Special and General Relativity

Boud Roukema

SR+GR

- SR+GR: construct spacetime from set theory

SR+GR

- SR+GR: construct spacetime from set theory
- start with X = set of points, no distances between points, angles, etc. defined

SR+GR

- SR+GR: construct spacetime from set theory
- start with X = set of points, no distances between points, angles, etc. defined
- no absolute simultaneity = “time is not absolute”

SR+GR

- SR+GR: construct spacetime from set theory
- start with X = set of points, no distances between points, angles, etc. defined
- no absolute simultaneity = “time is not absolute”
- add properties to X that satisfy theorems

SR+GR

- SR+GR: construct spacetime from set theory
- start with X = set of points, no distances between points, angles, etc. defined
- no absolute simultaneity = “time is not absolute”
- add properties to X that satisfy theorems
- $\rightarrow$ differentiable 4-(pseudo-)manifold

SR+GR

- SR+GR: construct spacetime from set theory
- start with X = set of points, no distances between points, angles, etc. defined
- no absolute simultaneity = “time is not absolute”
- add properties to X that satisfy theorems
- $\rightarrow$ differentiable 4-(pseudo-)manifold
- point particle in space $\rightarrow$ w:World line in spacetime

SR+GR

- SR+GR: construct spacetime from set theory
- start with X = set of points, no distances between points, angles, etc. defined
- no absolute simultaneity = "time is not absolute"
- add properties to X that satisfy theorems
- $\rightarrow$ differentiable 4-(pseudo-)manifold
- point particle in space $\rightarrow$ w:World line in spacetime
- point in spacetime $\rightarrow$ spacetime "event"

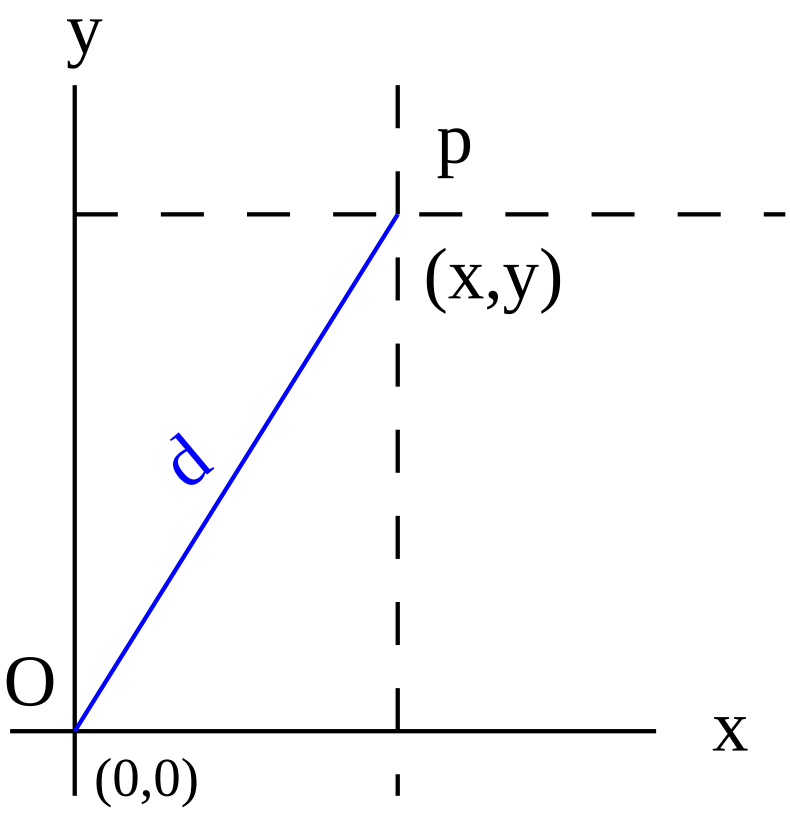
SR+GR

- SR+GR: construct spacetime from set theory
- start with X = set of points, no distances between points, angles, etc. defined
- no absolute simultaneity = “time is not absolute”
- add properties to X that satisfy theorems
- $\rightarrow$ differentiable 4-(pseudo-)manifold
- point particle in space $\rightarrow$ w:World line in spacetime
- point in spacetime $\rightarrow$ spacetime “event”
- SR: spacetime = w:Minkowski space

