



Special and General Relativity

Boud Roukema

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GR: intro

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= vector space (e.g. 4-momentum vectors)

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= dual vector space (think: contour map, gradients)

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duality in a basis of $T_x M$ and a basis of $T_x^* M$ usually defined using δ^μ_ν

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2+3. vector–one-form duality in a basis: δ^μ_ν

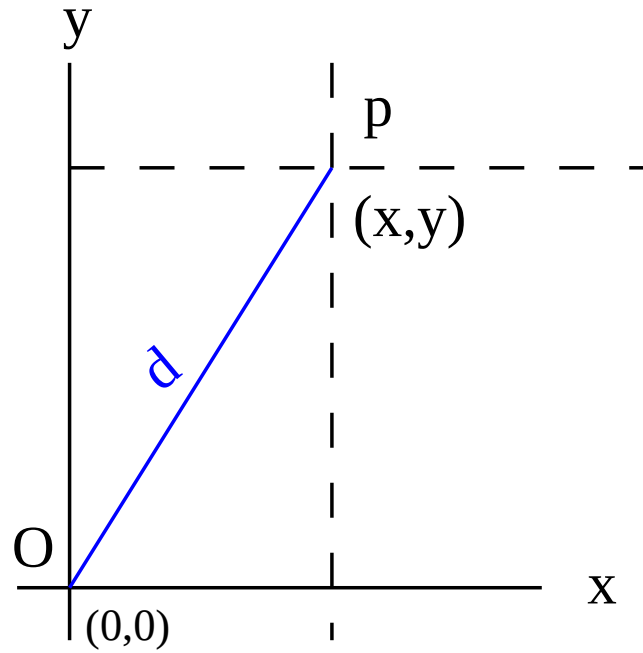
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4. w:Levi-Civita connection $\Leftarrow$ metric

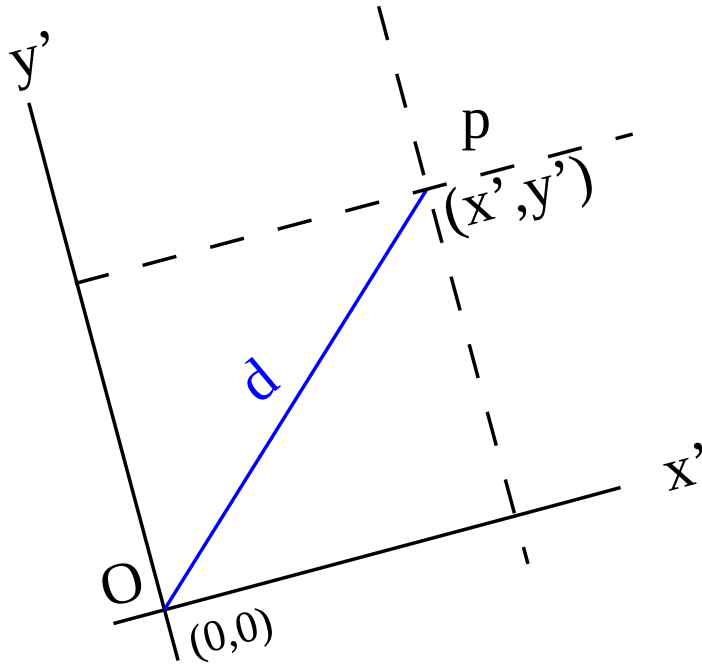
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5. metric $\Leftarrow$ Einstein field equations

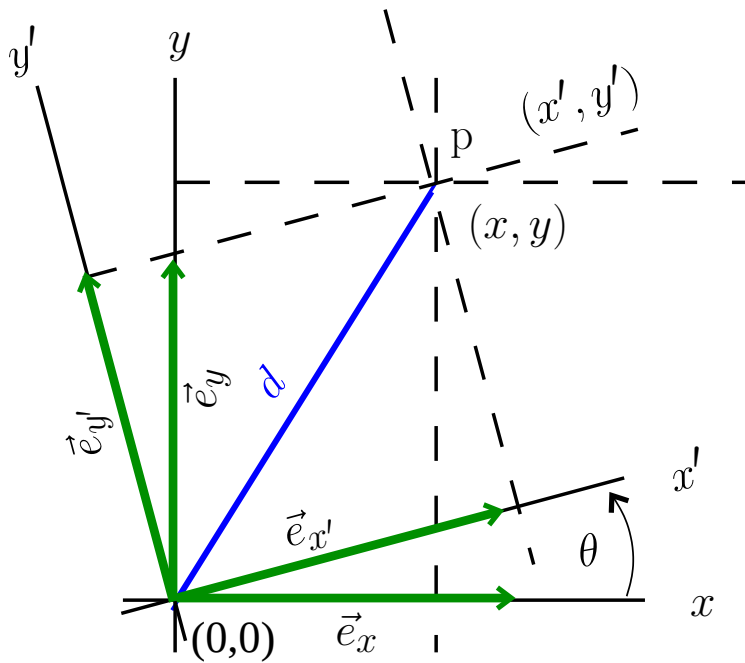
GR: coordinate transformations



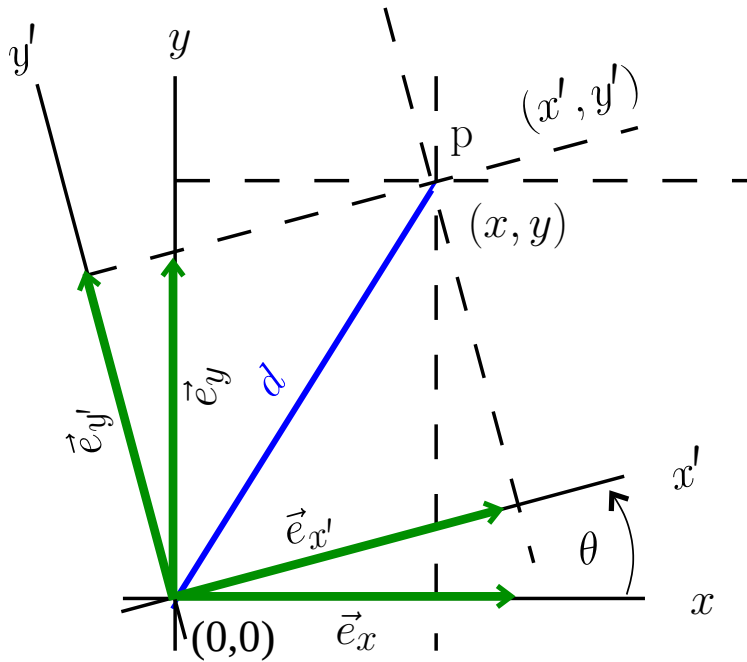
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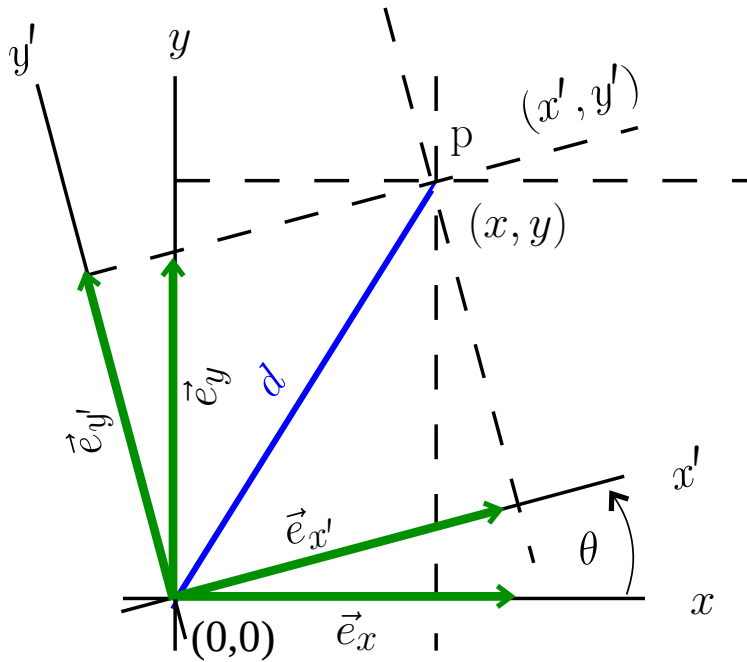
GR: coordinate transformations



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



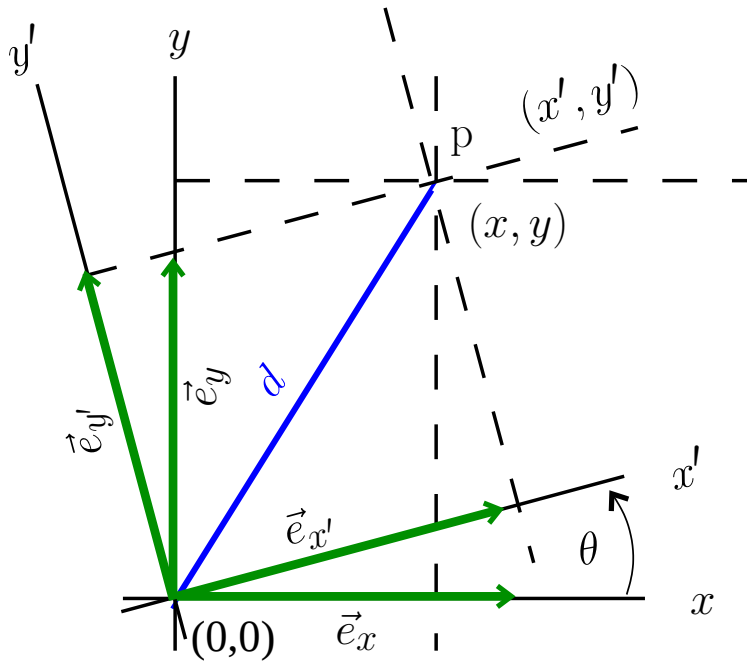
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$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$



