $\mathbf{x}$ $\exists$ 4D Minkowski tangent space $T_{\mathbf{x}}M$ at $\mathbf{x}$

= vector space, $\mathbf{g} \Rightarrow$ lengths of vectors in $T_{\mathbf{x}}M$

3. also, $\forall \mathbf{x} \in M$, $\exists$ 4D Mink. cotangent space $T^*_{\mathbf{x}}M$

= space of one-forms, $\mathbf{g}^{-1} \Rightarrow$ "lengths"

GR: intro

1. spacetime = 4D (curved) pseudo-Riemannian manifold M with metric $\mathbf{g}$

2. $\forall$ spacetime point $\mathbf{x}$ $\exists$ 4D Minkowski tangent space $T_{\mathbf{x}}M$ at $\mathbf{x}$

= vector space, $\mathbf{g} \Rightarrow$ lengths of vectors in $T_{\mathbf{x}}M$

3. also, $\forall \mathbf{x} \in M$, $\exists$ 4D Mink. cotangent space $T^*_{\mathbf{x}}M$

= space of one-forms, $\mathbf{g}^{-1} \Rightarrow$ "lengths"

duality in a basis of $T_{\mathbf{x}}M$ and a basis of $T^*_{\mathbf{x}}M$ usually defined using δ^{μ}_{ν}

GR: intro

1. spacetime = 4D (curved) pseudo-Riemannian manifold M with metric $\mathbf{g}$

2. $\forall$ spacetime point $\mathbf{x}$ $\exists$ 4D Minkowski tangent space $T_{\mathbf{x}}M$ at $\mathbf{x}$

= vector space, $\mathbf{g} \Rightarrow$ lengths of vectors in $T_{\mathbf{x}}M$

3. also, $\forall \mathbf{x} \in M$, $\exists$ 4D Mink. cotangent space $T^*_{\mathbf{x}}M$

= space of one-forms, $\mathbf{g}^{-1} \Rightarrow$ “lengths”

2+3. vector–one-form duality in a basis: δ^μ_ν

GR: intro

1. spacetime = 4D (curved) pseudo-Riemannian manifold M with metric $\mathbf{g}$

2. $\forall$ spacetime point $\mathbf{x}$ $\exists$ 4D Minkowski tangent space $T_{\mathbf{x}}M$ at $\mathbf{x}$

= vector space, $\mathbf{g} \Rightarrow$ lengths of vectors in $T_{\mathbf{x}}M$

3. also, $\forall \mathbf{x} \in M$, $\exists$ 4D Mink. cotangent space $T^*_{\mathbf{x}}M$

= space of one-forms, $\mathbf{g}^{-1} \Rightarrow$ "lengths"

2+3. vector–one-form duality in a basis: δ^μ_ν

4. w:Levi-Civita connection $\Leftarrow$ metric

GR: intro

1. spacetime = 4D (curved) pseudo-Riemannian manifold M with metric $\mathbf{g}$

2. $\forall$ spacetime point $\mathbf{x}$ $\exists$ 4D Minkowski tangent space $T_{\mathbf{x}}M$ at $\mathbf{x}$

= vector space, $\mathbf{g} \Rightarrow$ lengths of vectors in $T_{\mathbf{x}}M$

3. also, $\forall \mathbf{x} \in M$, $\exists$ 4D Mink. cotangent space $T^*_{\mathbf{x}}M$

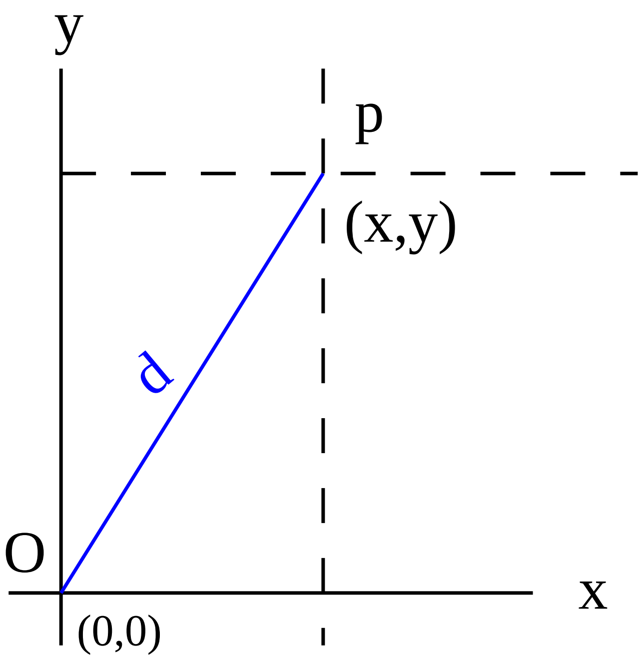
= space of one-forms, $\mathbf{g}^{-1} \Rightarrow$ “lengths”

2+3. vector–one-form duality in a basis: δ^{μ}_{ν}

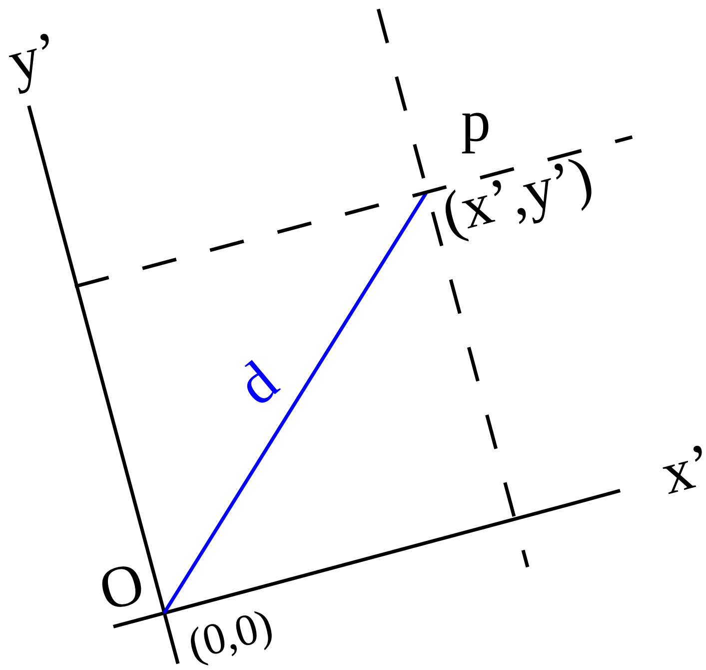
4. w:Levi-Civita connection $\Leftarrow$ metric

5. metric $\Leftarrow$ Einstein field equations

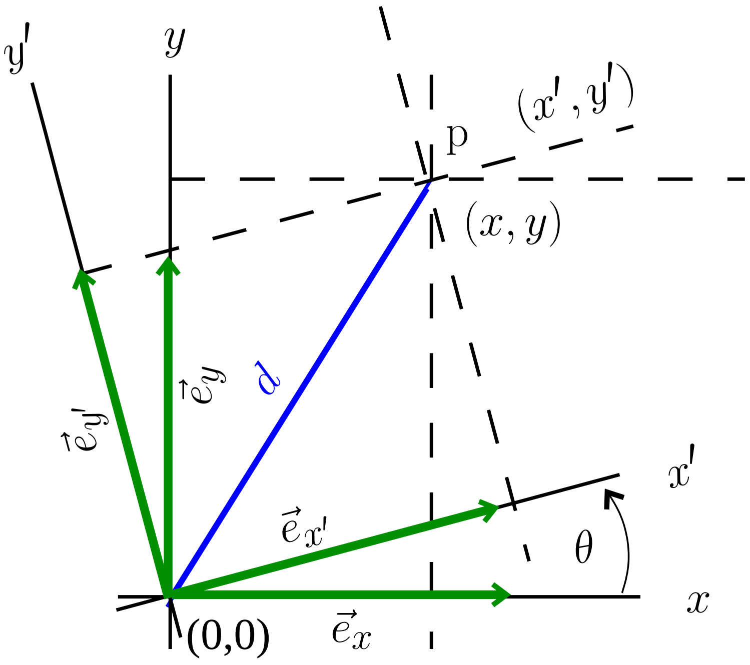
GR: coordinate transformations



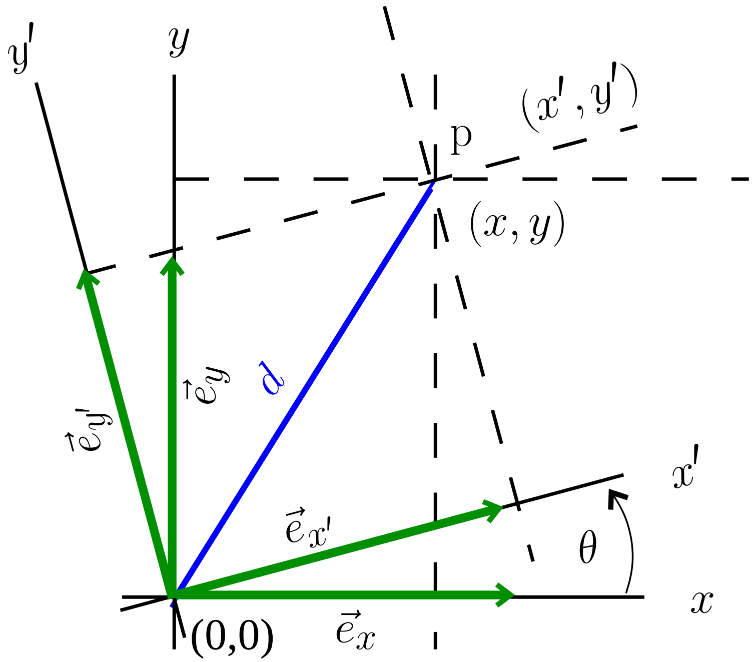
GR: coordinate transformations



GR: coordinate transformations

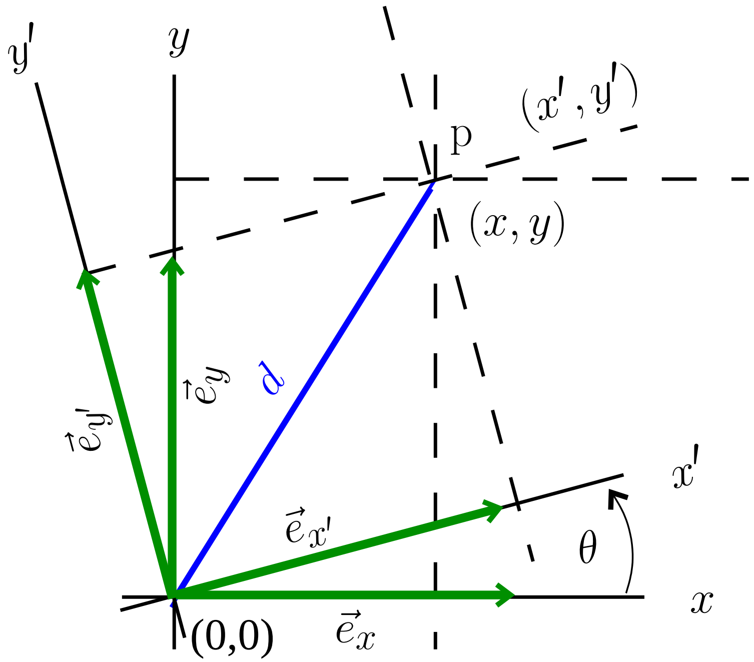


GR: coordinate transformations



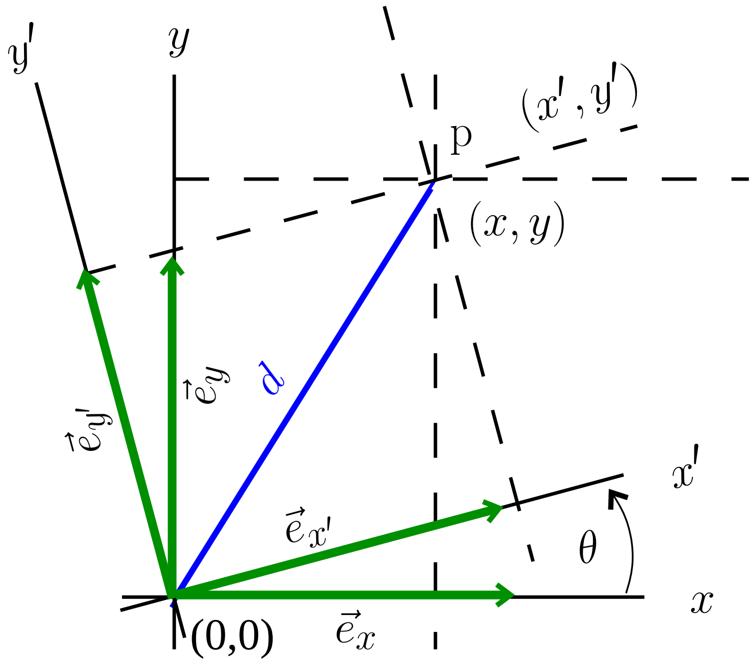
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

GR: coordinate transformations



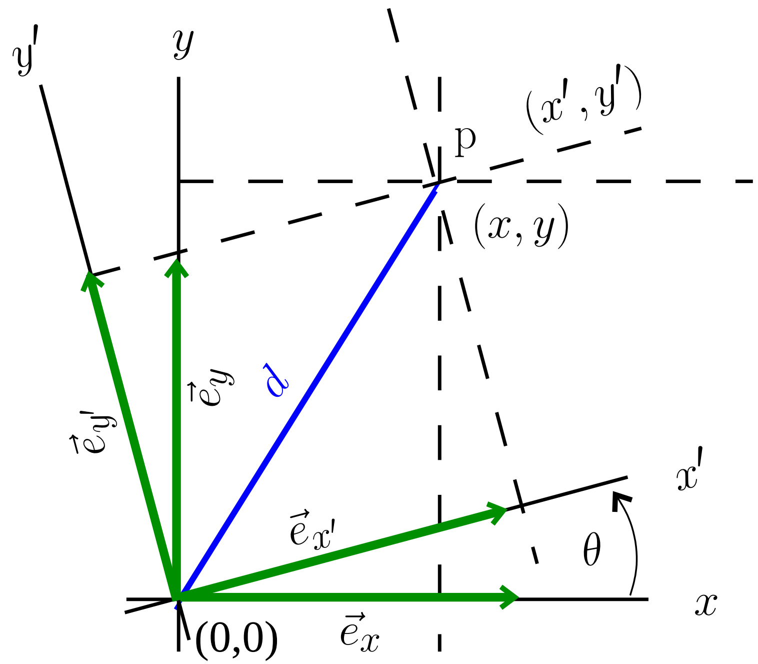
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

GR: coordinate transformations



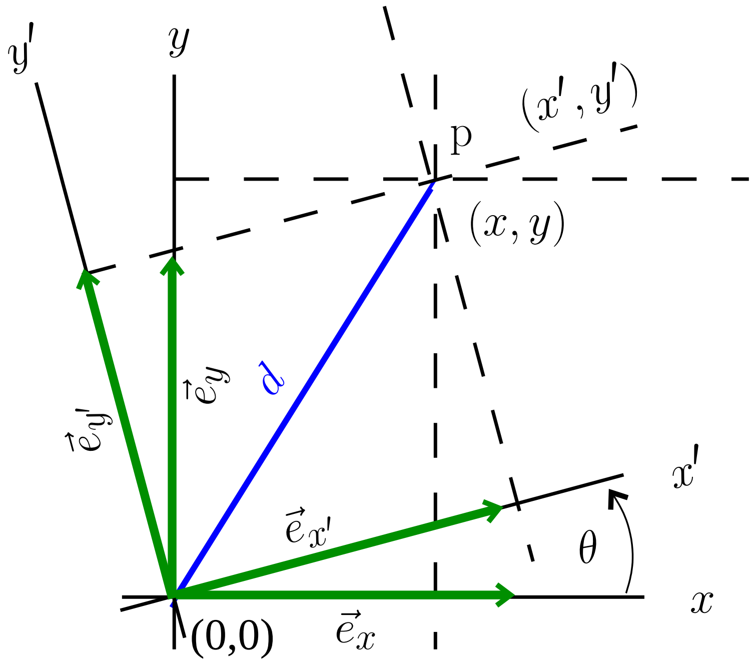
but $$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