SR+GR

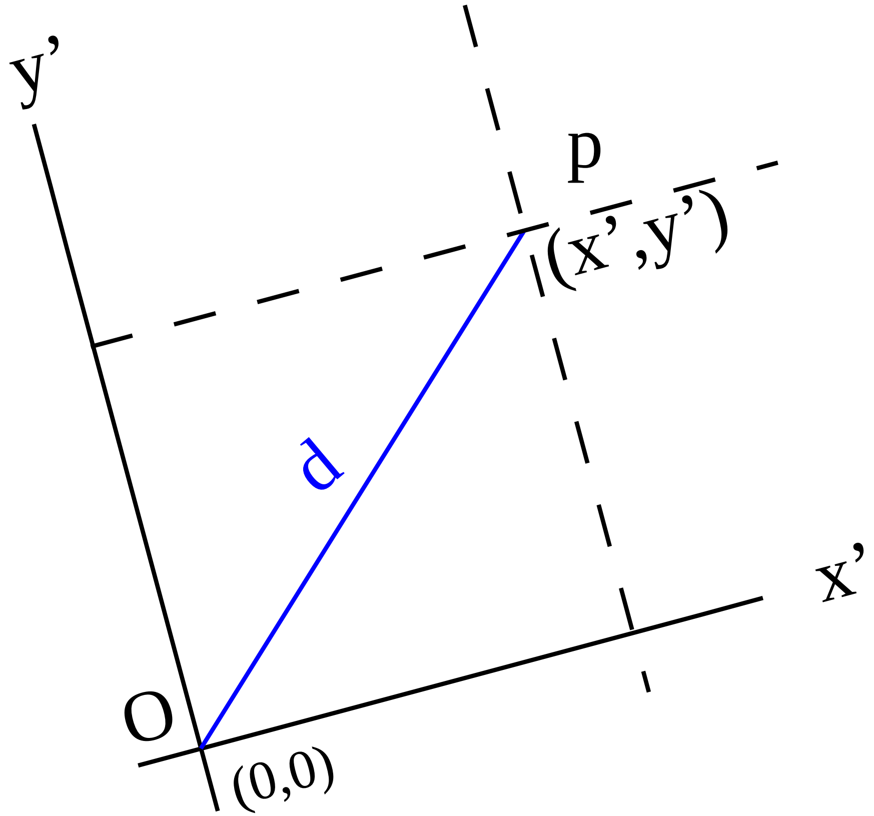
- SR+GR: construct spacetime from set theory
- start with X = set of points, no distances between points, angles, etc. defined
- no absolute simultaneity = “time is not absolute”
- add properties to X that satisfy theorems
- $\rightarrow$ differentiable 4-(pseudo-)manifold
- point particle in space $\rightarrow$ w:World line in spacetime
- point in spacetime $\rightarrow$ spacetime “event”
- SR: spacetime = w:Minkowski space
- GR: spacetime = a solution of the w:Einstein field equations