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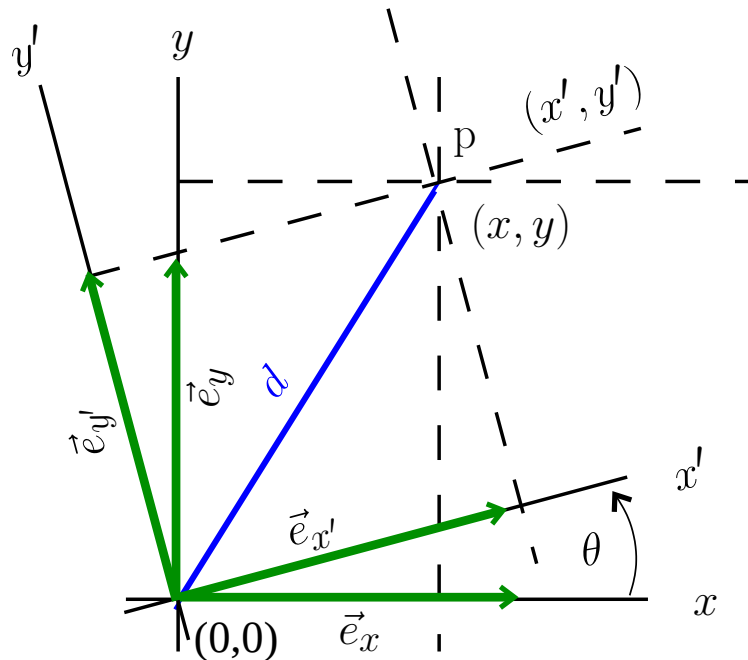


but

$$\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



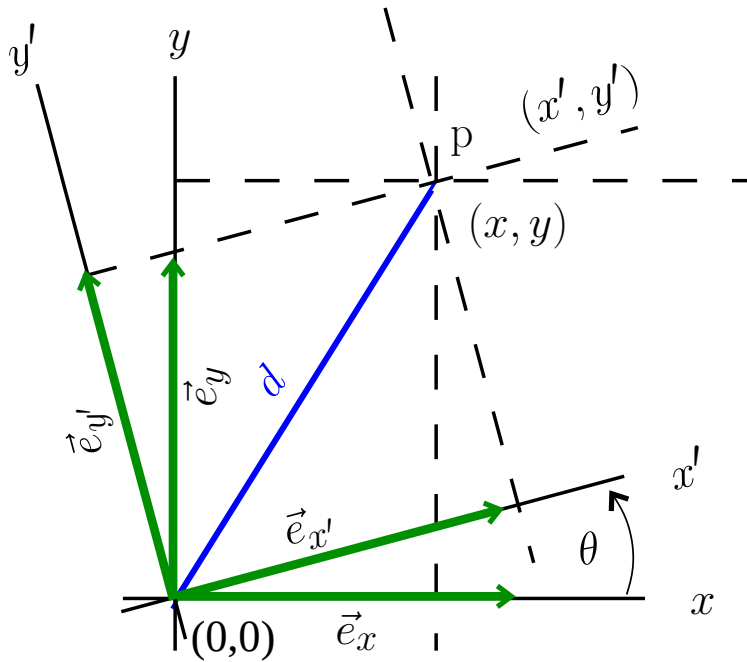
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$$\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



GR: coordinate transformations

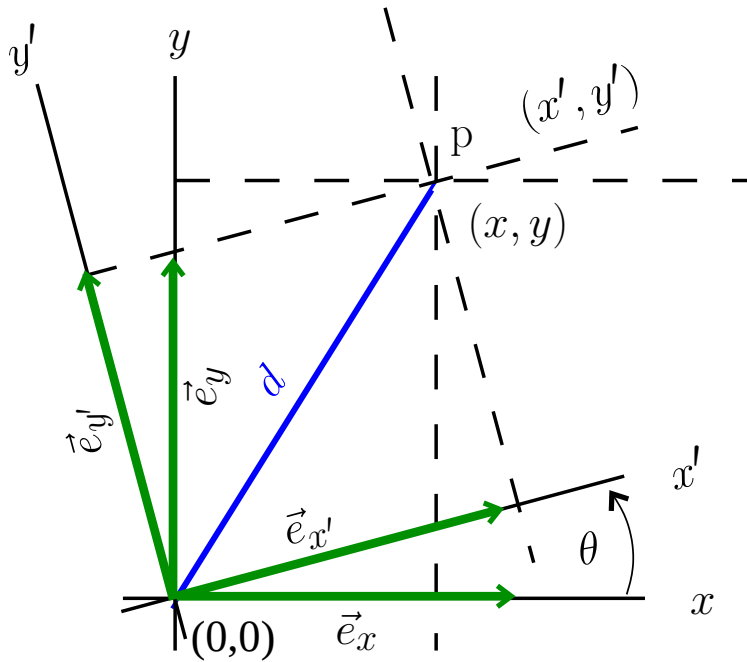


$$\vec{e}_{x'} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \vec{e}_x + \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \vec{e}_y$$

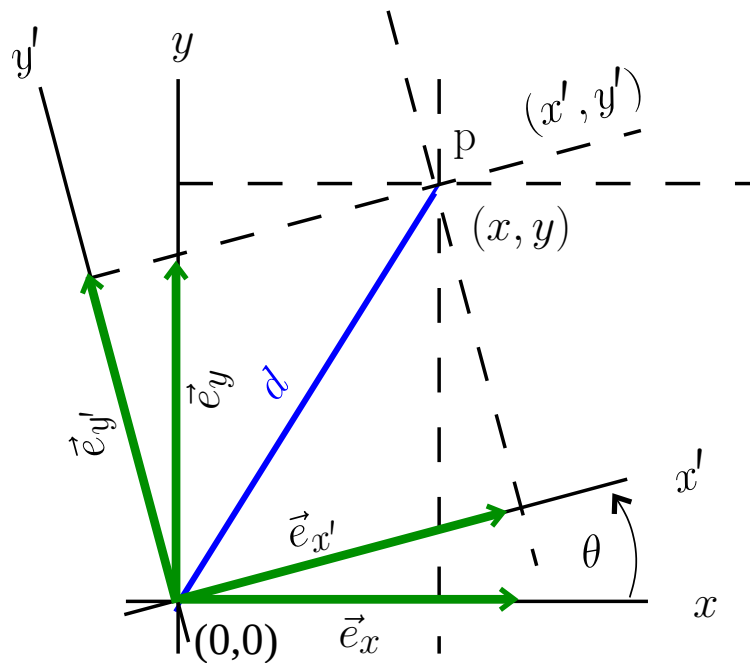


GR: coordinate transformations

$$\vec{e}_{x'} = \Lambda_{x'}^x \vec{e}_x + \Lambda_{x'}^y \vec{e}_y$$



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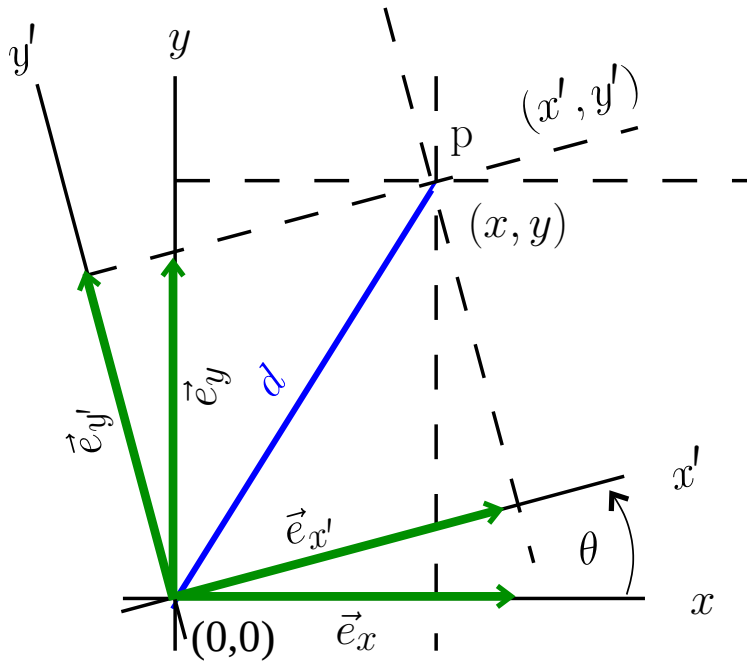


also:

$$\begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



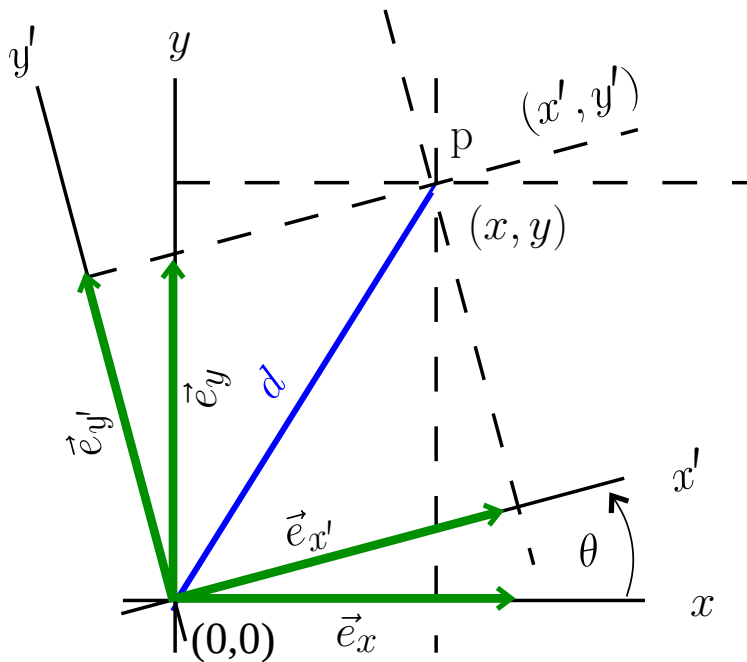
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$$\vec{e}_{y'} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \vec{e}_x + \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \vec{e}_y$$



GR: coordinate transformations



summary:

$$\vec{e}_{x'} = \Lambda_{x'}^x \vec{e}_x + \Lambda_{x'}^y \vec{e}_y,$$

$$\vec{e}_{y'} = \Lambda_{y'}^x \vec{e}_x + \Lambda_{y'}^y \vec{e}_y,$$

where $\Lambda_{\beta'}^{\alpha} :=$ element

of inverse of $\Lambda_{\beta}^{\alpha'}$,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$