GR: coordinate transformations



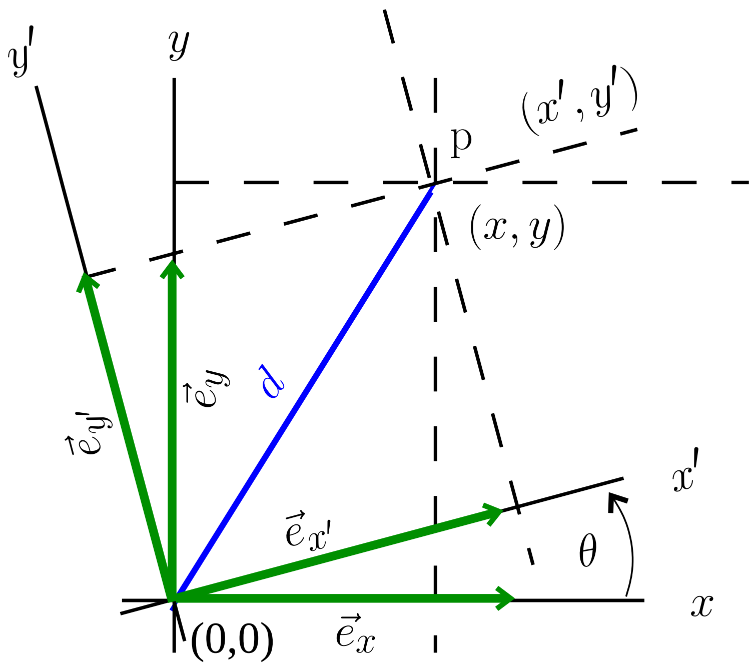
$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

GR: coordinate transformations



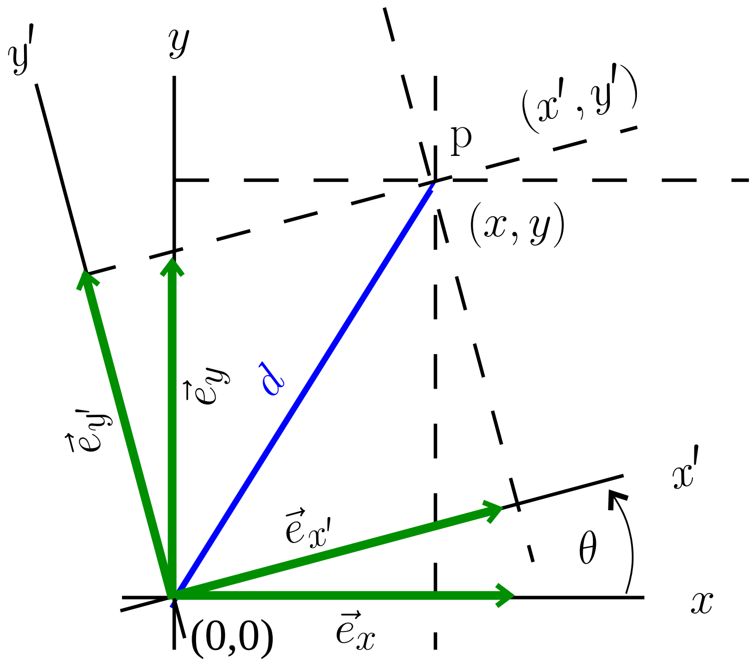
$$\vec{e}_{x'} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{e}_x + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{e}_y$$

GR: coordinate transformations



$$\vec{e}_{x'} = \Lambda^{x}_{x'}\,\vec{e}_x + \Lambda^{y}_{x'}\,\vec{e}_y$$

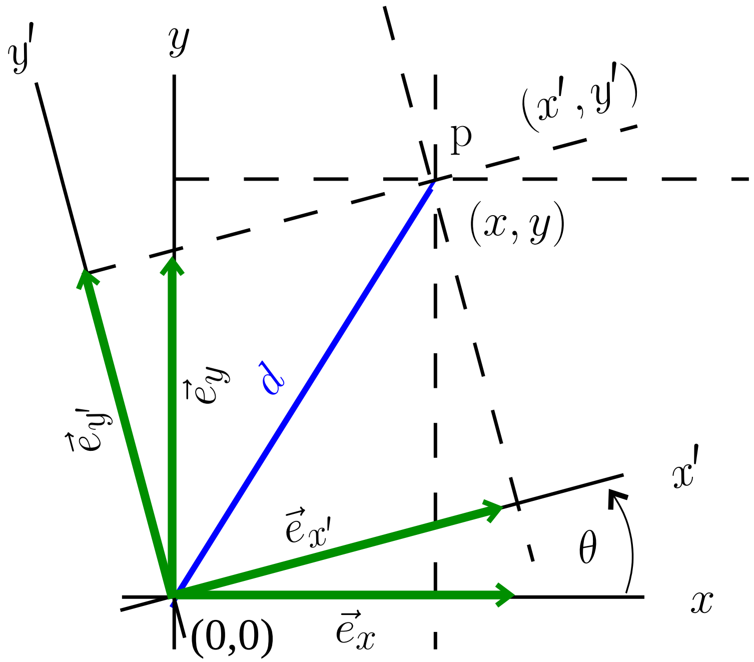
GR: coordinate transformations



also:

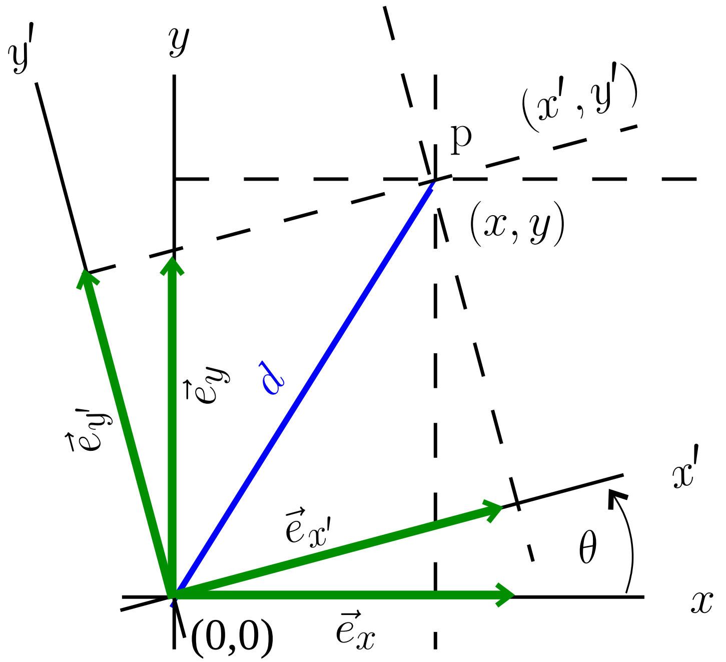
$$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

GR: coordinate transformations



$$\vec{e}_{y'} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{e}_x + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{e}_y$$

GR: coordinate transformations



summary:

$\vec{e}_{x'} = \Lambda^{x}_{x'}\,\vec{e}_x + \Lambda^{y}_{x'}\,\vec{e}_y,$

$\vec{e}_{y'} = \Lambda^{x}_{y'}\,\vec{e}_x + \Lambda^{y}_{y'}\,\vec{e}_y,$

where $\Lambda^{\alpha}_{\beta'}$:= element of inverse of $\Lambda^{\alpha'}_{\beta}$,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda^{x}_{x'}\,\vec{e}_x + \Lambda^{y}_{x'}\,\vec{e}_y,$$
$$\vec{e}_{y'} = \Lambda^{x}_{y'}\,\vec{e}_x + \Lambda^{y}_{y'}\,\vec{e}_y,$$

$$\vec{p} \to_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda^x_{x'}\, \vec{e}_x + \Lambda^y_{x'}\, \vec{e}_y,$$
$$\vec{e}_{y'} = \Lambda^x_{y'}\, \vec{e}_x + \Lambda^y_{y'}\, \vec{e}_y,$$

$$\vec{p} \rightarrow_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\vec{p} = \sum_i p^i \vec{e}_i$$

GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda^{x}_{x'}\,\vec{e}_x + \Lambda^{y}_{x'}\,\vec{e}_y,$$
$$\vec{e}_{y'} = \Lambda^{x}_{y'}\,\vec{e}_x + \Lambda^{y}_{y'}\,\vec{e}_y,$$

$$\vec{p} \rightarrow_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$\vec{p} = p^i \vec{e}_i$ (Einstein summation)

GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda^{x}_{x'}\,\vec{e}_x + \Lambda^{y}_{x'}\,\vec{e}_y,$$
$$\vec{e}_{y'} = \Lambda^{x}_{y'}\,\vec{e}_x + \Lambda^{y}_{y'}\,\vec{e}_y,$$

$$\vec{p} \to_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$\vec{p} = p^i \vec{e}_i$ (Einstein summation)

new basis vectors = sum of inverse $\Lambda \times$ old vectors

GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda^{x}_{x'}\,\vec{e}_x + \Lambda^{y}_{x'}\,\vec{e}_y,$$
$$\vec{e}_{y'} = \Lambda^{x}_{y'}\,\vec{e}_x + \Lambda^{y}_{y'}\,\vec{e}_y,$$

$$\vec{p} \rightarrow_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$\vec{p} = p^i \vec{e}_i$ (Einstein summation)

new basis vectors = sum of inverse $\Lambda \times$ old vectors

$\vec{e}_{\mu'} = \sum_\nu \Lambda^{\nu}{}_{\mu'} \vec{e}_\nu$

GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda^{x}_{x'}\,\vec{e}_x + \Lambda^{y}_{x'}\,\vec{e}_y,$$
$$\vec{e}_{y'} = \Lambda^{x}_{y'}\,\vec{e}_x + \Lambda^{y}_{y'}\,\vec{e}_y,$$

$$\vec{p} \rightarrow_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$\vec{p} = p^i \vec{e}_i$ (Einstein summation)

new basis vectors = sum of inverse $\Lambda\times$ old vectors

$$\vec{e}_{\mu'} = \Lambda^{\nu}{}_{\mu'}\vec{e}_{\nu}$$

GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda^x_{x'}\,\vec{e}_x + \Lambda^y_{x'}\,\vec{e}_y,$$
$$\vec{e}_{y'} = \Lambda^x_{y'}\,\vec{e}_x + \Lambda^y_{y'}\,\vec{e}_y,$$

$$\vec{p} \to_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$\vec{p} = p^i \vec{e}_i$ (Einstein summation)

new basis vectors = sum of inverse $\Lambda \times$ old vectors

new coords of vector $\vec{p}$ = $\Lambda \times$ old coords of same vector $\vec{p}$

GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda^x_{x'}\,\vec{e}_x + \Lambda^y_{x'}\,\vec{e}_y,$$
$$\vec{e}_{y'} = \Lambda^x_{y'}\,\vec{e}_x + \Lambda^y_{y'}\,\vec{e}_y, \qquad \vec{p} \rightarrow_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$\vec{p} = p^i \vec{e}_i$ (Einstein summation)

new basis vectors = sum of inverse $\Lambda \times$ old vectors

vector invariance requires contravariance of its coords
“contra” = inverse of change of basis vectors

GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda^x_{x'}\,\vec{e}_x + \Lambda^y_{x'}\,\vec{e}_y, \qquad \vec{e}_{y'} = \Lambda^x_{y'}\,\vec{e}_x + \Lambda^y_{y'}\,\vec{e}_y,$$

$$\vec{p} \rightarrow_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$\vec{p} = p^i \vec{e}_i$ (Einstein summation)

new basis vectors = sum of inverse $\Lambda \times$ old vectors

vector invariance requires contravariance of its coords
“contra” = inverse of change of basis vectors

- $\vec{p}$ is invariant: no dependence on coords
- $\vec{p}$ is contravariant: p^i change inversely to $\vec{e}_i$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x,y) \equiv \phi(x',y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

What is the relation between $(\phi_{,x'}, \phi_{,y'})$

and $(\phi_{,x}, \phi_{,y})$?

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

ϕ depends either on x and y, or on x' and y'

$\Rightarrow \frac{\partial \phi}{\partial x'} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial x'}$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

ϕ depends either on x and y, or on x' and y'

$\Rightarrow \phi_{,x'} = \phi_{,x}\, x_{,x'} + \phi_{,y}\, y_{,x'}$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

ϕ depends either on x and y, or on x' and y'

$\Rightarrow \phi_{,x'} = \phi_{,x}\, x_{,x'} + \phi_{,y}\, y_{,x'}$

$(\phi_{,x'}, \phi_{,y'}) =$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

ϕ depends either on x and y, or on x' and y'

$\Rightarrow \phi_{,x'} = \phi_{,x}\, x_{,x'} + \phi_{,y}\, y_{,x'}$

$(\phi_{,x'}, \phi_{,y'}) = \quad (\phi_{,x}\, x_{,x'} + \phi_{,y}\, y_{,x'}\ ,\ \phi_{,x}\, x_{,y'} + \phi_{,y}\, y_{,y'})$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

ϕ depends either on x and y, or on x' and y'

$\Rightarrow \phi_{,x'} = \phi_{,x}\, x_{,x'} + \phi_{,y}\, y_{,x'}$

$$(\phi_{,x'}, \phi_{,y'}) = \quad (\phi_{,x}, \phi_{,y}) \begin{pmatrix} x_{,x'} & x_{,y'} \\ y_{,x'} & y_{,y'} \end{pmatrix}$$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

ϕ depends either on x and y, or on x' and y'

$\Rightarrow \phi_{,x'} = \phi_{,x}\, x_{,x'} + \phi_{,y}\, y_{,x'}$

$$(\phi_{,x'}, \phi_{,y'}) = \quad (\phi_{,x}, \phi_{,y}) \begin{pmatrix} x_{,x'} & x_{,y'} \\ y_{,x'} & y_{,y'} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$ (example: rotation)

$$x_{,x'} = \frac{\partial x}{\partial x'} = \cos\theta$$

$$x_{,y'} = \frac{\partial x}{\partial y'} = -\sin\theta \ \ldots$$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

ϕ depends either on x and y, or on x' and y'

$\Rightarrow \phi_{,x'} = \phi_{,x}\, x_{,x'} + \phi_{,y}\, y_{,x'}$

$$(\phi_{,x'}, \phi_{,y'}) = \quad (\phi_{,x}, \phi_{,y}) \begin{pmatrix} x_{,x'} & x_{,y'} \\ y_{,x'} & y_{,y'} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_{,x'} & x_{,y'} \\ y_{,x'} & y_{,y'} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \text{(general)}$$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

ϕ depends either on x and y, or on x' and y'

$\Rightarrow \phi_{,x'} = \phi_{,x}\, x_{,x'} + \phi_{,y}\, y_{,x'}$

$$(\phi_{,x'}, \phi_{,y'}) = \quad (\phi_{,x}, \phi_{,y}) \begin{pmatrix} x_{,x'} & x_{,y'} \\ y_{,x'} & y_{,y'} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \text{(general)}$$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

ϕ depends either on x and y, or on x' and y'

$\Rightarrow \phi_{,x'} = \phi_{,x}\, x_{,x'} + \phi_{,y}\, y_{,x'}$

$$(\phi_{,x'}, \phi_{,y'}) = \quad (\phi_{,x}, \phi_{,y}) \begin{pmatrix} x_{,x'} & x_{,y'} \\ y_{,x'} & y_{,y'} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \text{(general)}$$

$\Rightarrow (\phi_{,x'}, \phi_{,y'}) = (\phi_{,x}, \phi_{,y})\, \Lambda^{-1}$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x,y) \equiv \phi(x',y')$

write $\phi_{,x} := \frac{\partial\phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

ϕ depends either on x and y, or on x' and y'

$\Rightarrow \phi_{,x'} = \phi_{,x}\, x_{,x'} + \phi_{,y}\, y_{,x'}$

$$(\phi_{,x'}, \phi_{,y'}) = \quad (\phi_{,x}, \phi_{,y}) \begin{pmatrix} x_{,x'} & x_{,y'} \\ y_{,x'} & y_{,y'} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix}$$ (general)

$$\tilde{\mathrm{d}}\phi = \left((\tilde{\mathrm{d}}\phi)_{x'}, (\tilde{\mathrm{d}}\phi)_{y'}\right) = \left((\tilde{\mathrm{d}}\phi)_{x}, (\tilde{\mathrm{d}}\phi)_{y}\right)\Lambda^{-1}$$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x,y) \equiv \phi(x',y')$

write $\phi_{,x} := \frac{\partial\phi}{\partial x} =: (\tilde{\mathrm{d}}\phi)_x$

ϕ depends either on x and y, or on x' and y'

$\Rightarrow \phi_{,x'} = \phi_{,x}\, x_{,x'} + \phi_{,y}\, y_{,x'}$

$$(\phi_{,x'}, \phi_{,y'}) = \quad (\phi_{,x}, \phi_{,y}) \begin{pmatrix} x_{,x'} & x_{,y'} \\ y_{,x'} & y_{,y'} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \text{(general)}$$

$$\tilde{\mathrm{d}}\phi = \left((\tilde{\mathrm{d}}\phi)_{x'}, (\tilde{\mathrm{d}}\phi)_{y'}\right) = \left((\tilde{\mathrm{d}}\phi)_{x}, (\tilde{\mathrm{d}}\phi)_{y}\right) \Lambda^{-1}$$

$$(\tilde{\mathrm{d}}\phi)_{\mu'} = (\tilde{\mathrm{d}}\phi)_{\nu} \Lambda^{\nu}{}_{\mu'}$$

GR: coord. transf.: 1-forms

basis vectors of different bases: $\vec{e}_{\mu'} = \Lambda^{\nu}{}_{\mu'} \vec{e}_{\nu}$

same vector: $(\vec{p})^{\mu'} = \Lambda^{\mu'}_{\ \nu} (\vec{p})^{\nu}$

same gradient (example 1-form): $(\tilde{\mathrm{d}}\phi)_{\mu'} = (\tilde{\mathrm{d}}\phi)_{\nu} \Lambda^{\nu}{}_{\mu'}$