SR: Minkowski spacetime



p at (x, y), distance from observer at O is d

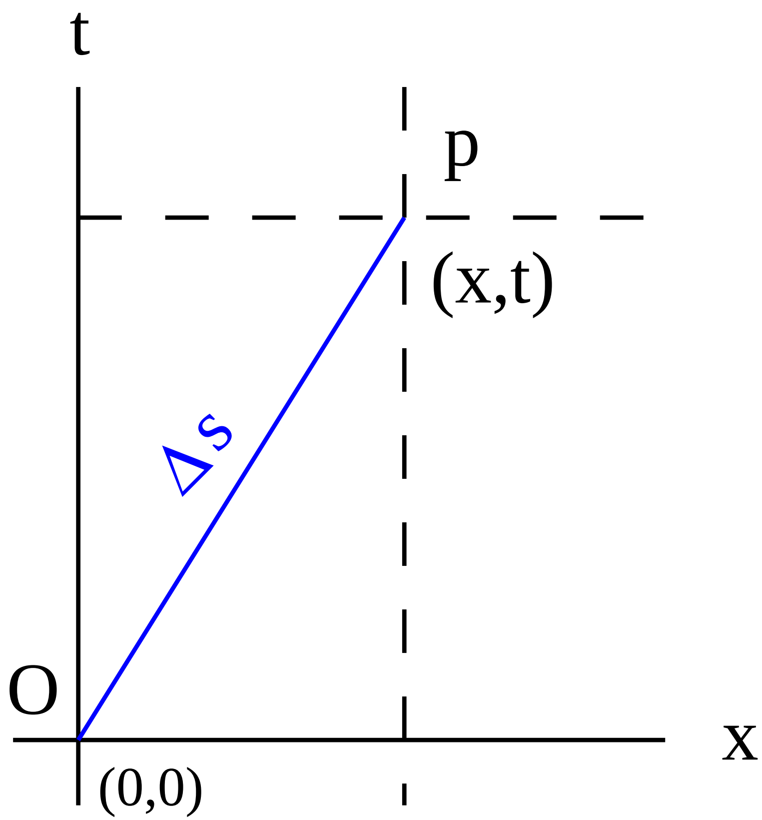
SR: Minkowski spacetime



$$\begin{pmatrix} p_{x'} \\ p_{y'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

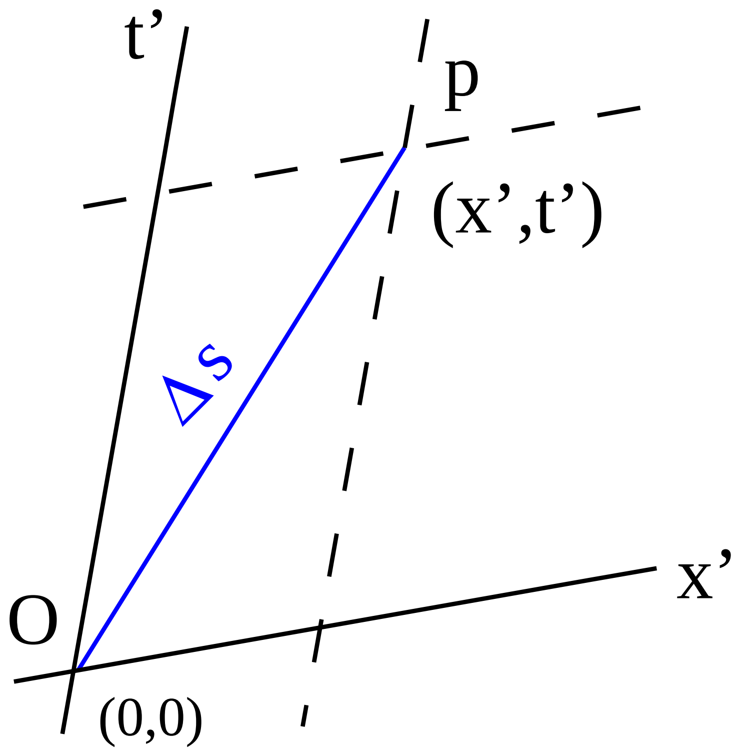
p at (x', y'), distance from observer at O is d = unchanged

SR: Minkowski spacetime



p at (x,t), w:invariant interval from observer at O is Δs where $(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2$

SR: Minkowski spacetime



$$\begin{pmatrix} p_{\mathrm{x}'} \\ p_{\mathrm{t}'} \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} p_{\mathrm{x}} \\ p_{\mathrm{t}} \end{pmatrix}$$

p at (x', t'), invariant interval from observer at O is Δs = $(\Delta s)^2 = -(\Delta t')^2 + (\Delta x')^2$ = unchanged