GR: coordinate transformations

$$\begin{aligned}\vec{e}_{x'} &= \Lambda_{x'}^x \vec{e}_x + \Lambda_{x'}^y \vec{e}_y, \\ \vec{e}_{y'} &= \Lambda_{y'}^x \vec{e}_x + \Lambda_{y'}^y \vec{e}_y,\end{aligned}\quad \vec{p} \rightarrow_{\mathcal{O}'} \begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

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$$\vec{p} = \sum_i p^i \vec{e}_i$$

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Einstein summation:

- coordinates like r, θ, x, y :

not a sum: $\Lambda_{y'}^x \vec{e}_x$

- repeated up-down coordinate indices like $i, j \in \{0, 1, 2\}$
or $\alpha, \beta, \gamma, \lambda, \mu, \nu \in \{0, 1, 2, 3\}$:

sum: $\Lambda_{j'}^i \vec{e}_i := \Lambda_{y'}^x \vec{e}_x + \Lambda_{y'}^y \vec{e}_y$ for a 2D manifold, coords x, y

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new basis vectors = sum of inverse Λ $\times$ **old** vectors

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new basis vectors = sum of inverse $\Lambda \times$ **old** vectors

new coords of vector $\vec{p} = \Lambda \times$ old coords of **same** vector $\vec{p}$

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vector invariance requires contravariance of its coords

“contra” = inverse of change of basis vectors

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new basis vectors = sum of inverse $\Lambda \times$ **old** vectors

vector invariance requires contravariance of its coords

“contra” = inverse of change of basis vectors

- $\vec{p}$ is invariant: no dependence on coords
- $\vec{p}$ is contravariant: p^i change inversely to $\vec{e}_i$

GR: coord. transf.: 1-forms

ϕ = scalar field = $\phi(x, y) \equiv \phi(x', y')$

write $\phi_{,x} := \frac{\partial \phi}{\partial x} =: (\tilde{d}\phi)_x$

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What is the relation between $(\phi_{,x'}, \phi_{,y'})$
and $(\phi_{,x}, \phi_{,y})$?

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ϕ depends either on x and y , or on x' and y'

$$\Rightarrow \frac{\partial \phi}{\partial x'} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial x'}$$

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$$(\phi_{,x'}, \phi_{,y'}) =$$

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$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \quad (\text{example: rotation})$$

$$x_{,x'} = \frac{\partial x}{\partial x'} = \cos \theta$$

$$x_{,y'} = \frac{\partial x}{\partial y'} = -\sin \theta \dots$$

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$$\tilde{d}\phi = \left((\tilde{d}\phi)_{x'}, (\tilde{d}\phi)_{y'} \right) = \left((\tilde{d}\phi)_x, (\tilde{d}\phi)_y \right) \Lambda^{-1}$$

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$$\begin{pmatrix} x \\ y \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} \quad (\text{general})$$

$$\tilde{d}\phi = \left((\tilde{d}\phi)_{x'}, (\tilde{d}\phi)_{y'} \right) = \left((\tilde{d}\phi)_x, (\tilde{d}\phi)_y \right) \Lambda^{-1}$$

$$(\tilde{d}\phi)_{\mu'} = (\tilde{d}\phi)_{\nu} \Lambda^{\nu}_{\mu'}$$

GR: coord. transf.: 1-forms

basis vectors of different bases: $\vec{e}_{\mu'} = \Lambda^{\nu}_{\mu'} \vec{e}_{\nu}$

same vector: $(\vec{p})^{\mu'} = \Lambda^{\mu'}_{\nu} (\vec{p})^{\nu}$

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basis vectors of different bases: $\vec{e}_{\mu'} = \Lambda_{\mu'}^{\nu} \vec{e}_{\nu}$

same vector: $p^{\mu'} = \Lambda^{\mu'}_{\nu} p^{\nu}$

same gradient (example 1-form): $(\tilde{d}\phi)_{\mu'} = (\tilde{d}\phi)_{\nu} \Lambda_{\mu'}^{\nu}$

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- vector $\vec{p}$ is **in**variant: no dependence on coords
- $\vec{p}$ is **contra**variant: components p^{ν} change inversely to how $\vec{e}_{\mu}$ change; inverses: matrix $\{\Lambda_{\mu'}^{\nu}\}$ vs $\{\Lambda^{\beta'}_{\alpha}\}$

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same gradient (example 1-form): $(\tilde{d}\phi)_{\mu'} = (\tilde{d}\phi)_{\nu} \Lambda^{\nu}_{\mu'}$

- vector $\vec{p}$ is **in**variant: no dependence on coords
- $\vec{p}$ is **contra**variant: components p^{ν} change inversely to

how $\vec{e}_{\mu}$ change; inverses: matrix $\{\Lambda^{\nu}_{\mu'}\}$ vs $\{\Lambda^{\beta'}_{\alpha}\}$

- 1-form $\tilde{d}\phi$ is **in**variant: no dependence on coords
- $\tilde{d}\phi$ is **cov**ariant: components $(\tilde{d}\phi)_{\mu}$ change like $\vec{e}_{\mu}$ (but left-multiply)

GR: coord. transf.: 1-forms

basis vectors of different bases: $\vec{e}_{\mu'} = \Lambda^{\nu}_{\mu'} \vec{e}_{\nu}$

same vector: $p^{\mu'} = \Lambda^{\mu'}_{\nu} p^{\nu}$

same gradient (example 1-form): $(\tilde{d}\phi)_{\mu'} = (\tilde{d}\phi)_{\nu} \Lambda^{\nu}_{\mu'}$

- vector $\vec{p}$ is **in**variant: no dependence on coords
- $\vec{p}$ is **contra**variant: components p^{ν} change inversely to

how $\vec{e}_{\mu}$ change; inverses: matrix $\{\Lambda^{\nu}_{\mu'}\}$ vs $\{\Lambda^{\beta'}_{\alpha}\}$

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w:Covariance and contravariance of vectors







$$\mathbf{GR}: \vec{p}, \tilde{q}, \langle \vec{p}, \tilde{q} \rangle, \mathbf{g}$$

GR tensors: two different scalar products
vector–1-form duality requirement:





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can be called I with components δ^μ_ν in a coordinate basis

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think: vector $\rightarrow$ column vector

1-form $\rightarrow$ row vector

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$\langle , \rangle = (1,1)$ -tensor = “row-column” matrix I with $I^\mu_\nu = \delta^\mu_\nu$

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ordinary linear algebra: column vectors, row vectors, matrices

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(m, n) -tensor algebra: m column n row $m + n$ -arrays

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e.g.: $(0, 2)$ -tensor: metric $g_{\mu\nu}$

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using $\langle, \rangle$, $(1, 0)$ -tensor = vector = function of 1-forms

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loosely speaking, the second $\otimes$ means “function of two vectors” (or 1-forms, or a vector and a 1-form) in *that particular left-to-right order*

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warning: the "rank" of tensors has two different meanings: w:Tensor_(intrinsic_definition)#Tensor_rank

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dimension of $V^* \otimes V^* = 16$ (for V = spacetime)

GR: g

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also written: $\vec{A} \cdot \vec{B}$ “dot product”

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in general, for a 2-form $\mathbf{T}$, $\mathbf{T}(\vec{A}, \vec{B}) \neq \mathbf{T}(\vec{B}, \vec{A})$

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g can be applied to basis vectors $\vec{e}_\mu$

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we can define components (used earlier): $g_{\mu\nu} := \mathbf{g}(\vec{e}_\mu, \vec{e}_\nu)$

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$$\text{e.g. } \mathbf{g} = g_{rr} \tilde{e}^r \otimes \tilde{e}^r + g_{r\theta} \tilde{e}^r \otimes \tilde{e}^\theta + g_{\theta r} \tilde{e}^\theta \otimes \tilde{e}^r + g_{\theta\theta} \tilde{e}^\theta \otimes \tilde{e}^\theta$$

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check: $\mathbf{g}(\vec{e}_r, \vec{e}_r) = g_{rr}$?

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GR: what is a coordinate?

a coordinate, e.g. x^0 or x^1 is a scalar field on the 4-manifold

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a coordinate system x^μ = set of four scalar fields on the 4-manifold

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(Bertschinger writes $x^\mu_{\mathbf{x}}$ to show dependence on position $\mathbf{x}$ in manifold $\neq$ vector space)

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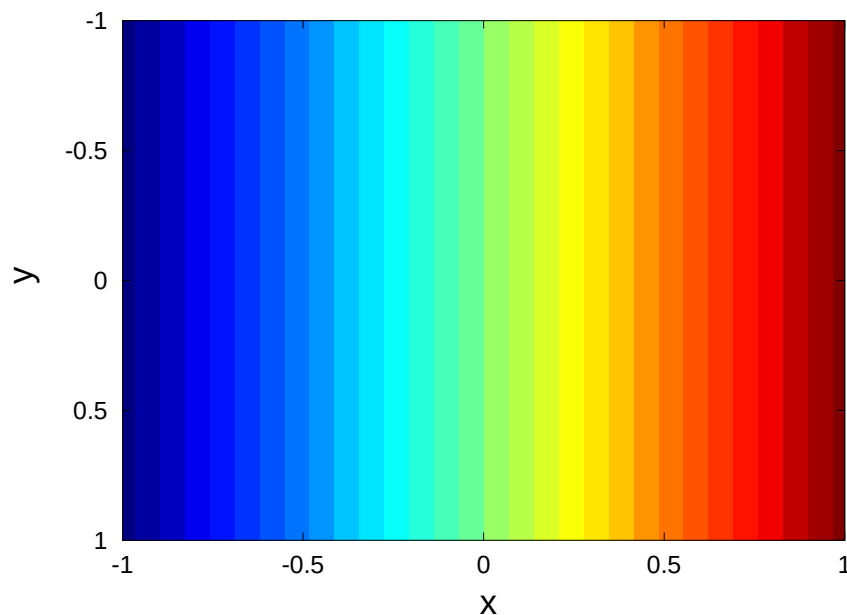
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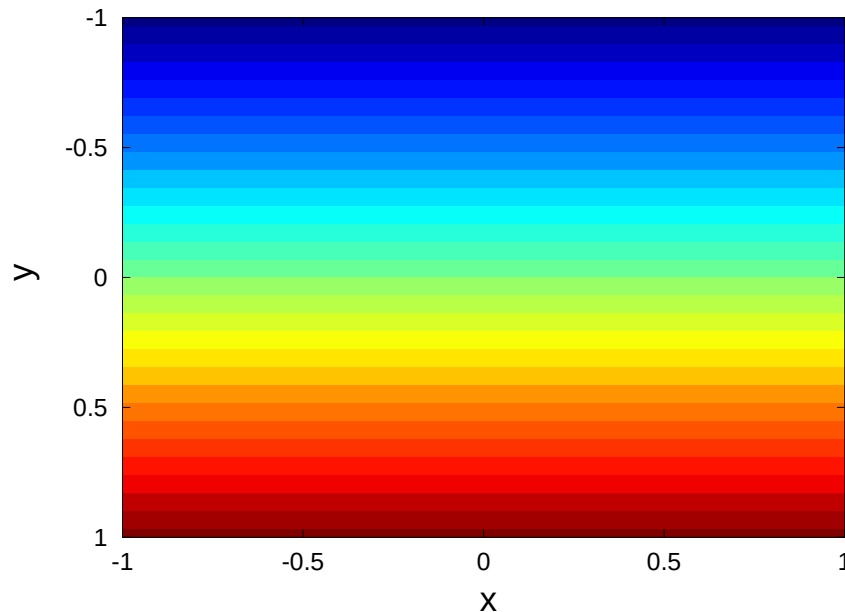