GR: coord. transf.: 1-forms

basis vectors of different bases: $\vec{e}_{\mu'} = \Lambda^{\nu}{}_{\mu'} \vec{e}_{\nu}$

same vector: $(\vec{p})^{\mu'} = \Lambda^{\mu'}_{\nu} (\vec{p})^{\nu}$

same gradient (example 1-form): $(\tilde{\mathrm{d}}\phi)_{\mu'} = (\tilde{\mathrm{d}}\phi)_{\nu} \Lambda^{\nu}{}_{\mu'}$

- vector $\vec{p}$ is invariant: no dependence on coords
- $\vec{p}$ is contravariant: components p^{ν} change inversely to

how $\vec{e}_{\mu}$ change;   inverses: matrix $\{\Lambda^{\nu}{}_{\mu'}\}$ vs $\{\Lambda^{\beta'}_{\alpha}\}$

GR: coord. transf.: 1-forms

basis vectors of different bases: $\vec{e}_{\mu'} = \Lambda^{\nu}{}_{\mu'}\vec{e}_{\nu}$

same vector: $(\vec{p})^{\mu'} = \Lambda^{\mu'}{}_{\nu}(\vec{p})^{\nu}$

same gradient (example 1-form): $(\tilde{\mathrm{d}}\phi)_{\mu'} = (\tilde{\mathrm{d}}\phi)_{\nu}\Lambda^{\nu}{}_{\mu'}$

- vector $\vec{p}$ is invariant: no dependence on coords
- $\vec{p}$ is contravariant: components p^{ν} change inversely to how $\vec{e}_{\mu}$ change; inverses: matrix $\{\Lambda^{\nu}{}_{\mu'}\}$ vs $\{\Lambda^{\beta'}{}_{\alpha}\}$
- 1-form $\tilde{\mathrm{d}}\phi$ is invariant: no dependence on coords
- $\tilde{\mathrm{d}}\phi$ is covariant: components $(\tilde{\mathrm{d}}\phi)_{\mu}$ change like $\vec{e}_{\mu}$ (but left-multiply)

GR: coord. transf.: 1-forms

basis vectors of different bases: $\vec{e}_{\mu'} = \Lambda^{\nu}{}_{\mu'} \vec{e}_{\nu}$

same vector: $(\vec{p})^{\mu'} = \Lambda^{\mu'}{}_{\nu} (\vec{p})^{\nu}$

same gradient (example 1-form): $(\tilde{\mathrm{d}}\phi)_{\mu'} = (\tilde{\mathrm{d}}\phi)_{\nu} \Lambda^{\nu}{}_{\mu'}$

- vector $\vec{p}$ is invariant: no dependence on coords
- $\vec{p}$ is contravariant: components p^{ν} change inversely to how $\vec{e}_{\mu}$ change; inverses: matrix $\{\Lambda^{\nu}{}_{\mu'}\}$ vs $\{\Lambda^{\beta'}{}_{\alpha}\}$
- 1-form $\tilde{\mathrm{d}}\phi$ is invariant: no dependence on coords
- $\tilde{\mathrm{d}}\phi$ is covariant: components $(\tilde{\mathrm{d}}\phi)_{\mu}$ change like $\vec{e}_{\mu}$ (but left-multiply)

w:Covariance and contravariance of vectors

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

$\langle \vec{p}, \tilde{q} \rangle = \sum_\mu p^\mu q_\mu$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

$$\langle \vec{p}, \tilde{q} \rangle = p^{\mu} q_{\mu}$$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

$$\langle \vec{p}, \tilde{q} \rangle = p^{\mu} q_{\mu} = \vec{p}(\tilde{q})$$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

$$\langle \vec{p}, \tilde{q} \rangle = p^\mu q_\mu = \vec{p}(\tilde{q}) = \tilde{q}(\vec{p})$$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

$\langle \vec{p}, \tilde{q} \rangle = p^\mu q_\mu = \vec{p}(\tilde{q}) = \tilde{q}(\vec{p})$

$\langle \;,\; \rangle$ is a (1,1) tensor

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

$\langle \vec{p}, \tilde{q} \rangle = p^{\mu} q_{\mu} = \vec{p}(\tilde{q}) = \tilde{q}(\vec{p})$

$\langle\ ,\ \rangle$ is a (1,1) tensor

can be called $I \equiv \delta^{\mu}_{\nu}$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

$$\langle \vec{p}, \tilde{q} \rangle = p^\mu q_\mu = \vec{p}(\tilde{q}) = \tilde{q}(\vec{p})$$

$\langle \ , \ \rangle$ is a (1,1) tensor

think: vector $\longrightarrow$ column vector
1-form $\longrightarrow$ row vector

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

$$\langle \vec{p}, \tilde{q} \rangle = p^{\mu} q_{\mu} = \vec{p}(\tilde{q}) = \tilde{q}(\vec{p})$$

$\langle \; , \; \rangle$ is a (1,1) tensor

$$(q_0, q_1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^0 \\ p^1 \end{pmatrix} =$$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

$\langle \vec{p}, \tilde{q} \rangle = p^\mu q_\mu = \vec{p}(\tilde{q}) = \tilde{q}(\vec{p})$

$\langle \ , \ \rangle$ is a (1,1) tensor

$$(q_0, q_1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^0 \\ p^1 \end{pmatrix} = (q_0, q_1) \begin{pmatrix} p^0 \\ p^1 \end{pmatrix}$$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

$$\langle \vec{p}, \tilde{q} \rangle = p^\mu q_\mu = \vec{p}(\tilde{q}) = \tilde{q}(\vec{p})$$

$\langle \; , \; \rangle$ is a (1,1) tensor

$$(q_0, q_1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^0 \\ p^1 \end{pmatrix} = (q_0, q_1) \begin{pmatrix} p^0 \\ p^1 \end{pmatrix} = p^\mu q_\mu$$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

vector–1-form duality defined:

$\langle \vec{p}, \tilde{q} \rangle = p^\mu q_\mu = \vec{p}(\tilde{q}) = \tilde{q}(\vec{p})$

$\langle \; , \; \rangle$ is a (1,1) tensor

$$(q_0, q_1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^0 \\ p^1 \end{pmatrix} = (q_0, q_1) \begin{pmatrix} p^0 \\ p^1 \end{pmatrix} = p^\mu q_\mu$$

$\langle \; , \; \rangle$ = (1,1)-tensor = “row-column” matrix I with $I^\mu_\nu = \delta^\mu_\nu$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

ordinary linear algebra: column vectors, row vectors, matrices

GR: $\vec{p}, \tilde{q}, \langle\vec{p}, \tilde{q}\rangle$, $\mathbf{g}$

GR tensors: two different scalar products

(m, n)-tensor algebra: m column n row $m + n$-arrays

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

(m, n)-tensor algebra: m column n row $m + n$-arrays

e.g.: $(0, 2)$-tensor: metric $g_{\mu\nu}$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

(m, n)-tensor algebra: m column n row $m + n$-arrays

using $\langle\ ,\ \rangle$, $(1, 0)$-tensor = vector = function of 1-forms

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

(m, n)-tensor algebra: m column n row $m + n$-arrays

using $\langle\ ,\ \rangle$, $(0, 1)$-tensor = 1-form = function of vectors

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

(m, n)-tensor algebra: m column n row $m + n$-arrays

(m, n)-tensor = function of m 1-forms and n vectors

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

(m, n)-tensor algebra: m column n row $m + n$-arrays

(m, n)-tensor = function of m 1-forms and n vectors

$V =$ space of vectors $\vec{p} = p^\mu \vec{e}_\mu$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

(m,n)-tensor algebra: m column n row $m+n$-arrays

(m,n)-tensor = function of m 1-forms and n vectors

$V =$ space of vectors $\vec{p} = p^\mu \vec{e}_\mu$

$V^* =$ dual space of 1-forms $\tilde{q} = q_\mu \tilde{e}^\mu$

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

(m, n)-tensor algebra: m column n row $m + n$-arrays

(m, n)-tensor = function of m 1-forms and n vectors

$V =$ space of vectors $\vec{p} = p^\mu \vec{e}_\mu$

$V^* =$ dual space of 1-forms $\tilde{q} = q_\mu \tilde{e}^\mu$

$V^* \otimes V^*$ = space of $(0, 2)$-tensors $T = T_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$ (e.g. metric) w:tensor product

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

(m, n)-tensor algebra: m column n row $m + n$-arrays

(m, n)-tensor = function of m 1-forms and n vectors

$V =$ space of vectors $\vec{p} = p^\mu \vec{e}_\mu$

$V^* =$ dual space of 1-forms $\tilde{q} = q_\mu \tilde{e}^\mu$

$V^* \otimes V^*$ = space of $(0, 2)$-tensors $T = T_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$ (e.g. metric) w:tensor product

loosely speaking, the second $\otimes$ means "function of two vectors" (or 1-forms, or a vector and a 1-form) in *that particular left-to-right order*

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

(m, n)-tensor algebra: m column n row $m + n$-arrays

(m, n)-tensor = function of m 1-forms and n vectors

$V =$ space of vectors $\vec{p} = p^\mu \vec{e}_\mu$

$V^* =$ dual space of 1-forms $\tilde{q} = q_\mu \tilde{e}^\mu$

$V^* \otimes V^*$ = space of $(0, 2)$-tensors $T = T_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$ (e.g. metric) w:tensor product

order of $V^* \otimes V^*$ = 2

GR: $\vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle$, $\mathbf{g}$

GR tensors: two different scalar products

(m, n)-tensor algebra: m column n row $m + n$-arrays

(m, n)-tensor = function of m 1-forms and n vectors

$V =$ space of vectors $\vec{p} = p^\mu \vec{e}_\mu$

$V^* =$ dual space of 1-forms $\tilde{q} = q_\mu \tilde{e}^\mu$

$V^* \otimes V^*$ = space of $(0, 2)$-tensors $T = T_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$ (e.g. metric) w:tensor product

order of $V^* \otimes V^*$ = 2

warning: the ”rank” of tensors has two different meanings: w:Tensor_(intrinsic_definition)#Tensor_rank

GR: $\vec{p}, \tilde{q}, \langle\vec{p}, \tilde{q}\rangle$, $\mathbf{g}$

GR tensors: two different scalar products

(m, n)-tensor algebra: m column n row $m + n$-arrays

(m, n)-tensor = function of m 1-forms and n vectors

$V =$ space of vectors $\vec{p} = p^\mu \vec{e}_\mu$

$V^* =$ dual space of 1-forms $\tilde{q} = q_\mu \tilde{e}^\mu$

$V^* \otimes V^*$ = space of $(0, 2)$-tensors $T = T_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$ (e.g. metric) w:tensor product

order of $V^* \otimes V^*$ = 2

dimension of $V^* \otimes V^*$ = 16 (for $V =$ spacetime)

GR: g

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where

$\otimes =$ w:tensor product

GR: $\mathbf{g}$

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^{\mu} \otimes \tilde{e}^{\nu}$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

GR: $\mathbf{g}$

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^{\mu} \otimes \tilde{e}^{\nu}$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

GR: g

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $\quad g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

GR: $\mathbf{g}$

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

GR: g

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g}(\vec{A}, \vec{B})$$

GR: g

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $\quad g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

also written: $\vec{A} \cdot \vec{B}$ “dot product”

GR: g

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g}(\vec{A}, \vec{B})$$

GR: g

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g}(\vec{A}, \vec{B}) = \left[\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \end{pmatrix} \right]^{\mathrm{T}} \begin{pmatrix} B_r \\ B_\theta \end{pmatrix}$$

GR: $\mathbf{g}$

$V^* \otimes V^*$ = space of $(0, 2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g}(\vec{A}, \vec{B}) \;= (A_r, A_\theta r^2) \begin{pmatrix} B_r \\ B_\theta \end{pmatrix}$$

GR: g

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g}(\vec{A}, \vec{B}) = A_r B_r + A_\theta B_\theta r^2$$

GR: g

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^{\mu} \otimes \tilde{e}^{\nu}$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $\quad g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g}(\vec{A}, \vec{B}) \quad = A_r B_r + A_\theta B_\theta r^2$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \right]^{\mathrm{T}} \begin{pmatrix} B_x \\ B_y \end{pmatrix}$$

GR: g

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g}(\vec{A}, \vec{B}) = A_r B_r + A_\theta B_\theta r^2 = (A_x, A_y) \begin{pmatrix} B_x \\ B_y \end{pmatrix}$$

GR: g

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= "row-row" matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g}(\vec{A}, \vec{B}) \quad = A_r B_r + A_\theta B_\theta r^2 \quad = A_x B_x + A_y B_y$$

GR: g

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^{\mu} \otimes \tilde{e}^{\nu}$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $\quad g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g}(\vec{A}, \vec{B}) \quad = A_r B_r + A_\theta B_\theta r^2 \quad = A_x B_x + A_y B_y$$

in general, for a 2-form T, $T(\vec{A}, \vec{B}) \neq T(\vec{B}, \vec{A})$

GR: g

$V^* \otimes V^*$ = space of $(0, 2)$-tensors $T = T_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= "row-row" matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g}(\vec{A}, \vec{B}) \quad = A_r B_r + A_\theta B_\theta r^2 \quad = A_x B_x + A_y B_y$$

$$\mathbf{g} = g_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$$

GR: $\mathbf{g}$

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g}(\vec{A}, \vec{B}) \quad = A_r B_r + A_\theta B_\theta r^2 \quad = A_x B_x + A_y B_y$$

$$\mathbf{g} = g_{r\theta}\tilde{e}^r \otimes \tilde{e}^\theta$$

GR: g

$V^* \otimes V^*$ = space of $(0,2)$-tensors $T = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$, where $\otimes =$ w:tensor product

e.g.: metric $\mathbf{g}$ = function of two vectors

= “row-row” matrix

e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$. $g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ or

$$g_{xy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{g}(\vec{A}, \vec{B}) \quad = A_r B_r + A_\theta B_\theta r^2 \quad = A_x B_x + A_y B_y$$

$$\mathbf{g} = g_{xy}\tilde{e}^x \otimes \tilde{e}^y$$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{r\theta}\tilde{e}^r \otimes \tilde{e}^\theta + g_{\theta r}\tilde{e}^\theta \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr} \tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta} \tilde{e}^\theta \otimes \tilde{e}^\theta$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr} \tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta} \tilde{e}^\theta \otimes \tilde{e}^\theta$

check: $\mathbf{g}(\vec{e}_r, \vec{e}_r) = g_{rr}$?