SR: Lorentz transformation

$$\Lambda(\phi) := \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

$$\cosh\phi := \frac{e^{\phi}+e^{-\phi}}{2}$$

$$\sinh\phi := \frac{e^{\phi}-e^{-\phi}}{2}$$

w:hyperbolic function

SR: Lorentz transformation

$$\cosh\phi := \frac{e^{\phi}+e^{-\phi}}{2}$$

$$\sinh\phi := \frac{e^{\phi}-e^{-\phi}}{2}$$

w:hyperbolic function

$$\Lambda(\phi) := \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

$$\begin{pmatrix} p_{\mathrm{x}'} \\ p_{\mathrm{t}'} \end{pmatrix} = \Lambda \begin{pmatrix} p_{\mathrm{x}} \\ p_{\mathrm{t}} \end{pmatrix}$$

SR: Lorentz transformation

$$\cosh\phi := \frac{e^{\phi}+e^{-\phi}}{2}$$

$$\sinh\phi := \frac{e^{\phi}-e^{-\phi}}{2}$$

$$\Lambda(\phi) := \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

w:hyperbolic function

$$\begin{pmatrix} p_{x'} \\ p_{t'} \end{pmatrix} = \Lambda \begin{pmatrix} p_x \\ p_t \end{pmatrix}$$

units: Λ only makes sense if same units for x, t, x', t'

SR: Lorentz transformation

$$\cosh\phi := \frac{e^{\phi}+e^{-\phi}}{2}$$

$$\sinh\phi := \frac{e^{\phi}-e^{-\phi}}{2}$$

w:hyperbolic function

$$\Lambda(\phi) := \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

$$\begin{pmatrix} p_{x'} \\ p_{t'} \end{pmatrix} = \Lambda \begin{pmatrix} p_x \\ p_t \end{pmatrix}$$

units: Λ only makes sense if same units for x, t, x', t'

Definition: 1 m := (1/2.99792458×10^8) s $\approx 10^{-8.5}$ s $\approx$ 3 ns

SR: Lorentz transformation

$$\cosh\phi := \frac{e^{\phi}+e^{-\phi}}{2}$$

$$\sinh\phi := \frac{e^{\phi}-e^{-\phi}}{2}$$

w:hyperbolic function

$$\Lambda(\phi) := \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

$$\begin{pmatrix} p_{x'} \\ p_{t'} \end{pmatrix} = \Lambda \begin{pmatrix} p_x \\ p_t \end{pmatrix}$$

units: Λ only makes sense if same units for x, t, x', t'

Definition: 1 m := (1/2.99792458×10^8) s $\approx 10^{-8.5}$ s $\approx$ 3 ns

speed of light in a vacuum = $c = \frac{2.99792458\times10^8 \text{ m}}{1 \text{ s}}$

SR: Lorentz transformation

$$\cosh\phi := \frac{e^{\phi}+e^{-\phi}}{2}$$

$$\sinh\phi := \frac{e^{\phi}-e^{-\phi}}{2}$$

w:hyperbolic function

$$\Lambda(\phi) := \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

$$\begin{pmatrix} p_{x'} \\ p_{t'} \end{pmatrix} = \Lambda \begin{pmatrix} p_x \\ p_t \end{pmatrix}$$

units: Λ only makes sense if same units for x, t, x', t'

Definition: 1 m := (1/2.99792458×10^8) s $\approx 10^{-8.5}$ s $\approx$ 3 ns

speed of light in a vacuum = $c = \frac{2.99792458\times10^8 \text{ m}}{1 \text{ s}}$

$$= \frac{2.99792458\times10^8\times(1/2.99792458\times10^8) \text{ s}}{1 \text{ s}}$$