e.g. on $\mathbb{R}^2$

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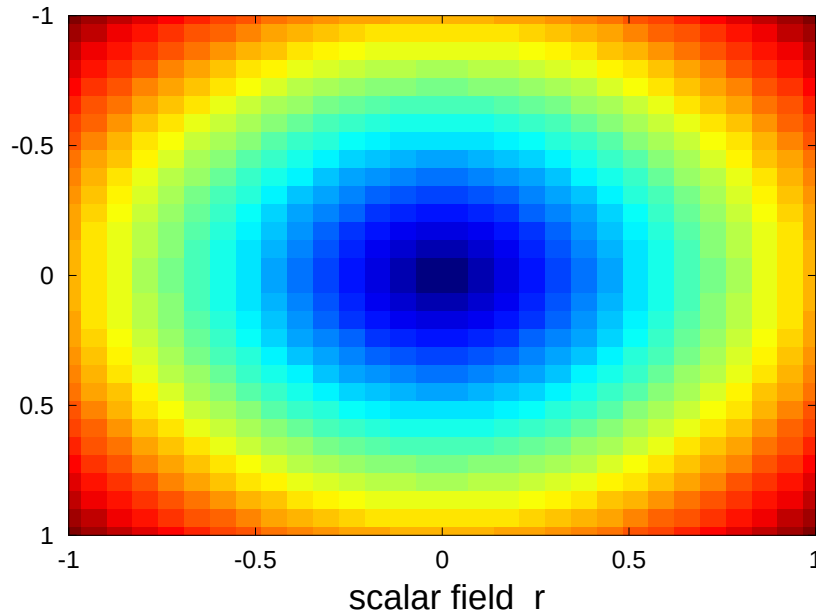


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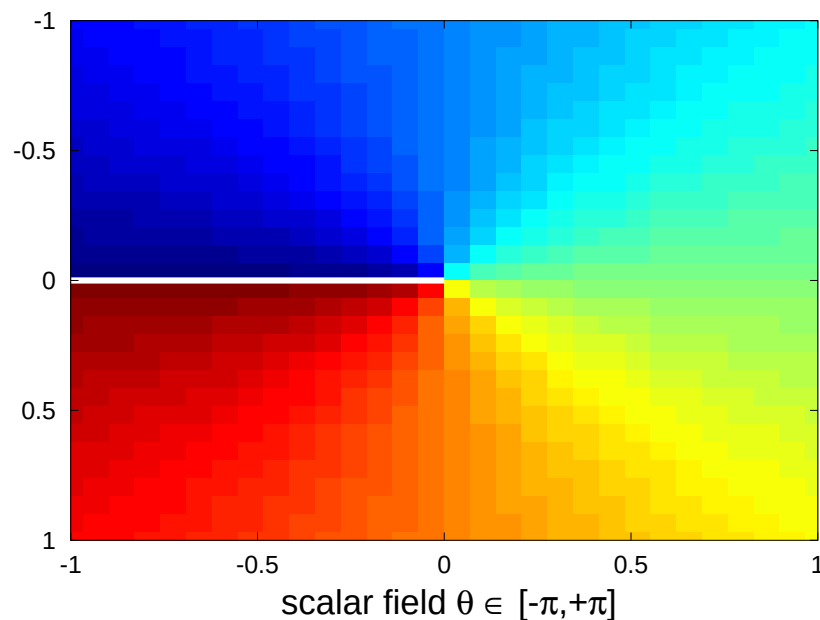


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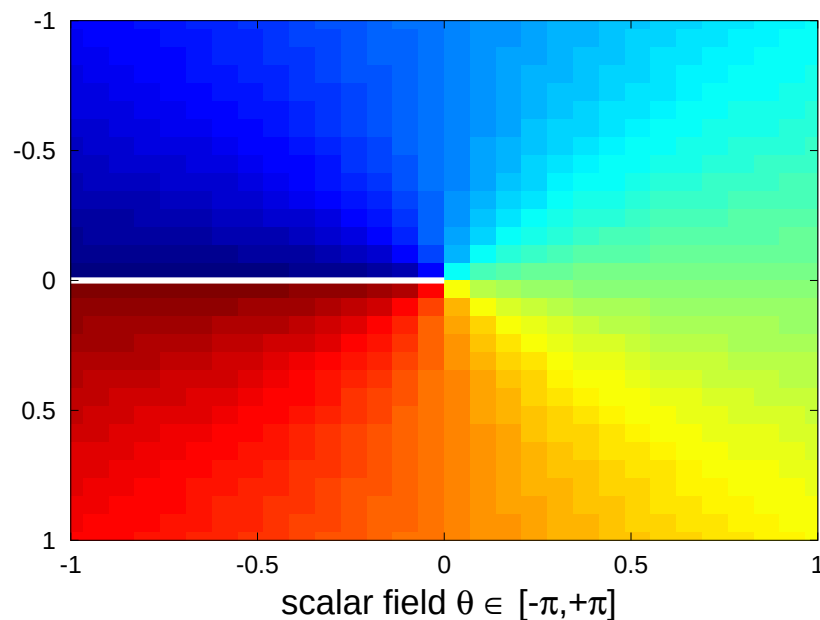


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coordinate singularity $\neq$ singularity in manifold

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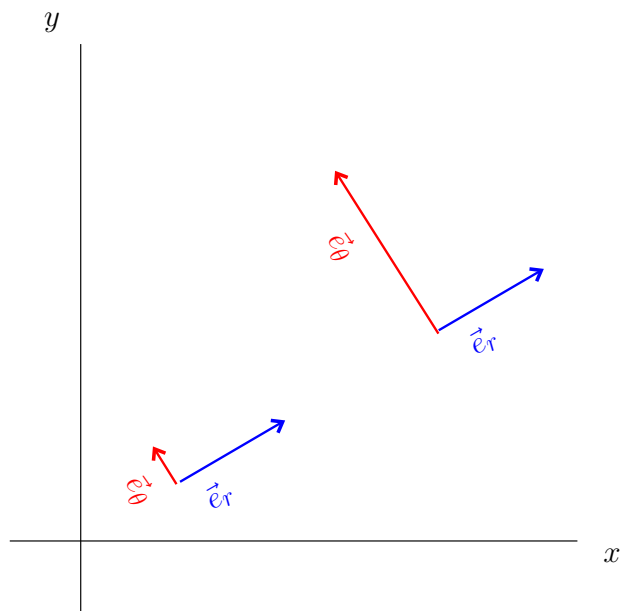
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$g_{r\theta}$ and g_{xy}

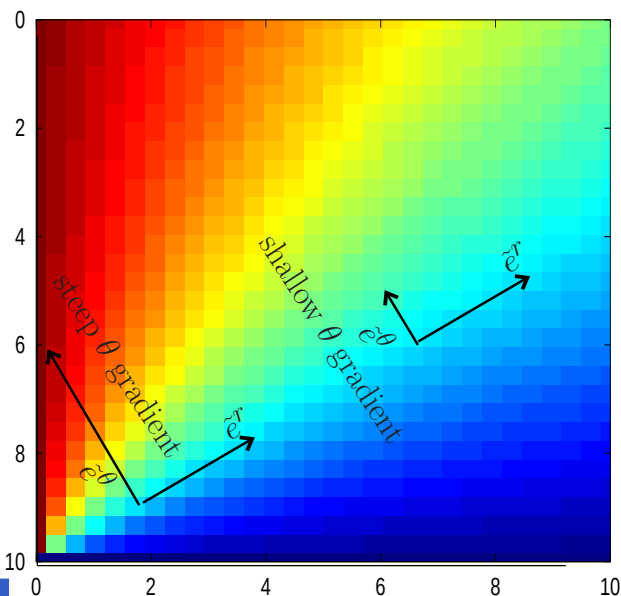
$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$

$$\vec{e}_x \cdot \vec{e}_x = 1 = \vec{e}_y \cdot \vec{e}_y, \text{ others zero}$$

$$\vec{e}_r \cdot \vec{e}_r = 1, \vec{e}_\theta \cdot \vec{e}_\theta = r^2 \neq 1$$

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \Rightarrow g^{xx} = 1 = g^{yy}, g^{xy} = 0 = g^{yx}$$

$$\text{but } g^{rr} = 1 \neq g^{\theta\theta} = r^{-2}, g^{r\theta} = 0 = g^{\theta r}$$



GR: e.g. Euclidean g on $\mathbb{R}^2$

$g_{r\theta}$ and g_{xy}

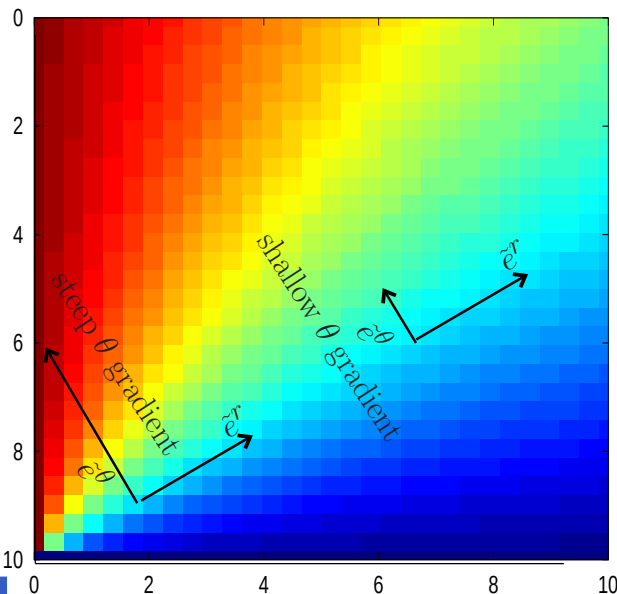
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$$\text{so } \tilde{e}^r \cdot \tilde{e}^r = 1, \tilde{e}^\theta \cdot \tilde{e}^\theta = r^{-2} \neq 1$$

GR: gradient of a vector: $\nabla \vec{A}$

gradient of scalar field: $\tilde{d}\phi \equiv \tilde{\nabla}\phi$

GR: gradient of a vector: $\nabla \vec{A}$

what is gradient of vector field $\tilde{\nabla} \vec{A}$?