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

$\mathbf{g}(\vec{e}_r, \vec{e}_r) = (g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta)(\vec{e}_r, \vec{e}_r)$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

$\mathbf{g}(\vec{e}_r, \vec{e}_r) = g_{rr}\tilde{e}^r \otimes \tilde{e}^r(\vec{e}_r, \vec{e}_r) + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta(\vec{e}_r, \vec{e}_r)$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

$\mathbf{g}(\vec{e}_r, \vec{e}_r) = g_{rr}\,\tilde{e}^r(\vec{e}_r)\,\tilde{e}^r(\vec{e}_r) + g_{\theta\theta}\,\tilde{e}^\theta(\vec{e}_r)\,\tilde{e}^\theta(\vec{e}_r)$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

$$\mathbf{g}(\vec{e}_r, \vec{e}_r) = g_{rr} \langle \tilde{e}^r, \vec{e}_r \rangle \langle \tilde{e}^r, \vec{e}_r \rangle + g_{\theta\theta} \left\langle \tilde{e}^\theta, \vec{e}_r \right\rangle \left\langle \tilde{e}^\theta, \vec{e}_r \right\rangle$$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr} \tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta} \tilde{e}^\theta \otimes \tilde{e}^\theta$

$\mathbf{g}(\vec{e}_r, \vec{e}_r) = g_{rr} \times 1 \times 1 + g_{\theta\theta} \times 0 \times 0$ by duality through scalar product $\langle\ ,\ \rangle$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr} \tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta} \tilde{e}^\theta \otimes \tilde{e}^\theta$

$\mathbf{g}(\vec{e}_r, \vec{e}_r) = g_{rr}$ self-consistent definition

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr} \tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta} \tilde{e}^\theta \otimes \tilde{e}^\theta$

inverse: $\mathbf{g}^{-1} = g^{\mu\nu} \vec{e}_\mu \otimes \vec{e}_\nu$,

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr} \tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta} \tilde{e}^\theta \otimes \tilde{e}^\theta$

inverse: $\mathbf{g}^{-1} = g^{\mu\nu} \vec{e}_\mu \otimes \vec{e}_\nu$,

where $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

inverse: $\mathbf{g}^{-1} = g^{\mu\nu}\vec{e}_\mu \otimes \vec{e}_\nu$,

where $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$

duality of associate vectors and 1-forms:
$\mathbf{g}(\vec{A}, \vec{B}) = \mathbf{g}^{-1}(\tilde{A}, \tilde{B})$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

inverse: $\mathbf{g}^{-1} = g^{\mu\nu}\vec{e}_\mu \otimes \vec{e}_\nu$,

where $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$

duality of associate vectors and 1-forms:
$\mathbf{g}(\vec{A}, \vec{B}) = \mathbf{g}^{-1}(\tilde{A}, \tilde{B}) \ = \vec{A} \cdot \vec{B}$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr} \tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta} \tilde{e}^\theta \otimes \tilde{e}^\theta$

inverse: $\mathbf{g}^{-1} = g^{\mu\nu} \vec{e}_\mu \otimes \vec{e}_\nu$,

where $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$

duality of associate vectors and 1-forms:
$\mathbf{g}(\vec{A}, \vec{B}) = \mathbf{g}^{-1}(\tilde{A}, \tilde{B}) \ = \vec{A} \cdot \vec{B} = g_{\mu\nu} A^\mu B^\nu$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr} \tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta} \tilde{e}^\theta \otimes \tilde{e}^\theta$

inverse: $\mathbf{g}^{-1} = g^{\mu\nu} \vec{e}_\mu \otimes \vec{e}_\nu$,

where $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$

duality of associate vectors and 1-forms:

$\mathbf{g}(\vec{A}, \vec{B}) = \mathbf{g}^{-1}(\tilde{A}, \tilde{B}) \ = \vec{A} \cdot \vec{B} = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr} \tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta} \tilde{e}^\theta \otimes \tilde{e}^\theta$

inverse: $\mathbf{g}^{-1} = g^{\mu\nu} \vec{e}_\mu \otimes \vec{e}_\nu$,

where $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$

duality of associate vectors and 1-forms:

$\mathbf{g}(\vec{A}, \vec{B}) = \mathbf{g}^{-1}(\tilde{A}, \tilde{B}) \; = \vec{A} \cdot \vec{B} = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu$

lower an index: $g_{\mu\nu} A^\mu = A_\nu$

GR: metric tensor $\mathbf{g}$, $\mathbf{g}^{-1}$, bases

$\mathbf{g}$ can be applied to basis vectors $\vec{e}_\mu$

we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

$\Rightarrow \mathbf{g} = g_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu$

e.g. $\mathbf{g} = g_{rr}\tilde{e}^r \otimes \tilde{e}^r + g_{\theta\theta}\tilde{e}^\theta \otimes \tilde{e}^\theta$

inverse: $\mathbf{g}^{-1} = g^{\mu\nu}\vec{e}_\mu \otimes \vec{e}_\nu$,

where $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$

duality of associate vectors and 1-forms:

$\mathbf{g}(\vec{A}, \vec{B}) = \mathbf{g}^{-1}(\tilde{A}, \tilde{B}) \ = \vec{A} \cdot \vec{B} = g_{\mu\nu}A^\mu B^\nu = g^{\mu\nu}A_\mu B_\nu$

lower an index: $g_{\mu\nu}A^\mu = A_\nu$

raise an index: $g^{\mu\nu}B_\nu = B^\mu$

GR: what is a coordinate?

a coordinate, e.g. x^0 or x^1 is a scalar field on the 4-manifold

GR: what is a coordinate?

a coordinate system x^μ = set of four scalar fields on the 4-manifold

GR: what is a coordinate?

a coordinate system x^μ = set of four scalar fields on the 4-manifold

(Bertschinger writes $x^\mu_{\mathbf{X}}$ to show dependence on position $\mathbf{X}$ in manifold $\neq$ vector space)

GR: what is a coordinate?

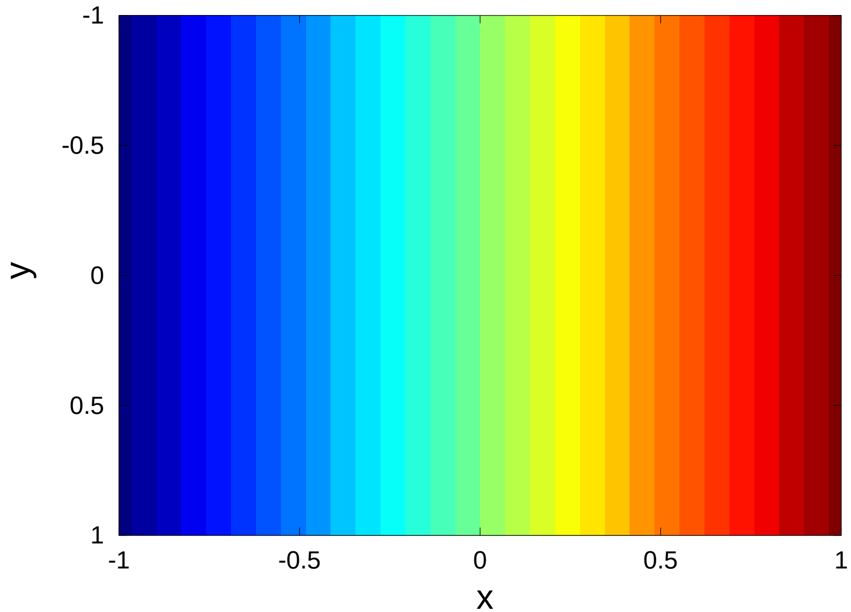
a coordinate system x^μ = set of four scalar fields on the 4-manifold

x^μ are differentiable *almost everywhere*

GR: what is a coordinate?

a coordinate system x^μ = set of four scalar fields on the 4-manifold

x^μ are differentiable *almost everywhere*

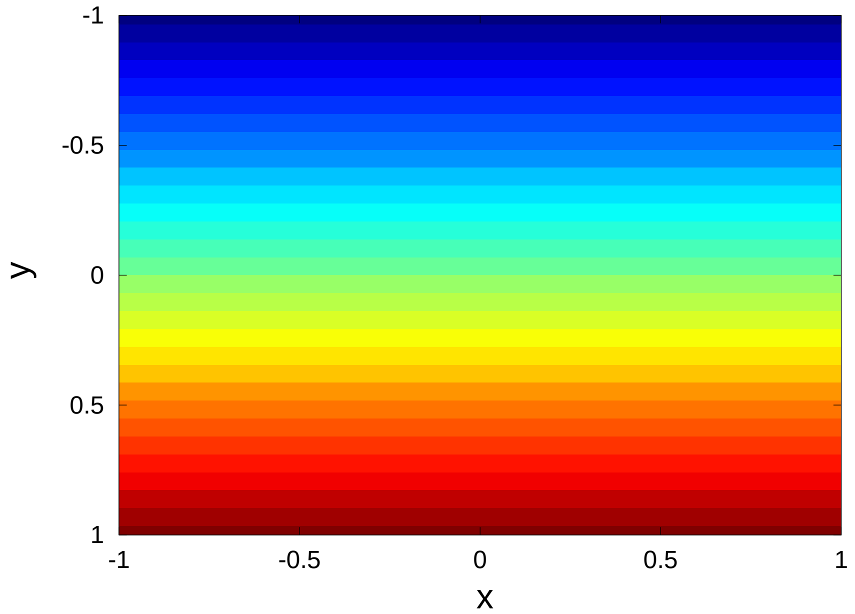


e.g. on $\mathbb{R}^2$

GR: what is a coordinate?

a coordinate system x^μ = set of four scalar fields on the 4-manifold

x^μ are differentiable *almost everywhere*

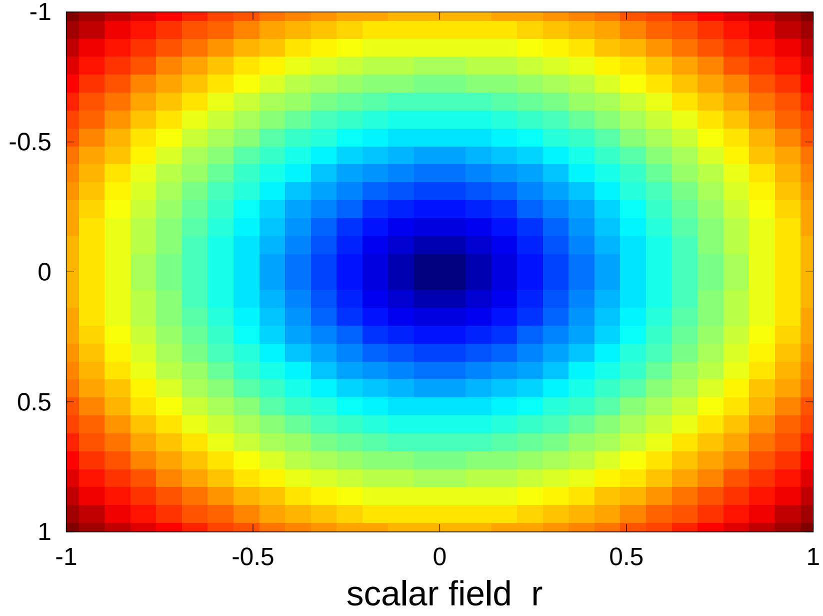


e.g. on $\mathbb{R}^2$

GR: what is a coordinate?

a coordinate system x^μ = set of four scalar fields on the 4-manifold

x^μ are differentiable *almost everywhere*

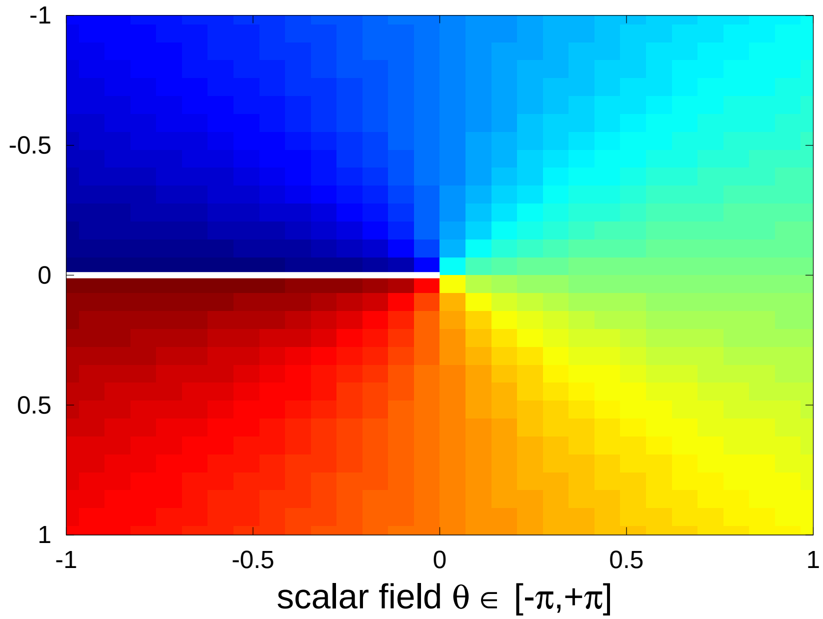


e.g. on $\mathbb{R}^2$

GR: what is a coordinate?

a coordinate system x^μ = set of four scalar fields on the 4-manifold

x^μ are differentiable *almost everywhere*

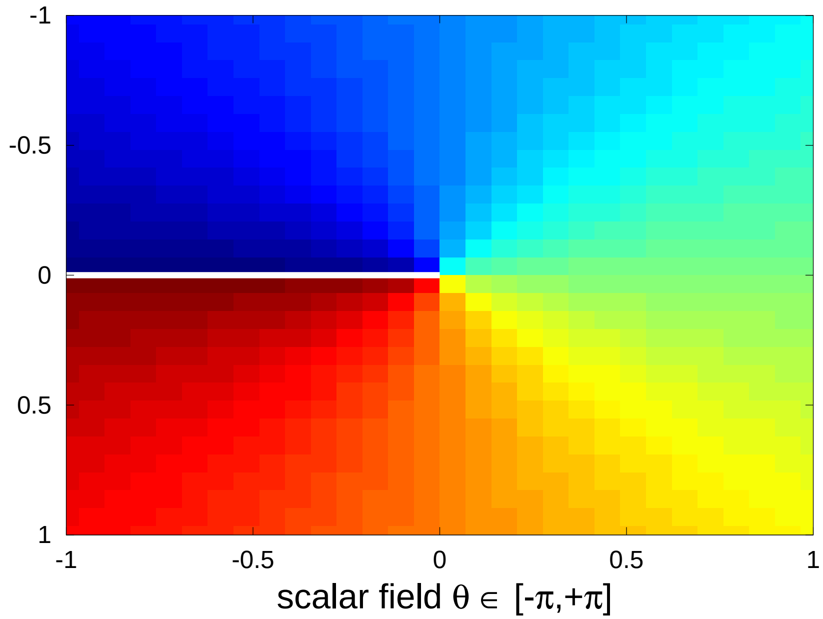


e.g. on $\mathbb{R}^2$

GR: what is a coordinate?

a coordinate system x^μ = set of four scalar fields on the 4-manifold

x^μ are differentiable *almost everywhere*



e.g. on $\mathbb{R}^2$

coordinate singularity $\neq$ singularity in manifold

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

$\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$ for any scalar field f coordinate-free

where $\tilde{\mathrm{d}} = \tilde{e}^\mu \partial_\mu$ in a coordinate basis

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

$\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$ for any scalar field f coordinate-free

where $\tilde{\mathrm{d}} = \tilde{e}^\mu \partial_\mu$ in a coordinate basis

(Bertschinger writes $\widetilde{\nabla}$ for the gradient $\tilde{\mathrm{d}}$)

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

$\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$ for any scalar field f coordinate-free

where $\tilde{\mathrm{d}} = \tilde{e}^\mu \partial_\mu$ in a coordinate basis

check: $\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

$\mathrm{d}f = \left\langle \tilde{\mathrm{d}}f, \mathrm{d}\vec{x} \right\rangle$ for any scalar field f coordinate-free

where $\tilde{\mathrm{d}} = \tilde{e}^\mu \partial_\mu$ in a coordinate basis

check: $\mathrm{d}f = \left\langle \tilde{\mathrm{d}}f, \mathrm{d}\vec{x} \right\rangle$

$= \langle \tilde{e}^\mu \partial_\mu f, \mathrm{d}x^\nu \vec{e}_\nu \rangle$

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

$\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$ for any scalar field f coordinate-free

where $\tilde{\mathrm{d}} = \tilde{e}^\mu \partial_\mu$ in a coordinate basis

check: $\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$

$= \langle \tilde{e}^\mu \partial_\mu f, \mathrm{d}x^\nu \vec{e}_\nu \rangle$

$= \partial_\mu f \, \mathrm{d}x^\nu \, \langle \tilde{e}^\mu, \vec{e}_\nu \rangle$ since scalars commute

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

$\mathrm{d}f = \left\langle \tilde{\mathrm{d}}f, \mathrm{d}\vec{x} \right\rangle$ for any scalar field f coordinate-free

where $\tilde{\mathrm{d}} = \tilde{e}^\mu \partial_\mu$ in a coordinate basis

check: $\mathrm{d}f = \left\langle \tilde{\mathrm{d}}f, \mathrm{d}\vec{x} \right\rangle$

$= \langle \tilde{e}^\mu \partial_\mu f, \mathrm{d}x^\nu \vec{e}_\nu \rangle$

$= \partial_\mu f \, \mathrm{d}x^\nu \, \langle \tilde{e}^\mu, \vec{e}_\nu \rangle$ since scalars commute

i.e. $\mathrm{d}f = \partial_\mu f \, \mathrm{d}x^\mu$ since $\langle \tilde{e}^\mu, \vec{e}_\nu \rangle = \delta^\mu_\nu$

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

$\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$ for any scalar field f coordinate-free

where $\tilde{\mathrm{d}} = \tilde{e}^\mu \partial_\mu$ in a coordinate basis

check: $\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$

$= \langle \tilde{e}^\mu \partial_\mu f, \mathrm{d}x^\nu \vec{e}_\nu \rangle$

$= \partial_\mu f \, \mathrm{d}x^\nu \, \langle \tilde{e}^\mu, \vec{e}_\nu \rangle$ since scalars commute

i.e. $\mathrm{d}f = \frac{\partial f}{\partial x^\mu} \, \mathrm{d}x^\mu$

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

$\mathrm{d}f = \left\langle \tilde{\mathrm{d}}f, \mathrm{d}\vec{x} \right\rangle$ for any scalar field f coordinate-free

where $\tilde{\mathrm{d}} = \tilde{e}^\mu \partial_\mu$ in a coordinate basis

check: $\mathrm{d}f = \left\langle \tilde{\mathrm{d}}f, \mathrm{d}\vec{x} \right\rangle$

$= \langle \tilde{e}^\mu \partial_\mu f, \mathrm{d}x^\nu \vec{e}_\nu \rangle$

$= \partial_\mu f \, \mathrm{d}x^\nu \, \langle \tilde{e}^\mu, \vec{e}_\nu \rangle$ since scalars commute

i.e. $\mathrm{d}f = \partial_\mu f \, \mathrm{d}x^\mu$

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

$\mathrm{d}f = \left\langle \tilde{\mathrm{d}}f, \mathrm{d}\vec{x} \right\rangle$ for any scalar field f coordinate-free

where $\tilde{\mathrm{d}} = \tilde{e}^\mu \partial_\mu$ in a coordinate basis

check: $\mathrm{d}f = \left\langle \tilde{\mathrm{d}}f, \mathrm{d}\vec{x} \right\rangle$