GR: gradient of a vector: $\nabla \vec{A}$

$$\tilde{\nabla} \vec{A} = \tilde{\nabla}(A^\nu \vec{e}_\nu)$$

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$$\begin{aligned}\tilde{\nabla} \vec{A} &= \tilde{\nabla}(A^\nu \vec{e}_\nu) \\ &= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)\end{aligned}$$

GR: gradient of a vector: $\nabla \vec{A}$

$$\begin{aligned}\tilde{\nabla} \vec{A} &= \tilde{\nabla}(A^\nu \vec{e}_\nu) \\ &= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu) \\ &= \tilde{e}^\mu \otimes [(\partial_\mu A^\nu) \vec{e}_\nu + A^\nu \partial_\mu \vec{e}_\nu] \text{ by product rule and linearity}\end{aligned}$$

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$$= \tilde{e}^\mu \partial_\mu (A^\nu \vec{e}_\nu)$$

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give a name to the second part: it must be a linear combination of basis vectors $\vec{e}_\lambda$

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define $\Gamma_{\nu\mu}^\lambda \vec{e}_\lambda := \partial_\mu \vec{e}_\nu$ Christoffel symbols of second kind
(symmetric defn)

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$$= \partial_\mu A^\nu \tilde{e}^\mu \otimes \vec{e}_\nu + A^\nu \Gamma_{\nu\mu}^\lambda \tilde{e}^\mu \otimes \vec{e}_\lambda \text{ since any } \Gamma_{\nu\mu}^\lambda \text{ is a scalar}$$

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since name of summation index is arbitrary, e.g.

$$\sum_\lambda x^{-2\lambda} = \sum_\mu x^{-2\mu} = \sum_\nu x^{-2\nu}$$

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$$\nabla_\mu A^\nu := A^\nu_{;\mu} := \partial_\mu A^\nu + A^\lambda \Gamma_{\lambda\mu}^\nu$$

w:covariant derivative of vector

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mathematically deeper: $\tilde{\nabla}$, usually written just as ∇ , is the w:Levi-Civita connection

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and $\tilde{\nabla}$ on a $(1, 0)$ -tensor field = vector field $\vec{A}$ gives a $(1, 1)$ -tensor with components $\nabla_\mu A^\nu = \partial_\mu A^\nu + A^\lambda \Gamma^\nu_{\lambda\mu}$

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GR: gradient of one-form $\tilde{\nabla} \tilde{A}$

how does a one-form change with position? $\tilde{\nabla} \tilde{A} = ?$

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evaluating $\tilde{\nabla} \tilde{A}$ as we did $\tilde{\nabla} \vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu_{\lambda\mu} \tilde{e}^\lambda$ for some coefficients $F^\nu_{\lambda\mu}$

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relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu_\lambda$

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$\partial_\mu \delta^\nu_\lambda = 0$ (obviously)

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can we use the product rule with this scalar product?

$$\partial_\mu (\langle \tilde{A}, \vec{B} \rangle) = ?$$

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$$\partial_\mu \left(\langle \tilde{A}, \vec{B} \rangle \right) = \partial_\mu (A_\nu B^\nu) \text{ in some coordinate basis}$$

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$$= (\partial_\mu A_\nu) B^\nu + A_\nu (\partial_\mu B^\nu) \text{ by product rule on functions}$$

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$$= \langle \partial_\mu \tilde{A}, \vec{B} \rangle + \langle \tilde{A}, \partial_\mu \vec{B} \rangle \text{ since}$$

$$\partial_\mu \tilde{A} = (\partial_\mu A_0, \partial_\mu A_1, \partial_\mu A_2, \partial_\mu A_3)$$

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$$= \langle F^\nu_{\kappa\mu} \tilde{e}^\kappa, \vec{e}_\lambda \rangle + \langle \tilde{e}^\nu, \Gamma^\kappa_{\lambda\mu} \vec{e}_\kappa \rangle$$

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$$= F^\nu_{\kappa\mu} \langle \tilde{e}^\kappa, \vec{e}_\lambda \rangle + \Gamma^\kappa_{\lambda\mu} \langle \tilde{e}^\nu, \vec{e}_\kappa \rangle$$

GR: gradient of one-form $\tilde{\nabla} \tilde{A}$

evaluating $\tilde{\nabla} \tilde{A}$ as we did $\tilde{\nabla} \vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu_{\lambda\mu} \tilde{e}^\lambda$ for some coefficients $F^\nu_{\lambda\mu}$

how can we relate $\Gamma^\nu_{\lambda\mu}$ to $F^\nu_{\lambda\mu}$?

relation between vectors and one-forms: $\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle = \delta^\nu_\lambda$

$$0 = \partial_\mu \delta^\nu_\lambda = \partial_\mu (\langle \tilde{e}^\nu, \vec{e}_\lambda \rangle)$$

$$\text{product rule holds: } \partial_\mu (\langle \tilde{A}, \vec{B} \rangle) = \langle \partial_\mu \tilde{A}, \vec{B} \rangle + \langle \tilde{A}, \partial_\mu \vec{B} \rangle$$

$$\text{so } 0 = \langle \partial_\mu \tilde{e}^\nu, \vec{e}_\lambda \rangle + \langle \tilde{e}^\nu, \partial_\mu \vec{e}_\lambda \rangle$$

$$= F^\nu_{\lambda\mu} + \Gamma^\nu_{\lambda\mu} \text{ since } \langle \tilde{e}^\kappa, \vec{e}_\lambda \rangle = \delta^\kappa_\lambda$$

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$$\text{hence, } \partial_\mu \tilde{e}^\nu =: F^\nu_{\lambda\mu} \tilde{e}^\lambda = -\Gamma^\nu_{\lambda\mu} \tilde{e}^\lambda$$

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$$\nabla_\mu A^\nu = \partial_\mu A^\nu + A^\lambda \Gamma^\nu_{\lambda\mu} \quad , \quad \nabla_\mu A_\nu = \partial_\mu A_\nu - A_\lambda \Gamma^\lambda_{\mu\nu}$$

GR: gradient of one-form $\tilde{\nabla} \tilde{A}$

evaluating $\tilde{\nabla} \tilde{A}$ as we did $\tilde{\nabla} \vec{A}$ shows that we again need $\partial_\mu \tilde{e}^\nu = F^\nu_{\lambda\mu} \tilde{e}^\lambda$ for some coefficients $F^\nu_{\lambda\mu}$

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$$= F^\nu_{\lambda\mu} + \Gamma^\nu_{\lambda\mu} \text{ since } \langle \tilde{e}^\kappa, \vec{e}_\lambda \rangle = \delta^\kappa_\lambda$$

$$\text{hence, } \partial_\mu \tilde{e}^\nu =: F^\nu_{\lambda\mu} \tilde{e}^\lambda = -\Gamma^\nu_{\lambda\mu} \tilde{e}^\lambda$$

$$A^\nu_{;\mu} = A^\nu_{,\mu} + A^\lambda \Gamma^\nu_{\lambda\mu} \quad , \quad A_{\nu;\mu} = A_{\nu,\mu} - A_\lambda \Gamma^\lambda_{\mu\nu}$$

GR: smooth manifold and $\tilde{\nabla}g$

similarly, we can write the $(0, 3)$ -tensor

$$\tilde{\nabla}g = (\nabla_\lambda g_{\mu\nu}) \tilde{e}^\lambda \otimes \tilde{e}^\mu \otimes \tilde{e}^\nu$$

$$\text{giving } \nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma_{\mu\lambda}^\kappa g_{\kappa\nu} - \Gamma_{\nu\lambda}^\kappa g_{\mu\kappa}$$

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$$\text{also } \tilde{\nabla}g^{-1} = (\nabla_\lambda g^{\mu\nu}) \tilde{e}^\lambda \otimes \tilde{e}_\mu \otimes \tilde{e}_\nu$$

$$\text{and } \nabla_\lambda g^{\mu\nu} = \partial_\lambda g^{\mu\nu} + \Gamma_{\kappa\lambda}^\mu g^{\kappa\nu} + \Gamma_{\kappa\lambda}^\nu g^{\mu\kappa}$$

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Do we know anything interesting about $\tilde{\nabla}g$ for the manifolds of interest to GR?