$= \langle \tilde{e}^\mu \partial_\mu f, \mathrm{d}x^\nu \vec{e}_\nu \rangle$

$= \partial_\mu f \, \mathrm{d}x^\nu \, \langle \tilde{e}^\mu, \vec{e}_\nu \rangle$ since scalars commute

i.e. $\mathrm{d}f = \partial_\mu f \, \mathrm{d}x^\mu$

check: $\tilde{\mathrm{d}}x^\mu = \tilde{e}^\nu \partial_\nu x^\mu$

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

$\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$ for any scalar field f coordinate-free

where $\tilde{\mathrm{d}} = \tilde{e}^\mu \partial_\mu$ in a coordinate basis

check: $\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$

$= \langle \tilde{e}^\mu \partial_\mu f, \mathrm{d}x^\nu \vec{e}_\nu \rangle$

$= \partial_\mu f \, \mathrm{d}x^\nu \, \langle \tilde{e}^\mu, \vec{e}_\nu \rangle$ since scalars commute

i.e. $\mathrm{d}f = \partial_\mu f \, \mathrm{d}x^\mu$

check: $\tilde{\mathrm{d}} x^\mu = \tilde{e}^\nu \partial_\nu x^\mu$

$= \sum_\nu \tilde{e}^\nu \partial_\nu x^\mu$

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

$\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$ for any scalar field f coordinate-free

where $\tilde{\mathrm{d}} = \tilde{e}^\mu \partial_\mu$ in a coordinate basis

check: $\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$

$= \langle \tilde{e}^\mu \partial_\mu f, \mathrm{d}x^\nu \vec{e}_\nu \rangle$

$= \partial_\mu f \, \mathrm{d}x^\nu \, \langle \tilde{e}^\mu, \vec{e}_\nu \rangle$ since scalars commute

i.e. $\mathrm{d}f = \partial_\mu f \, \mathrm{d}x^\mu$

check: $\tilde{\mathrm{d}} x^\mu = \tilde{e}^\nu \partial_\nu x^\mu$

$= \sum_\nu \tilde{e}^\nu \frac{\partial}{\partial x^\nu} x^\mu$

GR: what is a coordinate basis?

coordinate basis: $\vec{e}_\mu$, $\tilde{e}^\nu$ chosen so that:

$\mathrm{d}\vec{x} = \mathrm{d}x^\mu \vec{e}_\mu$ and

$\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$ for any scalar field f coordinate-free

where $\tilde{\mathrm{d}} = \tilde{e}^\mu \partial_\mu$ in a coordinate basis

check: $\mathrm{d}f = \left\langle \tilde{\mathrm{d}} f, \mathrm{d}\vec{x} \right\rangle$

$= \langle \tilde{e}^\mu \partial_\mu f, \mathrm{d}x^\nu \vec{e}_\nu \rangle$

$= \partial_\mu f \, \mathrm{d}x^\nu \, \langle \tilde{e}^\mu, \vec{e}_\nu \rangle$ since scalars commute

i.e. $\mathrm{d}f = \partial_\mu f \, \mathrm{d}x^\mu$

check: $\tilde{\mathrm{d}} x^\mu = \tilde{e}^\nu \partial_\nu x^\mu$

$= \tilde{e}^\mu$

GR: metric using $\mathrm{d}\vec{x}$

we now have

GR: metric using $\mathrm{d}\vec{x}$

we now have

$$\mathrm{d}s^2 := |\mathrm{d}\vec{x}|^2$$

GR: metric using $\mathrm{d}\vec{x}$

we now have

$$\mathrm{d}s^2 := |\mathrm{d}\vec{x}|^2 = \mathbf{g}(\mathrm{d}\vec{x}, \mathrm{d}\vec{x})$$

GR: metric using $\mathrm{d}\vec{x}$

we now have

$\mathrm{d}s^2 := |\mathrm{d}\vec{x}|^2 = \mathbf{g}(\mathrm{d}\vec{x}, \mathrm{d}\vec{x}) = \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x}$ coordinate-free

GR: metric using $\mathrm{d}\vec{x}$

we now have

$\mathrm{d}s^2 := |\mathrm{d}\vec{x}|^2 = \mathbf{g}(\mathrm{d}\vec{x}, \mathrm{d}\vec{x}) = \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x}$ coordinate-free

$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu x^\nu$ if x^μ are a coordinate basis

GR: e.g. Euclidean g on $\mathbb{R}^2$

$g_{r\theta}$ and g_{xy}

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$$

GR: e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$

$g_{r\theta}$ and g_{xy}

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$$

$\vec{e}_x \cdot \vec{e}_x = 1 = \vec{e}_y \cdot \vec{e}_y$, others zero

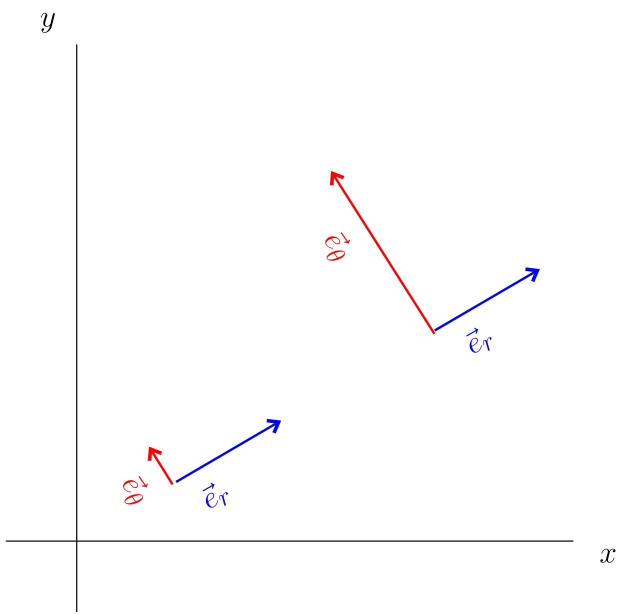
GR: e.g. Euclidean g on $\mathbb{R}^2$

$g_{r\theta}$ and g_{xy}

$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$

$\vec{e}_x \cdot \vec{e}_x = 1 = \vec{e}_y \cdot \vec{e}_y$, others zero

$\vec{e}_r \cdot \vec{e}_r = 1$, $\vec{e}_\theta \cdot \vec{e}_\theta = r^2 \neq 1$



GR: e.g. Euclidean g on $\mathbb{R}^2$

$g_{r\theta}$ and g_{xy}

$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$

$\vec{e}_x \cdot \vec{e}_x = 1 = \vec{e}_y \cdot \vec{e}_y$, others zero

$\vec{e}_r \cdot \vec{e}_r = 1$, $\vec{e}_\theta \cdot \vec{e}_\theta = r^2 \neq 1$

$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \Rightarrow g^{xx} = 1 = g^{yy}, g^{xy} = 0 = g^{yx}$

GR: e.g. Euclidean $\mathbf{g}$ on $\mathbb{R}^2$

$g_{r\theta}$ and g_{xy}

$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$

$\vec{e}_x \cdot \vec{e}_x = 1 = \vec{e}_y \cdot \vec{e}_y$, others zero

$\vec{e}_r \cdot \vec{e}_r = 1$, $\vec{e}_\theta \cdot \vec{e}_\theta = r^2 \neq 1$

$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \Rightarrow g^{xx} = 1 = g^{yy}, g^{xy} = 0 = g^{yx}$

but $g^{rr} = 1 \neq g^{\theta\theta} = r^{-2}, g^{r\theta} = 0 = g^{\theta r}$

GR: e.g. Euclidean g on $\mathbb{R}^2$

$g_{r\theta}$ and g_{xy}

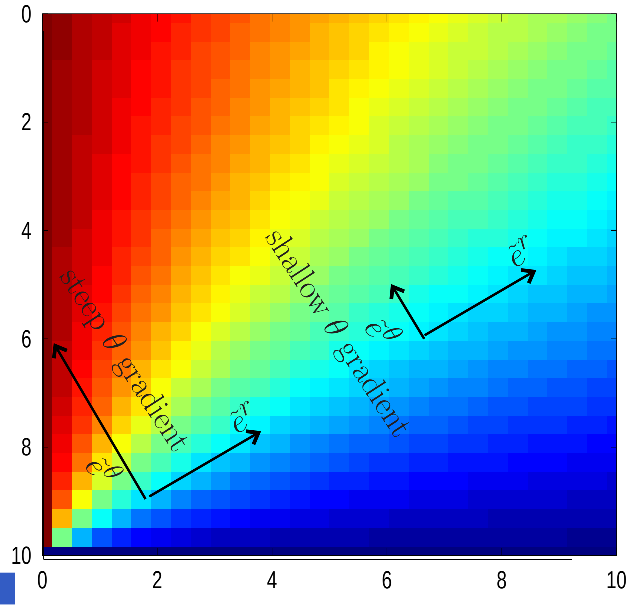
$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$

$\vec{e}_x \cdot \vec{e}_x = 1 = \vec{e}_y \cdot \vec{e}_y$, others zero

$\vec{e}_r \cdot \vec{e}_r = 1$, $\vec{e}_\theta \cdot \vec{e}_\theta = r^2 \neq 1$

$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \Rightarrow g^{xx} = 1 = g^{yy}, g^{xy} = 0 = g^{yx}$

but $g^{rr} = 1 \neq g^{\theta\theta} = r^{-2}, g^{r\theta} = 0 = g^{\theta r}$



GR: e.g. Euclidean g on $\mathbb{R}^2$

$g_{r\theta}$ and g_{xy}

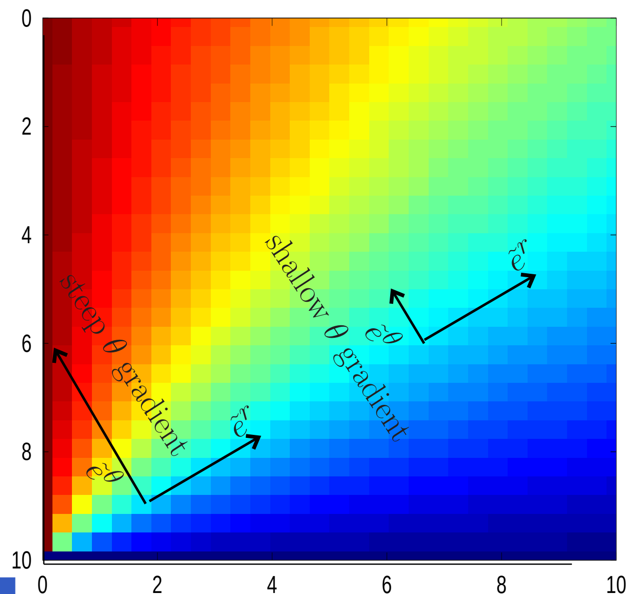
$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$

$\vec{e}_x \cdot \vec{e}_x = 1 = \vec{e}_y \cdot \vec{e}_y$, others zero

$\vec{e}_r \cdot \vec{e}_r = 1$, $\vec{e}_\theta \cdot \vec{e}_\theta = r^2 \neq 1$

$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \Rightarrow g^{xx} = 1 = g^{yy}, g^{xy} = 0 = g^{yx}$

but $g^{rr} = 1 \neq g^{\theta\theta} = r^{-2}, g^{r\theta} = 0 = g^{\theta r}$



so $\tilde{e}^r \cdot \tilde{e}^r = 1$, $\tilde{e}^\theta \cdot \tilde{e}^\theta = r^{-2} \neq 1$

GR: gradient of a vector: $\nabla\vec{A}$

gradient of scalar field: $\tilde{\mathrm{d}}\phi \equiv \widetilde{\nabla}\phi$

GR: gradient of a vector: $\nabla \vec{A}$

what is gradient of vector field $\widetilde{\nabla} \vec{A}$?

GR: gradient of a vector: $\nabla \vec{A}$

$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^{\nu}\vec{e}_{\nu})$$

GR: gradient of a vector: $\nabla\vec{A}$

$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$$

$$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$$

GR: gradient of a vector: $\nabla\vec{A}$

$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu\vec{e}_\nu)$$

$$= \tilde{e}^\mu\partial_\mu(A^\nu\vec{e}_\nu)$$

$$= \tilde{e}^\mu \otimes [(\partial_\mu A^\nu)\vec{e}_\nu + A^\nu\partial_\mu\vec{e}_\nu]$$ by product rule and linearity

GR: gradient of a vector: $\nabla\vec{A}$

$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$$

$$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$$

$$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \partial_\mu \vec{e}_\nu$$

GR: gradient of a vector: $\nabla\vec{A}$

$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$$

$$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$$

$$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \partial_\mu \vec{e}_\nu$$

give a name to the second part: it must be a linear combination of basis vectors $\vec{e}_\lambda$

GR: gradient of a vector: $\nabla \vec{A}$

$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$$

$$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$$

$$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \partial_\mu \vec{e}_\nu$$

define $\Gamma^\lambda{}_{\nu\mu} \vec{e}_\lambda := \partial_\mu \vec{e}_\nu$ Christoffel symbols of second kind (symmetric defn)

GR: gradient of a vector: $\nabla\vec{A}$

$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu\vec{e}_\nu)$$

$$= \tilde{e}^\mu\partial_\mu(A^\nu\vec{e}_\nu)$$

$$= \partial_\mu A^\nu\tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu\tilde{e}^\mu \otimes \partial_\mu\vec{e}_\nu$$

define $\Gamma^\lambda{}_{\nu\mu}\vec{e}_\lambda := \partial_\mu\vec{e}_\nu$ Christoffel symbols of second kind (symmetric defn)

so $\widetilde{\nabla}\vec{A} = \partial_\mu A^\nu\tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu\tilde{e}^\mu \otimes \Gamma^\lambda{}_{\nu\mu}\vec{e}_\lambda$

GR: gradient of a vector: $\nabla\vec{A}$

$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$

$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$

$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \partial_\mu \vec{e}_\nu$

define $\Gamma^\lambda{}_{\nu\mu} \vec{e}_\lambda := \partial_\mu \vec{e}_\nu$ Christoffel symbols of second kind (symmetric defn)

so $\widetilde{\nabla}\vec{A} = \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \Gamma^\lambda{}_{\nu\mu} \vec{e}_\lambda$

$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \Gamma^\lambda{}_{\nu\mu} \tilde{e}^\mu \otimes \vec{e}_\lambda$ since any $\Gamma^\lambda{}_{\nu\mu}$ is a scalar

GR: gradient of a vector: $\nabla\vec{A}$

$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$

$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$

$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \partial_\mu \vec{e}_\nu$

define $\Gamma^\lambda{}_{\nu\mu} \vec{e}_\lambda := \partial_\mu \vec{e}_\nu$ Christoffel symbols of second kind (symmetric defn)

so $\widetilde{\nabla}\vec{A} = \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \Gamma^\lambda{}_{\nu\mu} \vec{e}_\lambda$

$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\lambda \Gamma^\nu{}_{\lambda\mu} \tilde{e}^\mu \otimes \vec{e}_\nu$

since name of summation index is arbitrary, e.g.
$\sum_\lambda x^{-2\lambda} = \sum_\mu x^{-2\mu} = \sum_\nu x^{-2\nu}$

GR: gradient of a vector: $\nabla\vec{A}$

$$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$$

$$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$$

$$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \partial_\mu \vec{e}_\nu$$

define $\Gamma^\lambda{}_{\nu\mu} \vec{e}_\lambda := \partial_\mu \vec{e}_\nu$ Christoffel symbols of second kind (symmetric defn)

so $\widetilde{\nabla}\vec{A} = \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \Gamma^\lambda{}_{\nu\mu} \vec{e}_\lambda$

$$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\lambda \Gamma^\nu{}_{\lambda\mu} \tilde{e}^\mu \otimes \vec{e}_\nu$$

$$= (\partial_\mu A^\nu + A^\lambda \Gamma^\nu{}_{\lambda\mu}) \tilde{e}^\mu \otimes \vec{e}_\nu$$

GR: gradient of a vector: $\nabla\vec{A}$

$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$

$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$

$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \partial_\mu \vec{e}_\nu$

define $\Gamma^\lambda{}_{\nu\mu}\vec{e}_\lambda := \partial_\mu \vec{e}_\nu$ Christoffel symbols of second kind (symmetric defn)

so $\widetilde{\nabla}\vec{A} = \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \Gamma^\lambda{}_{\nu\mu}\vec{e}_\lambda$

$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\lambda \Gamma^\nu{}_{\lambda\mu} \tilde{e}^\mu \otimes \vec{e}_\nu$