GR: smooth manifold and $\tilde{\nabla}g$

similarly, we can write the $(0, 3)$ -tensor

$$\tilde{\nabla}g = (\nabla_{\lambda}g_{\mu\nu})\tilde{e}^{\lambda} \otimes \tilde{e}^{\mu} \otimes \tilde{e}^{\nu}$$

$$\text{giving } \nabla_{\lambda}g_{\mu\nu} = \partial_{\lambda}g_{\mu\nu} - \Gamma^{\kappa}_{\mu\lambda}g_{\kappa\nu} - \Gamma^{\kappa}_{\nu\lambda}g_{\mu\kappa}$$

$$\text{also } \tilde{\nabla}g^{-1} = (\nabla_{\lambda}g^{\mu\nu})\tilde{e}^{\lambda} \otimes \tilde{e}_{\mu} \otimes \tilde{e}_{\nu}$$

$$\text{and } \nabla_{\lambda}g^{\mu\nu} = \partial_{\lambda}g^{\mu\nu} + \Gamma^{\mu}_{\kappa\lambda}g^{\kappa\nu} + \Gamma^{\nu}_{\kappa\lambda}g^{\mu\kappa}$$

Do we know anything interesting about $\tilde{\nabla}g$ for the manifolds of interest to GR?

First, we need a rough description of the manifolds we need for GR.

GR: smooth manifold and $\tilde{\nabla}g$

topological manifold M

w:Manifold#Mathematical_definition

- only topological properties needed

GR: smooth manifold and $\tilde{\nabla}g$

topological manifold M

w:Manifold#Mathematical_definition

- only topological properties needed
- no differentiability, no metric needed

GR: smooth manifold and $\tilde{\nabla}g$

topological manifold M

w:Manifold#Mathematical_definition

- only topological properties needed

next: relation with $\mathbb{R}^4$ (or M^4)

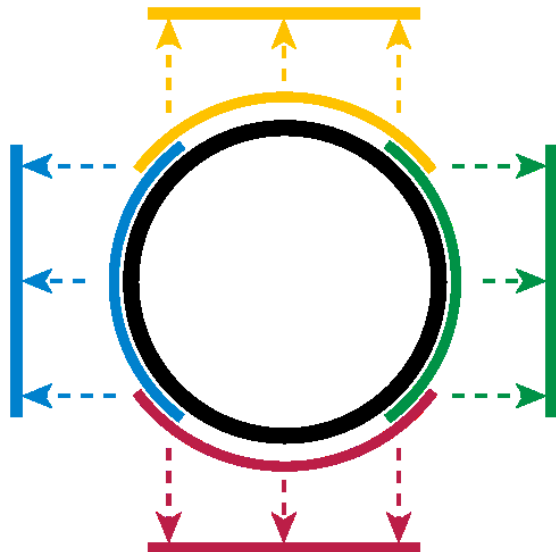
GR: smooth manifold and $\tilde{\nabla}g$

topological manifold M

w:Manifold#Mathematical_definition

- only topological properties needed

next: relation with $\mathbb{R}^4$ (or M^4)



GR: smooth manifold and $\tilde{\nabla}g$

topological manifold M

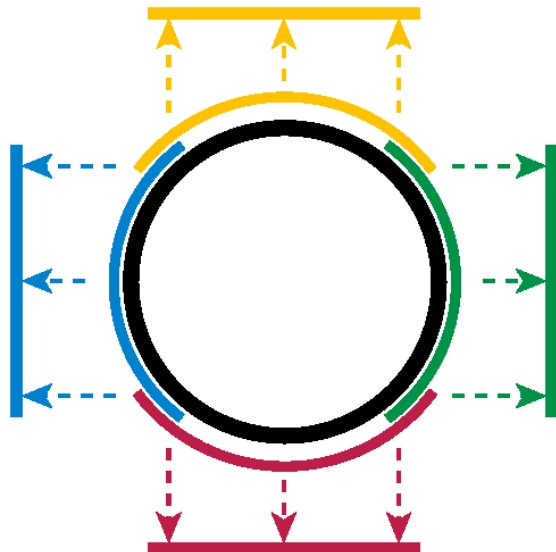
w:Manifold#Mathematical_definition

- only topological properties needed

next: relation with $\mathbb{R}^4$ (or M^4)

w:Manifold

- chart := function ϕ_α from part of pseudo-4-manifold M to part of M^4 (Minkowski)
- atlas := set of overlapping charts that cover M

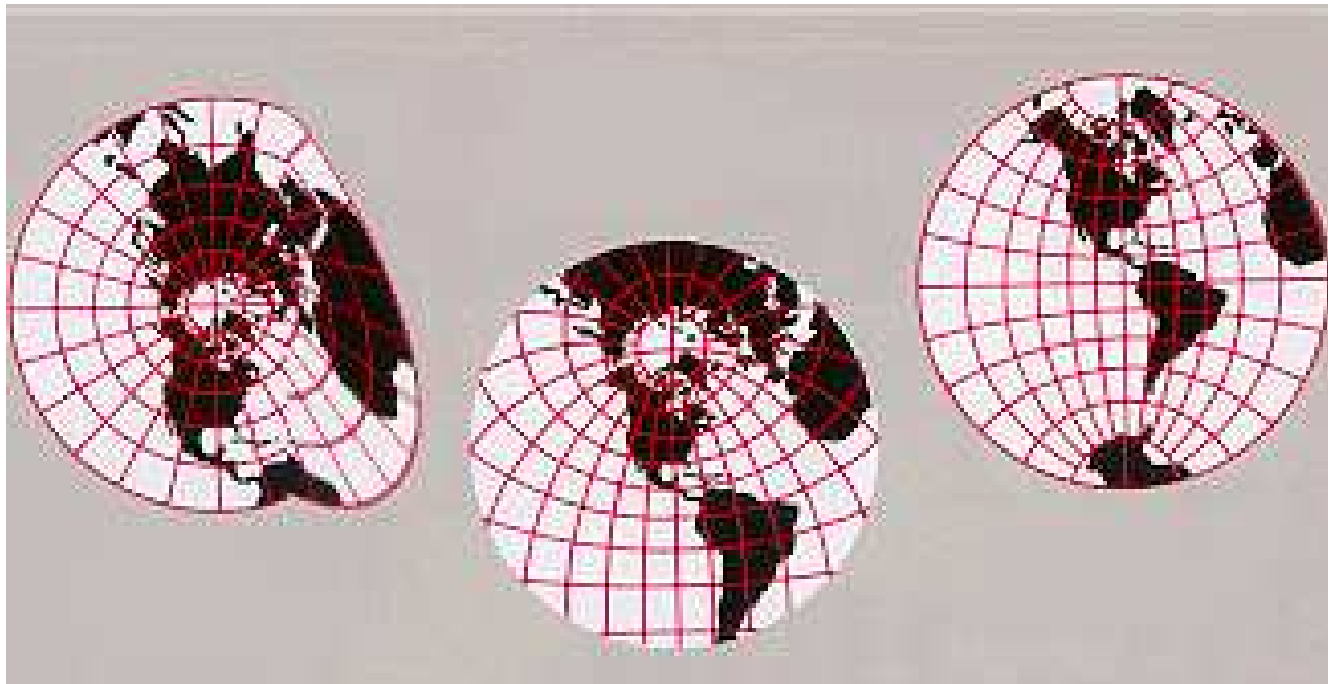


GR: smooth manifold and $\tilde{\nabla}g$

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

GR: smooth manifold and $\tilde{\nabla}g$

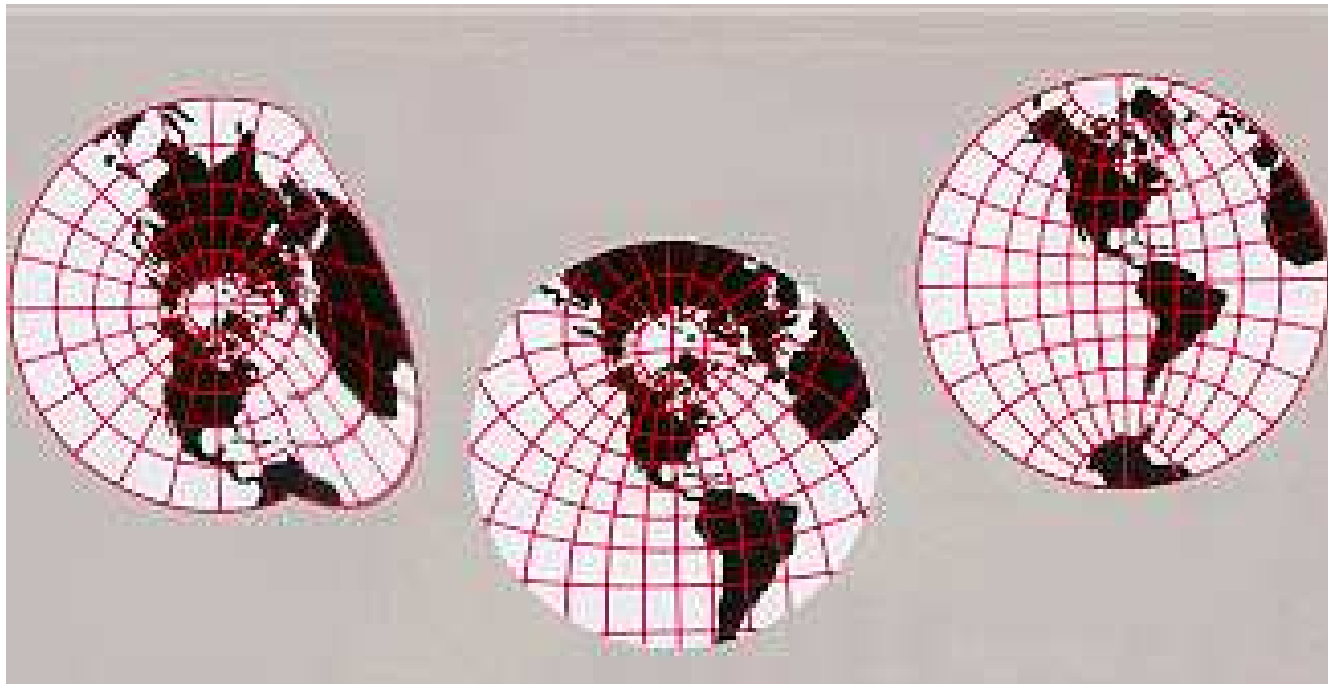
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W:

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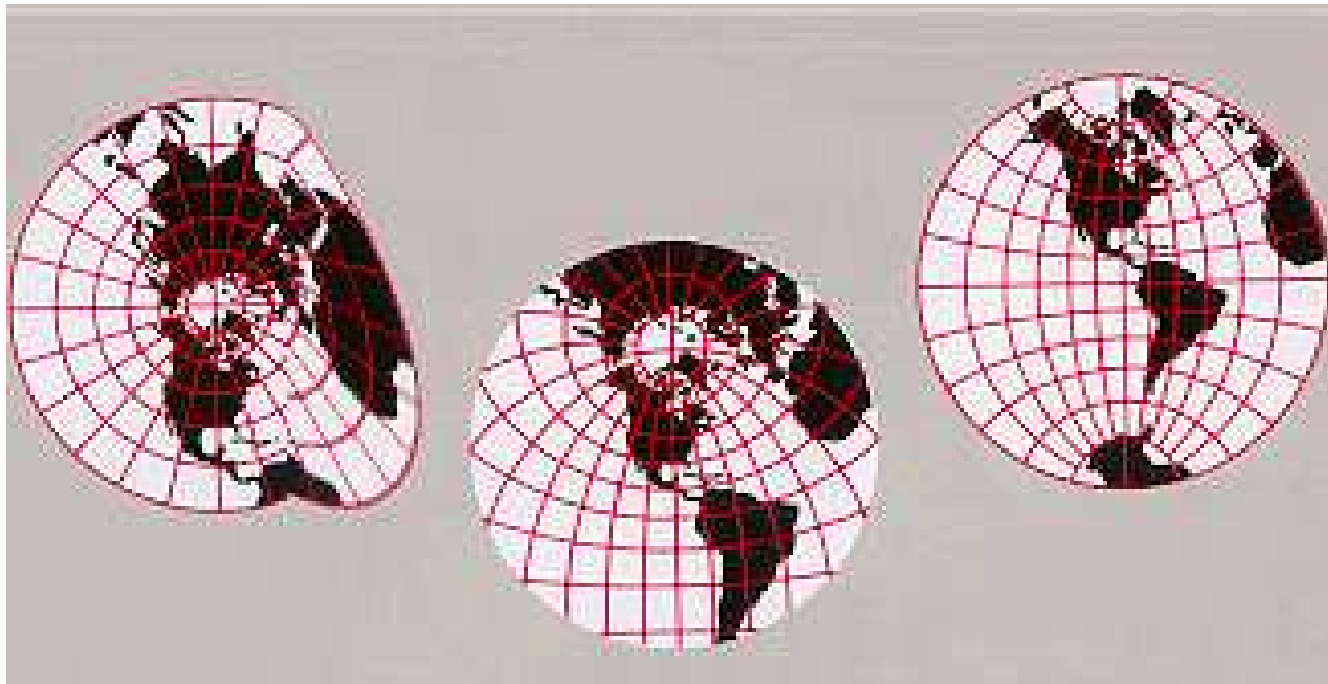


w:

projections (left-to-right) ϕ_1, ϕ_2, ϕ_3 from S^2 to $\mathbb{R}^2$

GR: smooth manifold and $\tilde{\nabla}g$

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

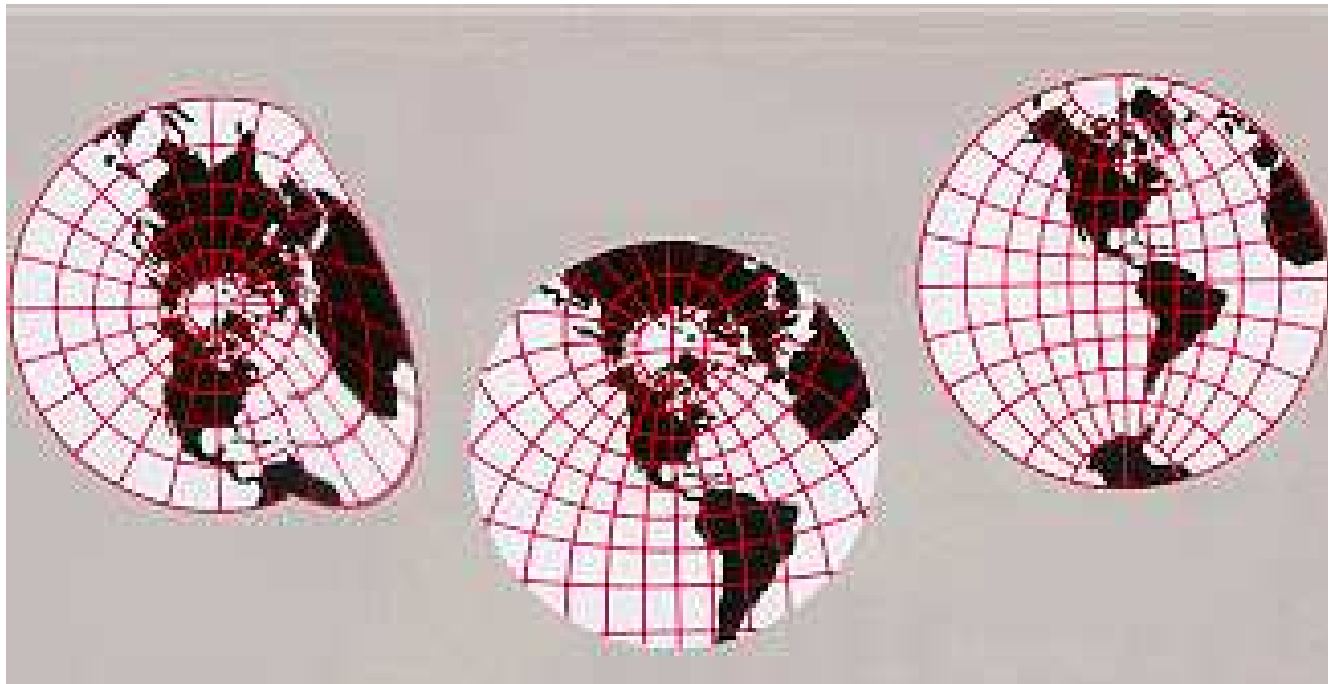


w:

ϕ_1 is not differentiable, so $\phi_1 \circ \phi_2^{-1}$ is not differentiable

GR: smooth manifold and $\tilde{\nabla}g$

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold



w:

atlas not enough to show that S^2 = differentiable 2-manifold

GR: smooth manifold and $\tilde{\nabla}g$

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

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if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

if $\forall k \geq 1, \exists k$ -th derivatives, then M is a smooth 4-(pseudo-)manifold

GR: smooth manifold and $\tilde{\nabla}g$

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

if $\forall k \geq 1, \exists k$ -th derivatives, then M is a smooth 4-(pseudo-)manifold

if a (pseudo-)w:Riemannian metric g can be added to M , then (M, g) is a (pseudo-)Riemannian 4-manifold

GR: smooth manifold and $\tilde{\nabla}g$

if every transition chart $:= \phi_\beta \circ \phi_\alpha^{-1}$ in an atlas for M is differentiable on $\mathbb{R}^4$ (or M^4), then M is a w:differentiable 4-(pseudo-)manifold

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if a (pseudo-)w:Riemannian metric g can be added to M , then (M, g) is a (pseudo-)Riemannian 4-manifold

if g has signature $(1, n-1)$ (i.e. $(-, +, +, +)$ or $(+, -, -, -)$, etc.), then (M, g) is a Lorentzian n -manifold



GR: smooth manifold and $\tilde{\nabla}g$

topological manifolds



GR: smooth manifold and $\tilde{\nabla}g$

topological manifolds

differentiable (pseudo-)manifolds

GR: smooth manifold and $\tilde{\nabla}g$



topological manifolds

differentiable (pseudo-)manifolds

smooth (pseudo-)manifolds



GR: smooth manifold and $\tilde{\nabla}g$



topological manifolds

differentiable (pseudo-)manifolds

smooth (pseudo-)manifolds

(pseudo-)Riemannian manifolds



GR: smooth manifold and $\tilde{\nabla}g$



topological manifolds

differentiable (pseudo-)manifolds

smooth (pseudo-)manifolds

(pseudo-)Riemannian manifolds

Lorentzian manifolds



GR: smooth manifold and $\tilde{\nabla}g$



topological manifolds

differentiable (pseudo-)manifolds

smooth (pseudo-)manifolds

(pseudo-)Riemannian manifolds

Lorentzian manifolds

Lorentzian 4-manifolds



GR: smooth manifold and $\tilde{\nabla}g$



topological manifolds

differentiable (pseudo-)manifolds

smooth (pseudo-)manifolds

(pseudo-)Riemannian manifolds

Lorentzian manifolds

Lorentzian 4-manifolds

GR: assume that spacetime is a Lorentzian 4-manifold



GR: smooth manifold and $\tilde{\nabla}g$

from above:

$$\nabla_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \Gamma_{\mu\lambda}^{\kappa} g_{\kappa\nu} - \Gamma_{\nu\lambda}^{\kappa} g_{\mu\kappa}$$

GR: smooth manifold and $\tilde{\nabla}g$

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$$\nabla_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \Gamma_{\mu\lambda}^{\kappa} g_{\kappa\nu} - \Gamma_{\nu\lambda}^{\kappa} g_{\mu\kappa}$$

in the tangent space at x , $\exists$ coordinate basis $\vec{e}_{\bar{\mu}}$ with

$$g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} = \text{diag}(-1, 1, 1, 1) = g^{\bar{\mu}\bar{\nu}}$$

$$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}}$$

GR: smooth manifold and $\tilde{\nabla} g$

from above:

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$$\Rightarrow \partial_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\lambda}} \eta_{\bar{\mu}\bar{\nu}} = 0$$

GR: smooth manifold and $\tilde{\nabla}g$

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$$\nabla_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \Gamma_{\mu\lambda}^{\kappa} g_{\kappa\nu} - \Gamma_{\nu\lambda}^{\kappa} g_{\mu\kappa}$$

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$$\text{also, } \Gamma_{\bar{\nu}\bar{\mu}}^{\bar{\lambda}} \vec{e}_{\bar{\lambda}} := \vec{e}_{\bar{\nu},\bar{\mu}} = \partial_{\bar{\mu}} \vec{e}_{\bar{\nu}}$$

GR: smooth manifold and $\tilde{\nabla}g$

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but in a Cartesian or Minkowski (vector) space, the basis vectors always point in the same direction and their lengths are fixed