$= (\partial_\mu A^\nu + A^\lambda \Gamma^\nu{}_{\lambda\mu}) \tilde{e}^\mu \otimes \vec{e}_\nu$

$$\boxed{\nabla_\mu A^\nu := A^\nu{}_{;\mu} := \partial_\mu A^\nu + A^\lambda \Gamma^\nu{}_{\lambda\mu}}$$

w:covariant derivative of vector

GR: gradient of a vector: $\nabla\vec{A}$

$\widetilde{\nabla}\vec{A} = \widetilde{\nabla}(A^\nu \vec{e}_\nu)$

$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$

$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \partial_\mu \vec{e}_\nu$

define $\Gamma^\lambda{}_{\nu\mu} \vec{e}_\lambda := \partial_\mu \vec{e}_\nu$ Christoffel symbols of second kind (symmetric defn)

so $\widetilde{\nabla}\vec{A} = \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \tilde{e}^\mu \otimes \Gamma^\lambda{}_{\nu\mu} \vec{e}_\lambda$

$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\lambda \Gamma^\nu{}_{\lambda\mu} \tilde{e}^\mu \otimes \vec{e}_\nu$

$= (\partial_\mu A^\nu + A^\lambda \Gamma^\nu{}_{\lambda\mu}) \tilde{e}^\mu \otimes \vec{e}_\nu$

$$\boxed{\nabla_\mu A^\nu := A^\nu{}_{;\mu} := A^\nu{}_{,\mu} + A^\lambda \Gamma^\nu{}_{\lambda\mu}}$$

w:covariant derivative of vector

GR: $\tilde{\nabla}$ and $\Gamma^{\nu}_{\lambda\mu}$

mathematically deeper: $\tilde{\nabla}$, usually written just as ∇, is the w:Levi-Civita connection

GR: $\tilde{\nabla}$ and $\Gamma^{\nu}{}_{\lambda\mu}$

mathematically deeper: $\tilde{\nabla}$, usually written just as ∇, is the w:Levi-Civita connection

warning: $\Gamma^{\nu}{}_{\lambda\mu}$ are NOT the components of a tensor

GR: $\widetilde{\nabla}$ and $\Gamma^{\nu}{}_{\lambda\mu}$

mathematically deeper: $\widetilde{\nabla}$, usually written just as ∇, is the w:Levi-Civita connection

warning: $\Gamma^{\nu}{}_{\lambda\mu}$ are NOT the components of a tensor

- $\widetilde{\nabla}$ applied to a (m, n)-tensor field on a manifold gives an $(m, n+1)$-tensor field

GR: $\widetilde{\nabla}$ and $\Gamma^{\nu}{}_{\lambda\mu}$

mathematically deeper: $\widetilde{\nabla}$, usually written just as ∇, is the w:Levi-Civita connection

warning: $\Gamma^{\nu}{}_{\lambda\mu}$ are NOT the components of a tensor

- $\widetilde{\nabla}$ applied to a (m, n)-tensor field on a manifold gives an $(m, n+1)$-tensor field

so far we showed how $\widetilde{\nabla}$ applied to a $(0, 0)$-tensor field = scalar field ϕ gives a $(0, 1)$-tensor field = one-form field = $(\tilde{\mathrm{d}}\phi)_\mu \vec{e}^{\mu}$

GR: $\tilde{\nabla}$ and $\Gamma^{\nu}{}_{\lambda\mu}$

mathematically deeper: $\tilde{\nabla}$, usually written just as ∇, is the w:Levi-Civita connection

warning: $\Gamma^{\nu}{}_{\lambda\mu}$ are NOT the components of a tensor

- $\tilde{\nabla}$ applied to a (m, n)-tensor field on a manifold gives an $(m, n+1)$-tensor field

so far we showed how $\tilde{\nabla}$ applied to a $(0, 0)$-tensor field = scalar field ϕ gives a $(0, 1)$-tensor field = one-form field = $\tilde{\nabla}_{\mu}\phi\vec{e}^{\mu} = \partial_{\mu}\phi\vec{e}^{\mu}$

GR: $\tilde{\nabla}$ and $\Gamma^{\nu}{}_{\lambda\mu}$

mathematically deeper: $\tilde{\nabla}$, usually written just as ∇, is the w:Levi-Civita connection

warning: $\Gamma^{\nu}{}_{\lambda\mu}$ are NOT the components of a tensor

- $\tilde{\nabla}$ applied to a (m, n)-tensor field on a manifold gives an $(m, n+1)$-tensor field

so far we showed how $\tilde{\nabla}$ applied to a $(0, 0)$-tensor field = scalar field ϕ gives a $(0, 1)$-tensor field = one-form field = $\tilde{\nabla}_{\mu}\phi\vec{e}^{\mu} = \partial_{\mu}\phi\vec{e}^{\mu}$

and $\tilde{\nabla}$ on a $(1, 0)$-tensor field = vector field $\vec{A}$ gives a $(1, 1)$-tensor with components $\nabla_{\mu}A^{\nu} = \partial_{\mu}A^{\nu} + A^{\lambda}\Gamma^{\nu}{}_{\lambda\mu}$

GR: $\widetilde{\nabla}$ and $\Gamma^{\nu}{}_{\lambda\mu}$

mathematically deeper: $\widetilde{\nabla}$, usually written just as ∇, is the w:Levi-Civita connection

warning: $\Gamma^{\nu}{}_{\lambda\mu}$ are NOT the components of a tensor

- $\widetilde{\nabla}$ applied to a (m,n)-tensor field on a manifold gives an $(m,n+1)$-tensor field

so far we showed how $\widetilde{\nabla}$ applied to a $(0,0)$-tensor field = scalar field ϕ gives a $(0,1)$-tensor field = one-form field = $\widetilde{\nabla}_{\mu}\phi\vec{e}^{\mu} = \partial_{\mu}\phi\vec{e}^{\mu}$

and $\widetilde{\nabla}$ on a $(1,0)$-tensor field = vector field $\vec{A}$ gives a $(1,1)$-tensor with components $\nabla_{\mu}A^{\nu} = \partial_{\mu}A^{\nu} + A^{\lambda}\Gamma^{\nu}{}_{\lambda\mu}$

- tensors: $\widetilde{\nabla}\phi = \widetilde{\nabla}_{\mu}\phi\vec{e}^{\mu}$,

GR: $\widetilde{\nabla}$ and ${\Gamma^\nu}_{\lambda\mu}$

mathematically deeper: $\widetilde{\nabla}$, usually written just as ∇, is the w:Levi-Civita connection

warning: ${\Gamma^\nu}_{\lambda\mu}$ are NOT the components of a tensor

- $\widetilde{\nabla}$ applied to a (m,n)-tensor field on a manifold gives an $(m,n+1)$-tensor field

so far we showed how $\widetilde{\nabla}$ applied to a $(0,0)$-tensor field = scalar field ϕ gives a $(0,1)$-tensor field = one-form field = $\widetilde{\nabla}_\mu \phi \vec{e}^\mu = \partial_\mu \phi \vec{e}^\mu$

and $\widetilde{\nabla}$ on a $(1,0)$-tensor field = vector field $\vec{A}$ gives a $(1,1)$-tensor with components $\nabla_\mu A^\nu = \partial_\mu A^\nu + A^\lambda {\Gamma^\nu}_{\lambda\mu}$

- tensors: $\widetilde{\nabla}\phi = \widetilde{\nabla}_\mu \phi \vec{e}^\mu$, $\quad \widetilde{\nabla}\vec{A} = \left(\partial_\mu A^\nu + A^\lambda {\Gamma^\nu}_{\lambda\mu}\right) \tilde{e}^\mu \otimes \vec{e}^\nu$

GR: $\widetilde{\nabla}$ and $\Gamma^{\nu}{}_{\lambda\mu}$

mathematically deeper: $\widetilde{\nabla}$, usually written just as ∇, is the w:Levi-Civita connection

warning: $\Gamma^{\nu}{}_{\lambda\mu}$ are NOT the components of a tensor

- $\widetilde{\nabla}$ applied to a (m,n)-tensor field on a manifold gives an $(m,n+1)$-tensor field

so far we showed how $\widetilde{\nabla}$ applied to a $(0,0)$-tensor field = scalar field ϕ gives a $(0,1)$-tensor field = one-form field = $\widetilde{\nabla}_{\mu}\phi\vec{e}^{\mu} = \partial_{\mu}\phi\vec{e}^{\mu}$

and $\widetilde{\nabla}$ on a $(1,0)$-tensor field = vector field $\vec{A}$ gives a $(1,1)$-tensor with components $\nabla_{\mu}A^{\nu} = \partial_{\mu}A^{\nu} + A^{\lambda}\Gamma^{\nu}{}_{\lambda\mu}$

- tensors: $\widetilde{\nabla}\phi = \widetilde{\nabla}_{\mu}\phi\vec{e}^{\mu}, \quad \widetilde{\nabla}\vec{A} = \left(\partial_{\mu}A^{\nu} + A^{\lambda}\Gamma^{\nu}{}_{\lambda\mu}\right)\tilde{e}^{\mu}\otimes\vec{e}^{\nu}$
- not components of tensor: $\Gamma^{\nu}{}_{\lambda\mu}$

GR: gradient of one-form $\tilde{\nabla}\tilde{A}$

how does a one-form change with position? $\tilde{\nabla}\tilde{A} =$?

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu} \tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$\partial_\mu \delta^\nu{}_\lambda = 0$ (obviously)

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu} \tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$$

can we use the product rule with this scalar product?

$$\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = ?$$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu} \tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$

can we use the product rule with this scalar product?

$\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \partial_\mu \left(A_\nu B^\nu \right)$ in some coordinate basis

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$

can we use the product rule with this scalar product?

$\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \partial_\mu \left(A_\nu B^\nu \right)$

$= (\partial_\mu A_\nu) B^\nu + A_\nu (\partial_\mu B^\nu)$ by product rule on functions

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$$

can we use the product rule with this scalar product?

$$\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \partial_\mu \left(A_\nu B^\nu \right)$$

$$= (\partial_\mu A_\nu) B^\nu + A_\nu (\partial_\mu B^\nu)$$

$= \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$ since

$$\partial_\mu \tilde{A} = (\partial_\mu A_0, \partial_\mu A_1, \partial_\mu A_2, \partial_\mu A_3)$$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$

product rule holds: $\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$

product rule holds: $\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$

so $0 = \langle \partial_\mu \tilde{e}^\nu, \vec{e}_\lambda \rangle + \langle \tilde{e}^\nu, \partial_\mu \vec{e}_\lambda \rangle$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$$

product rule holds: $\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$

so $0 = \langle \partial_\mu \tilde{e}^\nu, \vec{e}_\lambda \rangle + \langle \tilde{e}^\nu, \partial_\mu \vec{e}_\lambda \rangle$

$$= \left\langle F^\nu{}_{\kappa\mu}\tilde{e}^\kappa, \vec{e}_\lambda \right\rangle + \left\langle \tilde{e}^\nu, \Gamma^\kappa{}_{\lambda\mu}\vec{e}_\kappa \right\rangle$$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$$

product rule holds: $\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$

so $0 = \langle \partial_\mu \tilde{e}^\nu, \vec{e}_\lambda \rangle + \langle \tilde{e}^\nu, \partial_\mu \vec{e}_\lambda \rangle$

$$= F^\nu{}_{\kappa\mu} \langle \tilde{e}^\kappa, \vec{e}_\lambda \rangle + \Gamma^\kappa{}_{\lambda\mu} \langle \tilde{e}^\nu, \vec{e}_\kappa \rangle$$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$

product rule holds: $\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$

so $0 = \langle \partial_\mu \tilde{e}^\nu, \vec{e}_\lambda \rangle + \langle \tilde{e}^\nu, \partial_\mu \vec{e}_\lambda \rangle$

$= F^\nu{}_{\lambda\mu} + \Gamma^\nu{}_{\lambda\mu}$ since $\langle \tilde{e}^\kappa, \vec{e}_\lambda \rangle = \delta^\kappa{}_\lambda$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu} \tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$

product rule holds: $\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$

so $0 = \langle \partial_\mu \tilde{e}^\nu, \vec{e}_\lambda \rangle + \langle \tilde{e}^\nu, \partial_\mu \vec{e}_\lambda \rangle$

$= F^\nu{}_{\lambda\mu} + \Gamma^\nu{}_{\lambda\mu}$ since $\langle \tilde{e}^\kappa, \vec{e}_\lambda \rangle = \delta^\kappa{}_\lambda$

hence, $\partial_\mu \tilde{e}^\nu =: F^\nu{}_{\lambda\mu} \tilde{e}^\lambda = -\Gamma^\nu{}_{\lambda\mu} \tilde{e}^\lambda$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$

product rule holds: $\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$

so $0 = \langle \partial_\mu \tilde{e}^\nu, \vec{e}_\lambda \rangle + \langle \tilde{e}^\nu, \partial_\mu \vec{e}_\lambda \rangle$

$= F^\nu{}_{\lambda\mu} + \Gamma^\nu{}_{\lambda\mu}$ since $\langle \tilde{e}^\kappa, \vec{e}_\lambda \rangle = \delta^\kappa{}_\lambda$

hence, $\partial_\mu \tilde{e}^\nu =: F^\nu{}_{\lambda\mu}\tilde{e}^\lambda = -\Gamma^\nu{}_{\lambda\mu}\tilde{e}^\lambda$

$$\boxed{\widetilde{\nabla}_\mu A^\nu = \partial_\mu A^\nu + A^\lambda \Gamma^\nu{}_{\lambda\mu} \quad , \quad \widetilde{\nabla}_\mu A_\nu = \partial_\mu A_\nu - A_\lambda \Gamma^\lambda{}_{\mu\nu}}$$

GR: gradient of one-form $\widetilde{\nabla}\tilde{A}$

evaluating $\widetilde{\nabla}\tilde{A}$ as we did $\widetilde{\nabla}\vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu{}_{\lambda\mu}\tilde{e}^\lambda$ for some coefficients $F^\nu{}_{\lambda\mu}$

how can we relate $\Gamma^\nu{}_{\lambda\mu}$ to $F^\nu{}_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu{}_\lambda$

$0 = \partial_\mu \delta^\nu{}_\lambda = \partial_\mu \left(\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle \right)$

product rule holds: $\partial_\mu \left(\left\langle \tilde{A}, \vec{B} \right\rangle \right) = \left\langle \partial_\mu \tilde{A}, \vec{B} \right\rangle + \left\langle \tilde{A}, \partial_\mu \vec{B} \right\rangle$

so $0 = \langle \partial_\mu \tilde{e}^\nu, \vec{e}_\lambda \rangle + \langle \tilde{e}^\nu, \partial_\mu \vec{e}_\lambda \rangle$

$= F^\nu{}_{\lambda\mu} + \Gamma^\nu{}_{\lambda\mu}$ since $\langle \tilde{e}^\kappa, \vec{e}_\lambda \rangle = \delta^\kappa{}_\lambda$

hence, $\partial_\mu \tilde{e}^\nu =: F^\nu{}_{\lambda\mu}\tilde{e}^\lambda = -\Gamma^\nu{}_{\lambda\mu}\tilde{e}^\lambda$

$$A^\nu{}_{;\mu} = A^\nu{}_{,\mu} + A^\lambda \Gamma^\nu{}_{\lambda\mu} \quad , \quad A_{\nu;\mu} = A_{\nu,\mu} - A_\lambda \Gamma^\lambda{}_{\mu\nu}$$

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

similarly, we can write the $(0,3)$-tensor

$$\widetilde{\nabla}\mathbf{g} = (\nabla_\lambda g_{\mu\nu})\tilde{e}^\lambda \otimes \tilde{e}^\mu \otimes \tilde{e}^\nu$$

giving $\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

similarly, we can write the $(0,3)$-tensor

$$\widetilde{\nabla}\mathbf{g} = (\nabla_\lambda g_{\mu\nu})\tilde{e}^\lambda \otimes \tilde{e}^\mu \otimes \tilde{e}^\nu$$

giving $\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$

also $\widetilde{\nabla}\mathbf{g}^{-1} = (\nabla_\lambda g^{\mu\nu})\tilde{e}^\lambda \otimes \vec{e}_\mu \otimes \vec{e}_\nu$

and $\nabla_\lambda g^{\mu\nu} = \partial_\lambda g^{\mu\nu} + \Gamma^\mu{}_{\kappa\lambda} g_{\kappa\nu} + \Gamma^\nu{}_{\kappa\lambda} g_{\mu\kappa}$

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

similarly, we can write the $(0,3)$-tensor

$\widetilde{\nabla}\mathbf{g} = (\nabla_\lambda g_{\mu\nu})\tilde{e}^\lambda \otimes \tilde{e}^\mu \otimes \tilde{e}^\nu$

giving $\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$

also $\widetilde{\nabla}\mathbf{g}^{-1} = (\nabla_\lambda g^{\mu\nu})\tilde{e}^\lambda \otimes \vec{e}_\mu \otimes \vec{e}_\nu$

and $\nabla_\lambda g^{\mu\nu} = \partial_\lambda g^{\mu\nu} + \Gamma^\mu{}_{\kappa\lambda} g_{\kappa\nu} + \Gamma^\nu{}_{\kappa\lambda} g_{\mu\kappa}$

Do we know anything interesting about $\widetilde{\nabla}\mathbf{g}$ for the manifolds of interest to GR?

GR: smooth manifold and $\tilde{\nabla}\mathbf{g}$

similarly, we can write the $(0,3)$-tensor

$$\tilde{\nabla}\mathbf{g} = (\nabla_\lambda g_{\mu\nu})\tilde{e}^\lambda \otimes \tilde{e}^\mu \otimes \tilde{e}^\nu$$

giving $\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$

also $\tilde{\nabla}\mathbf{g}^{-1} = (\nabla_\lambda g^{\mu\nu})\tilde{e}^\lambda \otimes \vec{e}_\mu \otimes \vec{e}_\nu$

and $\nabla_\lambda g^{\mu\nu} = \partial_\lambda g^{\mu\nu} + \Gamma^\mu{}_{\kappa\lambda} g_{\kappa\nu} + \Gamma^\nu{}_{\kappa\lambda} g_{\mu\kappa}$

Do we know anything interesting about $\tilde{\nabla}\mathbf{g}$ for the manifolds of interest to GR?

First, we need a rough description of the manifolds we need for GR.

GR: smooth manifold and $\widetilde{\nabla \mathbf{g}}$

topological 4-(pseudo-)manifold M
w:Manifold#Mathematical_definition

- only topological properties needed

GR: smooth manifold and $\widetilde{\nabla \mathbf{g}}$

topological 4-(pseudo-)manifold M
w:Manifold#Mathematical_definition

- only topological properties needed
- no differentiability, no metric needed

GR: smooth manifold and $\widetilde{\nabla} \mathbf{g}$

topological 4-(pseudo-)manifold M
w:Manifold#Mathematical_definition

- only topological properties needed

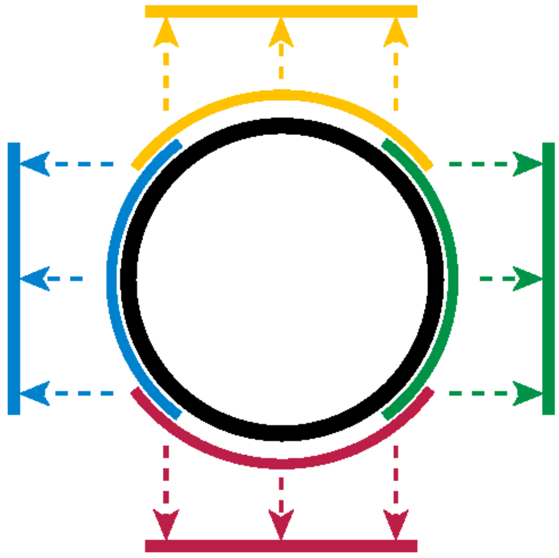
next: relation with $\mathbb{R}^4$ (or M^4)

GR: smooth manifold and $\widetilde{\nabla}$g

topological 4-(pseudo-)manifold M
w:Manifold#Mathematical_definition

- only topological properties needed

next: relation with $\mathbb{R}^4$ (or M^4)

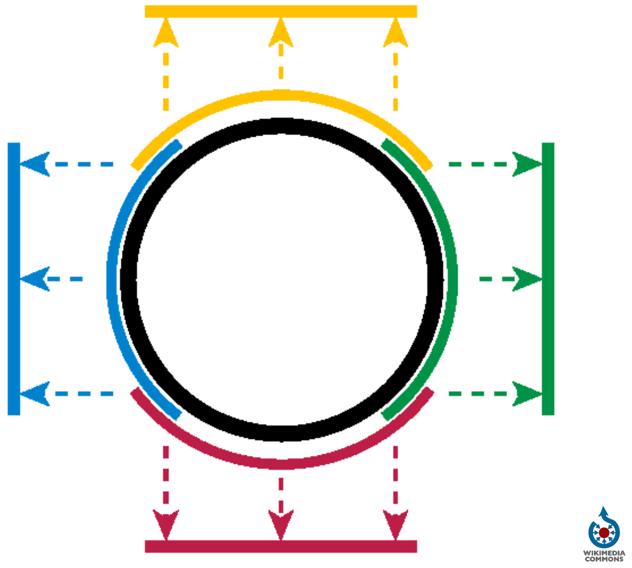


GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

topological 4-(pseudo-)manifold M
w:Manifold#Mathematical_definition

- only topological properties needed

next: relation with $\mathbb{R}^4$ (or M^4)



w:Manifold

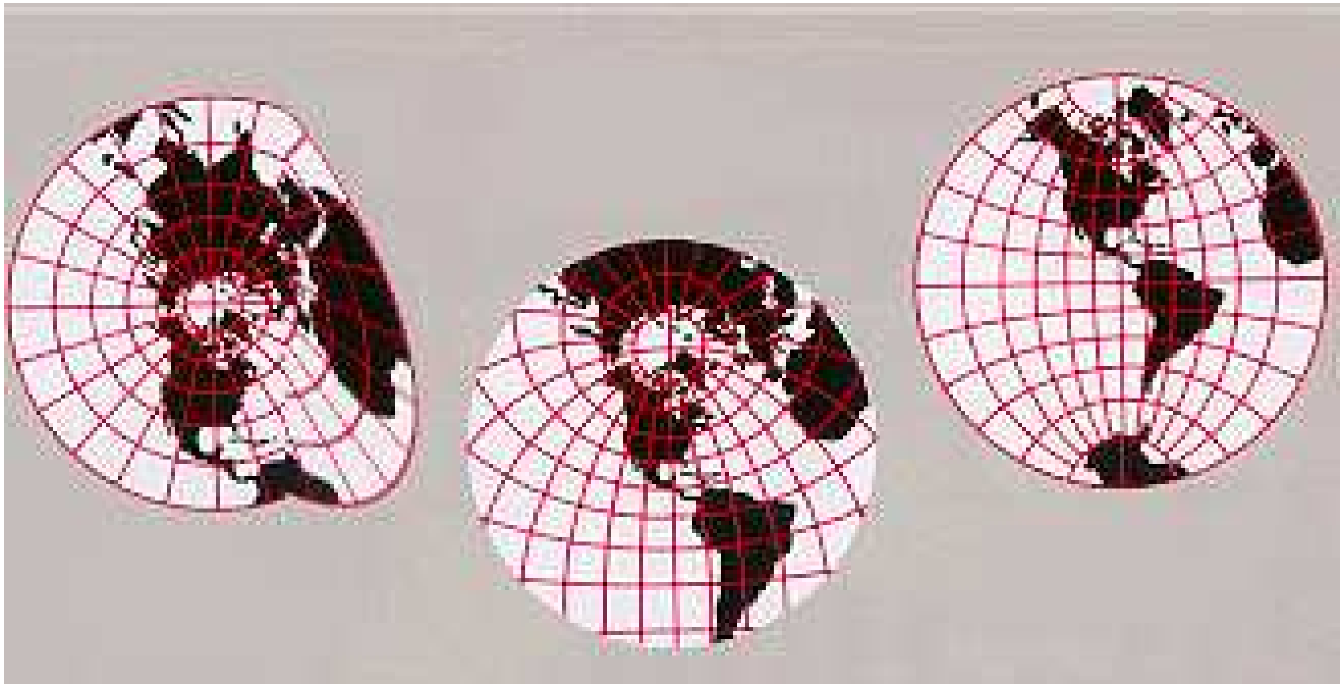
- chart := function ϕ_α from part of pseudo-4-manifold M to part of M^4 (Minkowski)
- atlas := set of overlapping charts that cover M

GR: smooth manifold and $\widetilde{\nabla}$g

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

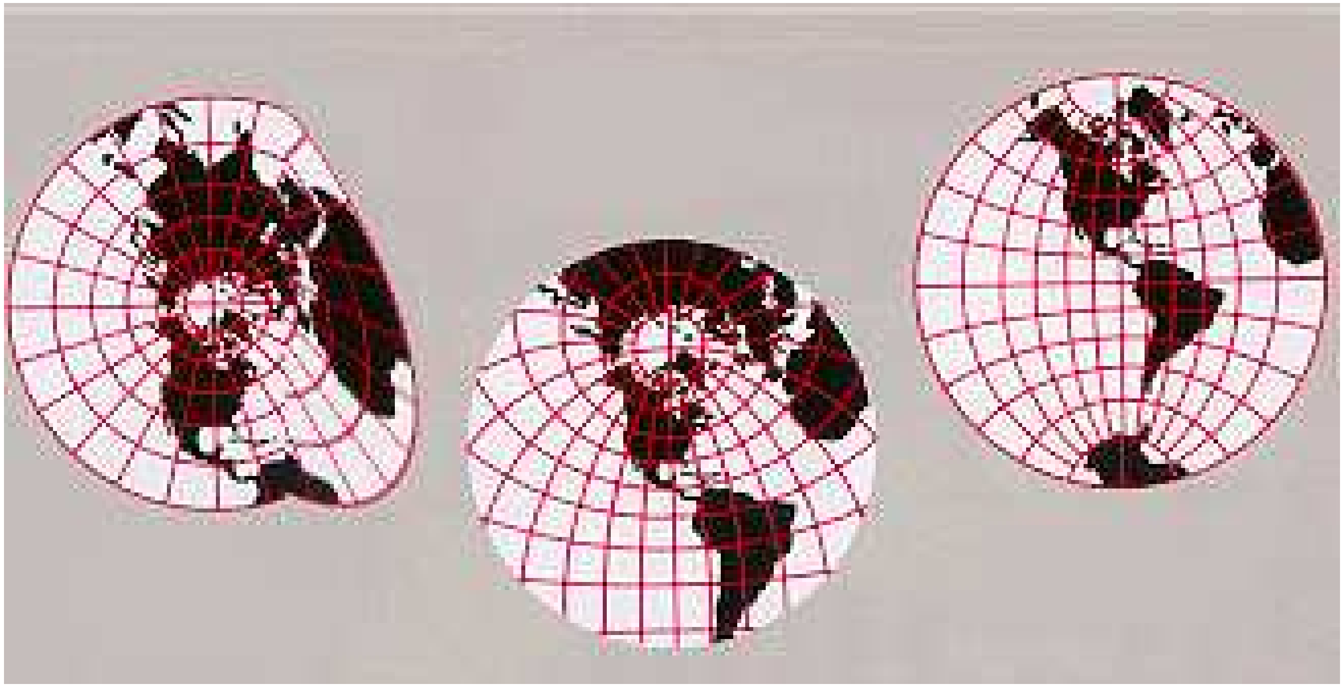
if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold



w:

GR: smooth manifold and $\widetilde{\nabla}$g

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

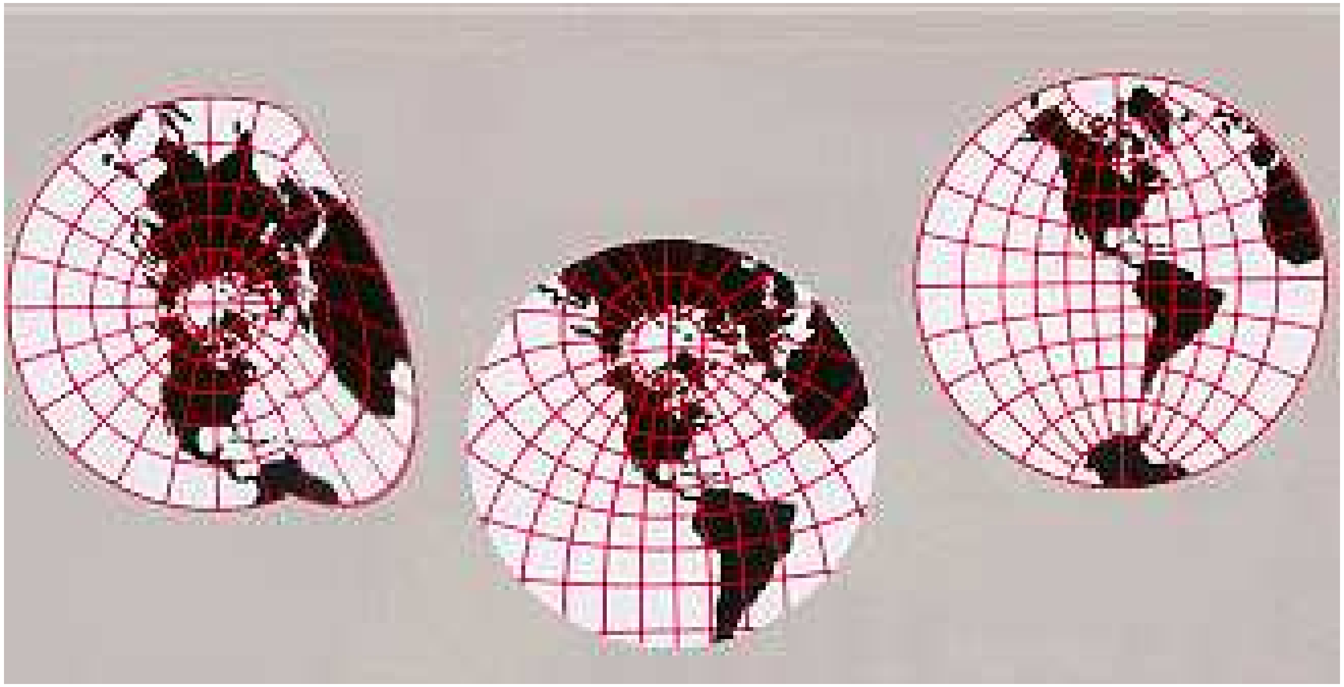


w:

projections (left-to-right) ϕ_1, ϕ_2, ϕ_3 from S^2 to $\mathbb{R}^2$

GR: smooth manifold and $\widetilde{\nabla}$g

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

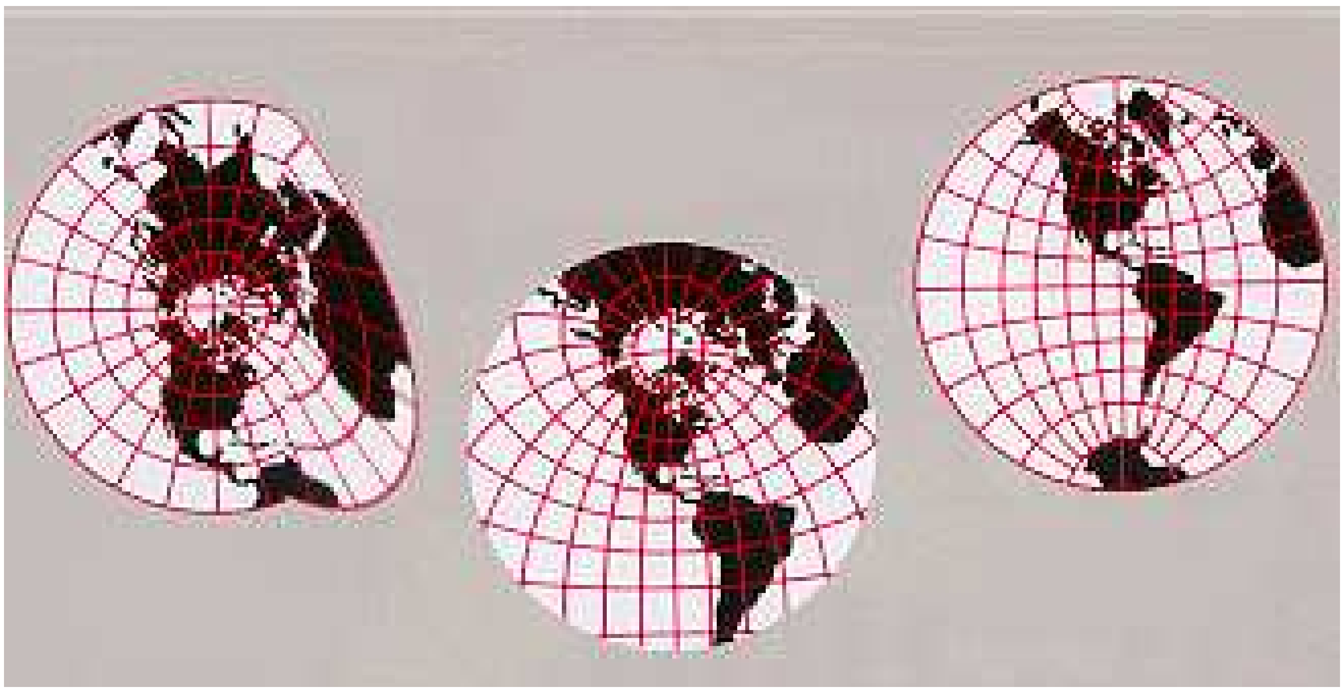


w:

ϕ_1 is not differentiable, so $\phi_1 \circ \phi_2^{-1}$ is not differentiable

GR: smooth manifold and $\widetilde{\nabla} \mathbf{g}$

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold



w:

atlas not enough to show that $S^2 =$ differentiable 2-manifold

GR: smooth manifold and $\widetilde{\nabla}$g

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

if $\forall k \geq 1$, $\exists k$-th derivatives, then M is a smooth 4-(pseudo-)manifold