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$$M^4 \Rightarrow \Gamma_{\bar{\nu}\bar{\mu}}^{\bar{\lambda}} \vec{e}_{\bar{\lambda}} = 0$$

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$$\text{so } \nabla_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = 0$$

GR: smooth manifold and $\tilde{\nabla} \mathbf{g}$

from above:

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$$\text{so } \nabla_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = 0$$

so $\tilde{\nabla} \mathbf{g} = 0$ (also $\tilde{\nabla} \mathbf{g}^{-1} = 0$) on the tangent space, since if true in one coord system, also true in others

GR: smooth manifold and $\tilde{\nabla} \mathbf{g}$

from above:

$$\nabla_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \Gamma^{\kappa}_{\mu\lambda} g_{\kappa\nu} - \Gamma^{\kappa}_{\nu\lambda} g_{\mu\kappa}$$

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$$\text{so } \nabla_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = 0$$

$$\text{so } \tilde{\nabla} \mathbf{g} = 0 = \tilde{\nabla} \mathbf{g}^{-1} \text{ on tangent space}$$

$$\dots \tilde{\nabla} \mathbf{g} = 0 = \tilde{\nabla} \mathbf{g}^{-1} \text{ on } M$$

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from above:

$$\nabla_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \Gamma_{\mu\lambda}^{\kappa} g_{\kappa\nu} - \Gamma_{\nu\lambda}^{\kappa} g_{\mu\kappa}$$

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$$\text{so } \nabla_{\bar{\lambda}} g_{\bar{\mu}\bar{\nu}} = 0$$

$$\text{so } \tilde{\nabla} \mathbf{g} = 0 = \tilde{\nabla} \mathbf{g}^{-1} \text{ on tangent space}$$

$$\dots \tilde{\nabla} \mathbf{g} = 0 = \tilde{\nabla} \mathbf{g}^{-1} \text{ on } M$$

$$\dots \Gamma_{\mu\nu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda} \text{ in any coord. basis (symmetric defn)}$$

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from above:

$$\nabla_{\lambda} g_{\mu\nu} = \partial_{\lambda} g_{\mu\nu} - \Gamma^{\kappa}_{\mu\lambda} g_{\kappa\nu} - \Gamma^{\kappa}_{\nu\lambda} g_{\mu\kappa}$$

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$$\text{so } \tilde{\nabla} \mathbf{g} = 0 = \tilde{\nabla} \mathbf{g}^{-1} \text{ on tangent space}$$

$$\dots \tilde{\nabla} \mathbf{g} = 0 = \tilde{\nabla} \mathbf{g}^{-1} \text{ on } M$$

$$\dots \Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu} \text{ in any coord. basis (symmetric defn)}$$

...

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} (\partial_{\mu} g_{\nu\kappa} + \partial_{\nu} g_{\mu\kappa} - \partial_{\kappa} g_{\mu\nu}) \text{ in a coordinate basis}$$

- 
- A blue L-shaped polygon on a white background. It consists of a vertical rectangle on the left and a horizontal rectangle extending to the right from the bottom of the vertical one.

GR: directional deriv.: $\langle \tilde{\nabla} \vec{A}, \vec{V} \rangle$

- $\tilde{\nabla} \phi, \tilde{\nabla} \vec{A}, \tilde{\nabla} \tilde{A}$ gave how the fields $\phi, \vec{A}$, or $\tilde{A}$ change around the manifold in general
- how about moving along a specific curve?

GR: directional deriv.: $\langle \tilde{\nabla} \vec{A}, \vec{V} \rangle$

- $\tilde{\nabla} \phi, \tilde{\nabla} \vec{A}, \tilde{\nabla} \tilde{A}$ gave how the fields ϕ , $\vec{A}$, or $\tilde{A}$ change around the manifold in general
- how about moving along a specific curve?
- curve on manifold parametrised by a continuously changing real parameter λ : $x(\lambda)$

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warning: $\{x^\mu(\lambda)\}$ at some λ on the manifold is a point on the manifold but NOT a vector; while $d\vec{x}$ — in the tangent space — IS a vector

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∇_V written by Bertschinger without $\vec{}$ or $\sim$ because $\nabla_V \mathbf{T}$ of tensor $\mathbf{T}$ has the same tensor order as $\mathbf{T}$

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so in a coord basis,

$$\nabla_V \vec{A} = \left(\frac{dA^\nu}{d\lambda} + V^\mu A^\kappa \Gamma^\nu_{\kappa\mu} \right) \vec{e}_\nu$$

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special (interesting) case: vector field $\vec{A}$ and curve with tangents $\vec{V} := \frac{d\vec{x}}{d\lambda}$ where $\vec{A}$ “locally does not change direction”

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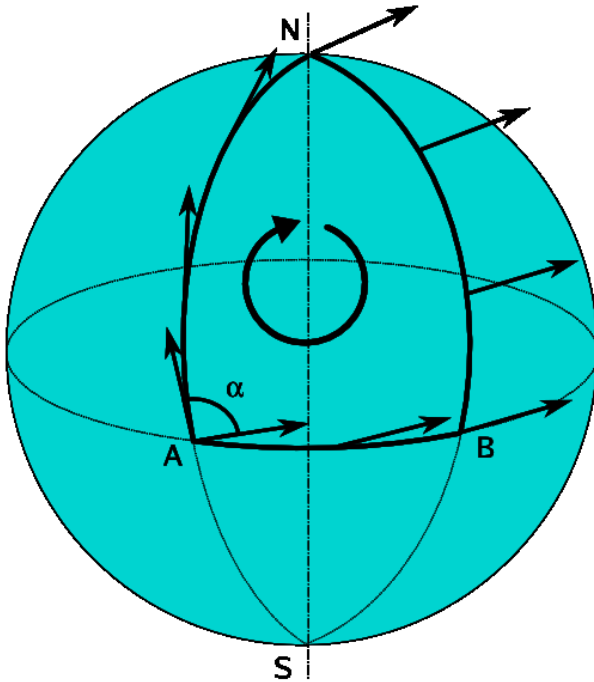
i.e. $\nabla_V \vec{A} = 0$

$\nabla_V \vec{A} = 0$ defn: parallel transport of $\vec{A}$ along path $\mathbf{x}(\lambda)$

where $\vec{V} := \frac{d\vec{x}}{d\lambda}$

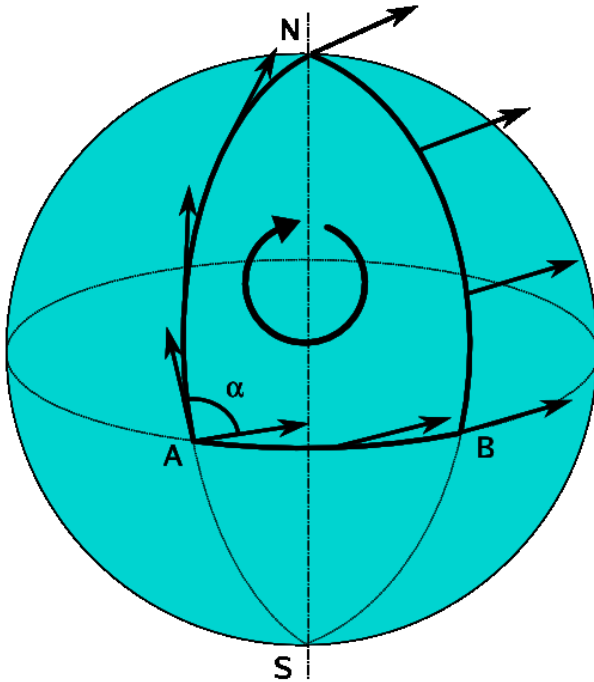
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example:



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example:



on S^2 , parallel transport of $\vec{A}$ around a closed loop does not conserve $\vec{A}$'s direction



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cf w:Euler-Lagrange equation

GR: parallel transp. @closed curve

parallel transport around “small” parallelogram in two directions $d\vec{x}_1, d\vec{x}_2$,

(“1” and “2” are not component indices here)

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What is the change in $\vec{A}$ after parallel transport around the closed loop $d\vec{x}_1, d\vec{x}_2, -d\vec{x}_1, -d\vec{x}_2$?

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use covariant derivatives of covariant derivatives ...



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also written with commutator $[\ , \]$

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using $\nabla_\alpha A^\mu$ from above and similar formulae, ...

$$R^\mu_{\nu\alpha\beta} A^\nu = \Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta} + \Gamma^\mu_{\kappa\alpha} \Gamma^\kappa_{\nu\beta} - \Gamma^\mu_{\kappa\beta} \Gamma^\kappa_{\nu\alpha}$$

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- $\Gamma^\mu_{\nu\beta}$: sum over first order partial derivatives of $g_{\nu\kappa}, \dots$
- so R has second order partial derivatives of $g_{\nu\kappa}, \dots$

GR: Riemann tensor

- first order ∂ :

(pseudo-)manifold locally like $\mathbb{R}^3$ (M^4), $\exists$ coords where $\Gamma^\mu_{\nu\beta} = 0$ locally

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- first order ∂ :

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- second order ∂ :

(pseudo-)manifold globally like $\mathbb{R}^3$ (M^4) $\Leftrightarrow R^\mu_{\nu\alpha\beta}(\mathbf{x}) = 0 \ \forall \mathbf{x}$

GR: Bianchi, Ricci, Einstein

...second Bianchi identity:

$$\nabla_{\sigma} R^{\mu}_{\nu\kappa\lambda} + \nabla_{\kappa} R^{\mu}_{\nu\lambda\sigma} + \nabla_{\lambda} R^{\mu}_{\nu\sigma\kappa} = 0$$

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warning: “R” written **coordinate-free** may mean:

- an order 4, dimension 64 tensor $\mathbf{R}$;
- an order 2, dimension 16 tensor $\mathbf{R}$ or R ; or
- an order 0, dimension 1 tensor $\equiv$ scalar R
- all three are fields over a spacetime 4-manifold

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w:Proofs involving covariant derivatives

$$\dots \nabla_{\nu} (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0$$

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... second Bianchi identity:

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$$R_{\mu\nu} := R^{\alpha}_{\mu\alpha\nu}$$

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defn Einstein tensor (by components):

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Cactus - <http://cactuscode.org>