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

if $\forall k \geq 1$, $\exists k$-th derivatives, then M is a smooth 4-(pseudo-)manifold

if a (pseudo-)w:Riemannian metric $\mathbf{g}$ can be added to M, then $(M, \mathbf{g})$ is a (pseudo-)Riemannian 4-manifold

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

if $\forall k \geq 1$, $\exists k$-th derivatives, then M is a smooth 4-(pseudo-)manifold

if a (pseudo-)w:Riemannian metric $\mathbf{g}$ can be added to M, then $(M, \mathbf{g})$ is a (pseudo-)Riemannian 4-manifold

if $\mathbf{g}$ has signature $(1, n-1)$ (i.e. $(-,+,+,+)$ or $(+,-,-,-)$, etc.), then $(M, \mathbf{g})$ is a Lorentzian n-manifold

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

topological manifolds

GR: smooth manifold and $\widetilde{\nabla}$g

topological manifolds
differentiable (pseudo-)manifolds

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

topological manifolds

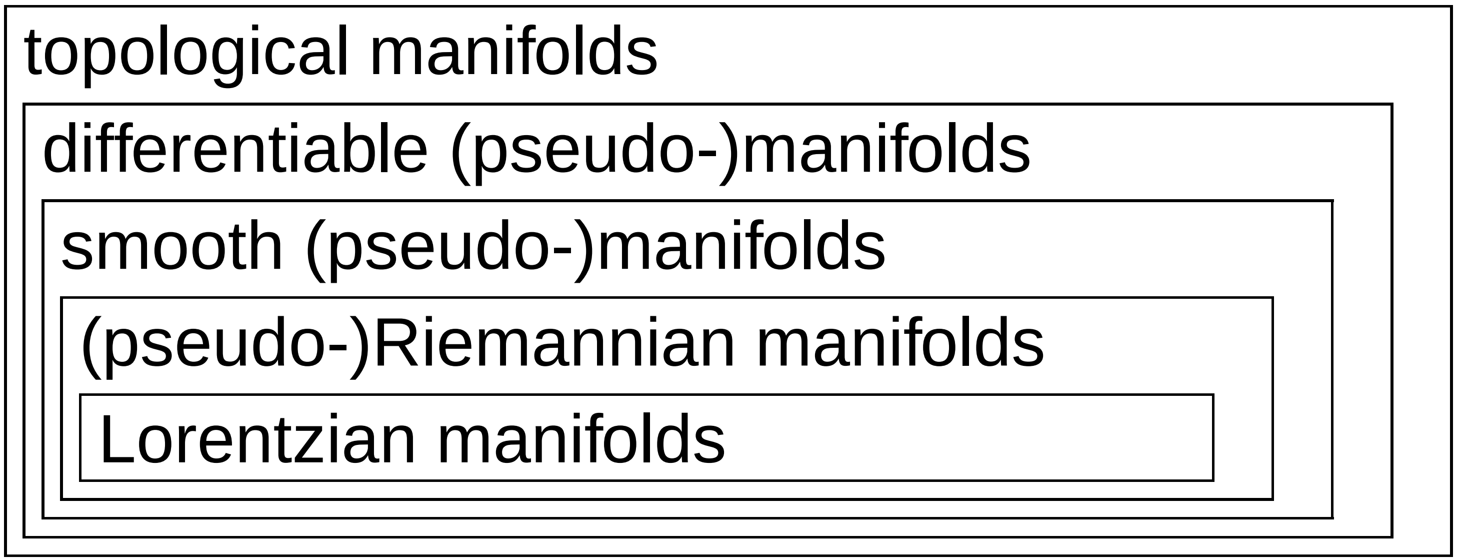
differentiable (pseudo-)manifolds

smooth (pseudo-)manifolds

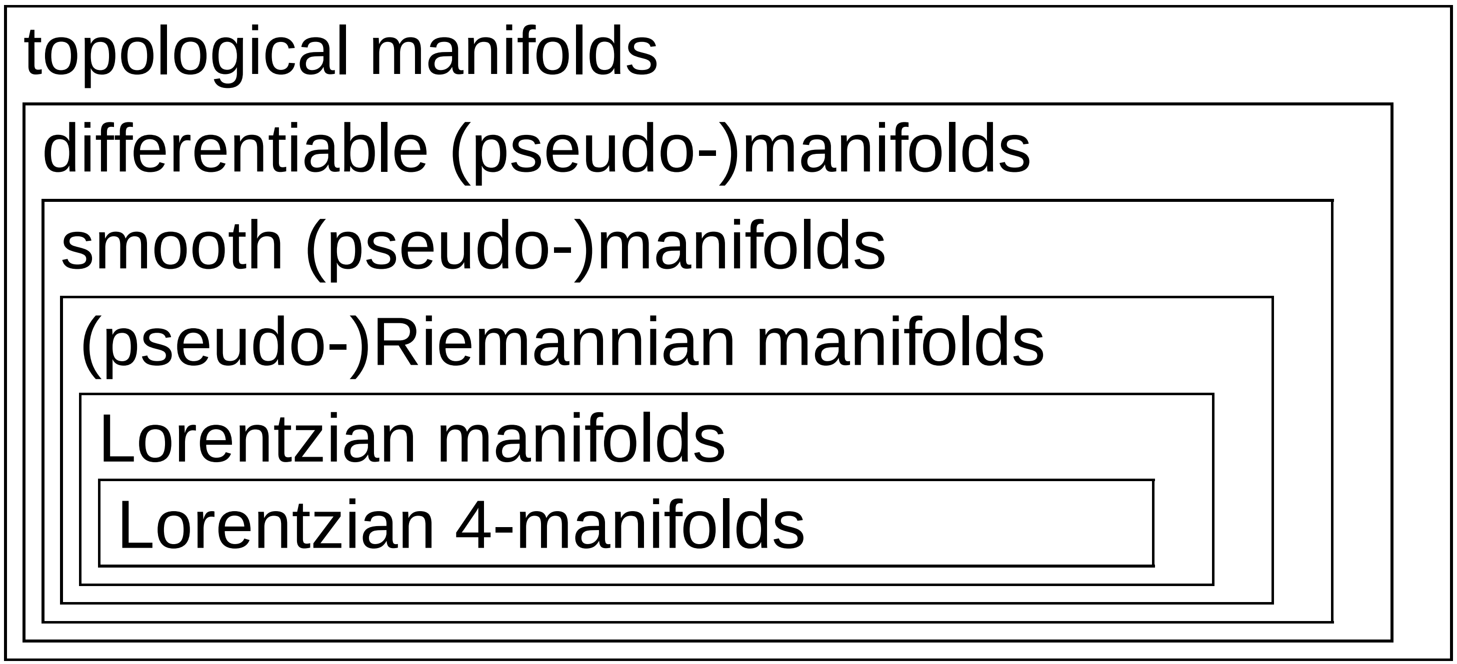
GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

- topological manifolds
  - differentiable (pseudo-)manifolds
    - smooth (pseudo-)manifolds
      - (pseudo-)Riemannian manifolds

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$



GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$



GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

- topological manifolds
  - differentiable (pseudo-)manifolds
    - smooth (pseudo-)manifolds
      - (pseudo-)Riemannian manifolds
        - Lorentzian manifolds
          - Lorentzian 4-manifolds

GR: assume that spacetime is a Lorentzian 4-manifold

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

from above:

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$$

GR: smooth manifold and $\widetilde{\nabla \mathbf{g}}$

from above:

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$$

in the tangent space at $\mathbf{x}$, $\exists$ coordinate basis $\vec{e}^{\bar{\mu}}$ with

$$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(-1, 1, 1, 1) = g^{\bar{\mu}\bar{\nu}}$$

$$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}}$$

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

from above:

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$$

in the tangent space at $\mathbf{x}$, $\exists$ coordinate basis $\vec{e}^{\bar{\mu}}$ with

$$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(-1,1,1,1) = g^{\bar{\mu}\bar{\nu}}$$

$$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}} \;= 0$$

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

from above:

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$$

in the tangent space at $\mathbf{x}$, $\exists$ coordinate basis $\vec{e}^{\bar{\mu}}$ with

$$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(-1,1,1,1) = g^{\bar{\mu}\bar{\nu}}$$

$$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}} \ = 0$$

also, $\Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} \vec{e}_{\bar{\lambda}} := \vec{e}_{\bar{\nu},\bar{\mu}} = \partial_{\bar{\mu}} \vec{e}_{\bar{\nu}}$

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

from above:

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$$

in the tangent space at $\mathbf{x}$, $\exists$ coordinate basis $\vec{e}^{\bar{\mu}}$ with

$$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(-1, 1, 1, 1) = g^{\bar{\mu}\bar{\nu}}$$

$$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}} \;= 0$$

also, $\Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} \vec{e}_{\bar{\lambda}} := \vec{e}_{\bar{\nu},\bar{\mu}} = \partial_{\bar{\mu}} \vec{e}_{\bar{\nu}}$

but in a Cartesian or Minkowski (vector) space, the basis vectors always point in the same direction and their lengths are fixed

GR: smooth manifold and $\widetilde{\nabla}$g

from above:

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$$

in the tangent space at $\mathbf{x}$, $\exists$ coordinate basis $\vec{e}^{\bar{\mu}}$ with

$$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(-1,1,1,1) = g^{\bar{\mu}\bar{\nu}}$$

$$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}} = 0$$

also, $\Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} \vec{e}_{\bar{\lambda}} := \vec{e}_{\bar{\nu},\bar{\mu}} = \partial_{\bar{\mu}} \vec{e}_{\bar{\nu}}$

M^4 $\Rightarrow \Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} \vec{e}_{\bar{\lambda}} = 0$

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

from above:

$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$

in the tangent space at $\mathbf{x}$, $\exists$ coordinate basis $\vec{e}^{\bar{\mu}}$ with

$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(-1, 1, 1, 1) = g^{\bar{\mu}\bar{\nu}}$

$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}} \ = 0$

also, $\Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} \vec{e}_{\bar{\lambda}} := \vec{e}_{\bar{\nu},\bar{\mu}} = \partial_{\bar{\mu}} \vec{e}_{\bar{\nu}}$

so $\nabla_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = 0$

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

from above:

$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$

in the tangent space at $\mathbf{x}$, $\exists$ coordinate basis $\vec{e}^{\bar{\mu}}$ with

$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(-1,1,1,1) = g^{\bar{\mu}\bar{\nu}}$

$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}} \ = 0$

also, $\Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} \vec{e}_{\bar{\lambda}} := \vec{e}_{\bar{\nu},\bar{\mu}} = \partial_{\bar{\mu}} \vec{e}_{\bar{\nu}}$

so $\nabla_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = 0$

so $\widetilde{\nabla}\mathbf{g} = \mathbf{0}$ (also $\widetilde{\nabla}\mathbf{g}^{-1} = 0$) on the tangent space, since if true in one coord system, also true in others

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

from above:

$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$

in the tangent space at $\mathbf{x}$, $\exists$ coordinate basis $\vec{e}^{\bar{\mu}}$ with

$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(-1, 1, 1, 1) = g^{\bar{\mu}\bar{\nu}}$

$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}} \ = 0$

also, $\Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} \vec{e}_{\bar{\lambda}} := \vec{e}_{\bar{\nu},\bar{\mu}} = \partial_{\bar{\mu}} \vec{e}_{\bar{\nu}}$

so $\nabla_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = 0$

so $\widetilde{\nabla}\mathbf{g} = \mathbf{0} = \widetilde{\nabla}\mathbf{g}^{-1}$ on tangent space

$\ldots \widetilde{\nabla}\mathbf{g} = \mathbf{0} = \widetilde{\nabla}\mathbf{g}^{-1}$ on M

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

from above:

$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$

in the tangent space at $\mathbf{x}$, $\exists$ coordinate basis $\vec{e}^{\bar{\mu}}$ with

$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(-1, 1, 1, 1) = g^{\bar{\mu}\bar{\nu}}$

$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}} \;\; = 0$

also, $\Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} \vec{e}_{\bar{\lambda}} := \vec{e}_{\bar{\nu},\bar{\mu}} = \partial_{\bar{\mu}} \vec{e}_{\bar{\nu}}$

so $\nabla_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = 0$

so $\widetilde{\nabla}\mathbf{g} = \mathbf{0} = \widetilde{\nabla}\mathbf{g}^{-1}$ on tangent space

$\ldots \widetilde{\nabla}\mathbf{g} = \mathbf{0} = \widetilde{\nabla}\mathbf{g}^{-1}$ on M

$\ldots \Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$ in any coord. basis (symmetric defn)

GR: smooth manifold and $\widetilde{\nabla}\mathbf{g}$

from above:

$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa}$

in the tangent space at $\mathbf{x}$, $\exists$ coordinate basis $\vec{e}^{\bar{\mu}}$ with

$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(-1,1,1,1) = g^{\bar{\mu}\bar{\nu}}$

$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}} \; = 0$

also, $\Gamma^{\bar{\lambda}}{}_{\bar{\nu}\bar{\mu}} \vec{e}_{\bar{\lambda}} := \vec{e}_{\bar{\nu},\bar{\mu}} = \partial_{\bar{\mu}} \vec{e}_{\bar{\nu}}$

so $\nabla_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = 0$

so $\widetilde{\nabla}\mathbf{g} = \mathbf{0} = \widetilde{\nabla}\mathbf{g}^{-1}$ on tangent space

$\ldots \widetilde{\nabla}\mathbf{g} = \mathbf{0} = \widetilde{\nabla}\mathbf{g}^{-1}$ on M

$\ldots \Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$ in any coord. basis (symmetric defn)

$\ldots$

$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} (\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})$ in a coordinate basis

GR:

w:Einstein field equations

GR:

w:Einstein field equations

GR:

w:Einstein field equations

GR:

w:Einstein field equations

GR:

w:Einstein field equations

GR:

w:Einstein field equations

GR:

w:Einstein field equations

GR:

w:Einstein field equations

GR:

w:Equivalence principle

can be thought of as a *consequence* of the model

GR:

w:Equivalence principle

can be thought of as a *consequence* of the model

GR:

w:Equivalence principle

can be thought of as a *consequence* of the model

GR:

w:Equivalence principle

can be thought of as a *consequence* of the model

GR:

w:Equivalence principle

can be thought of as a *consequence* of the model

GR:

w:Equivalence principle

can be thought of as a *consequence* of the model

GR:

w:Equivalence principle

can be thought of as a *consequence* of the model

GR:

w:Equivalence principle

can be thought of as a *consequence* of the model

GR:

w:Schwarzschild metric

GR:

w:Schwarzschild metric

GR:

w:Schwarzschild metric

GR:

w:Schwarzschild metric

GR:

w:Schwarzschild metric

GR:

w:Schwarzschild metric

GR:

w:Schwarzschild metric

GR:

w:Schwarzschild metric

GR:

w:Friedmann-Lemaître-Robertson-Walker metric

GR:

w:Friedmann-Lemaître-Robertson-Walker metric

GR:

w:Friedmann-Lemaître-Robertson-Walker metric

GR:

w:Friedmann-Lemaître-Robertson-Walker metric

GR:

w:Friedmann-Lemaître-Robertson-Walker metric

GR:

w:Friedmann-Lemaître-Robertson-Walker metric

GR:

w:Friedmann-Lemaître-Robertson-Walker metric

GR:

w:Friedmann-Lemaître-Robertson-Walker metric

GR: maxima

maxima - component tensor packet ctensor; itensor

GR: maxima

maxima - component tensor packet ctensor; itensor

GR: maxima

maxima - component tensor packet ctensor; itensor

GR: maxima

maxima - component tensor packet ctensor; itensor

GR: maxima

maxima - component tensor packet ctensor; itensor

GR: maxima

maxima - component tensor packet ctensor; itensor

GR: maxima

maxima - component tensor packet ctensor; itensor

GR: maxima

maxima - component tensor packet ctensor; itensor

GR: an approximation method: ADM

+ w:ADM formalism

GR: an approximation method: ADM

+ w:ADM formalism

GR: an approximation method: ADM

+ w:ADM formalism

GR: an approximation method: ADM

+ w:ADM formalism

GR: an approximation method: ADM

+ w:ADM formalism

GR: an approximation method: ADM

+ w:ADM formalism

GR: an approximation method: ADM

+ w:ADM formalism

GR: an approximation method: ADM

+ w:ADM formalism

GR: a numerical method: cactus

+ Cactus - http://cactuscode.org

GR: a numerical method: cactus

+ Cactus - http://cactuscode.org

GR: a numerical method: cactus

+ Cactus - http://cactuscode.org

GR: a numerical method: cactus

+ Cactus - http://cactuscode.org

GR: a numerical method: cactus

+ Cactus - http://cactuscode.org

GR: a numerical method: cactus

+ Cactus - http://cactuscode.org

GR: a numerical method: cactus

+ Cactus - http://cactuscode.org

GR: a numerical method: cactus

+ Cactus - http://cactuscode.org