SR: Lorentz transformation

$$\cosh\phi := \frac{e^{\phi}+e^{-\phi}}{2}$$

$$\sinh\phi := \frac{e^{\phi}-e^{-\phi}}{2}$$

w:hyperbolic function

$$\Lambda(\phi) := \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

$$\begin{pmatrix} p_{\mathrm{x}'} \\ p_{\mathrm{t}'} \end{pmatrix} = \Lambda \begin{pmatrix} p_{\mathrm{x}} \\ p_{\mathrm{t}} \end{pmatrix}$$

units: Λ only makes sense if same units for x, t, x', t'

Definition: 1 m := (1/2.99792458×10^{8}) s $\approx 10^{-8.5}$ s $\approx$ 3 ns

speed of light in a vacuum = $c = \frac{2.99792458\times10^{8}\ \mathrm{m}}{1\ \mathrm{s}}$

$= \frac{1\ \mathrm{s}}{1\ \mathrm{s}}$

SR: Lorentz transformation

$$\Lambda(\phi) := \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

$\cosh\phi := \frac{e^{\phi}+e^{-\phi}}{2}$

$\sinh\phi := \frac{e^{\phi}-e^{-\phi}}{2}$

w:hyperbolic function

$$\begin{pmatrix} p_{x'} \\ p_{t'} \end{pmatrix} = \Lambda \begin{pmatrix} p_x \\ p_t \end{pmatrix}$$

units: Λ only makes sense if same units for x, t, x', t'

Definition: 1 m := (1/2.99792458×10^8) s $\approx 10^{-8.5}$ s $\approx$ 3 ns

speed of light in a vacuum = $c = \frac{2.99792458\times10^8 \text{ m}}{1 \text{ s}}$

$= \frac{1 \text{ s}}{1 \text{ s}} = 1$ (dimensionless)

SR: rapidity ϕ vs velocity β

What is ϕ ?

SR: rapidity ϕ vs velocity β

What is ϕ ?

observer A has worldline $(x, t) = (0, t)$

SR: rapidity ϕ vs velocity β

What is ϕ ?

observer B has worldline $(x', t') = (0, t')$

SR: rapidity ϕ vs velocity β

What is ϕ ?

observer B has worldline $(x', t') = (0, t')$

in A's coordinate system, B's worldline is:

$$\begin{pmatrix} 0 \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ t \end{pmatrix}$$

SR: rapidity ϕ vs velocity β

What is ϕ ?

observer B has worldline $(x', t') = (0, t')$

in A's coordinate system, B's worldline is:

$$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} 0 \\ t' \end{pmatrix} = \begin{pmatrix} t' \sinh\phi \\ t' \cosh\phi \end{pmatrix}$$

SR: rapidity ϕ vs velocity β

What is ϕ ?

observer B has worldline $(x', t') = (0, t')$

in A's coordinate system, B's worldline is:

$$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} 0 \\ t' \end{pmatrix} = \begin{pmatrix} t' \sinh \phi \\ t' \cosh \phi \end{pmatrix}$$

$$\Rightarrow x = t' \sinh \phi$$

SR: rapidity ϕ vs velocity β

What is ϕ ?

observer B has worldline $(x', t') = (0, t')$

in A's coordinate system, B's worldline is:

$$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} 0 \\ t' \end{pmatrix} = \begin{pmatrix} t' \sinh\phi \\ t' \cosh\phi \end{pmatrix}$$

$$\Rightarrow x = t' \sinh\phi = t' \cosh\phi \frac{\sinh\phi}{\cosh\phi}$$

SR: rapidity ϕ vs velocity β

What is ϕ ?

observer B has worldline $(x', t') = (0, t')$

in A's coordinate system, B's worldline is:

$$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} 0 \\ t' \end{pmatrix} = \begin{pmatrix} t' \sinh\phi \\ t' \cosh\phi \end{pmatrix}$$

$$\Rightarrow x = t' \sinh\phi = t' \cosh\phi \tanh\phi$$

SR: rapidity ϕ vs velocity β

What is ϕ ?

observer B has worldline $(x', t') = (0, t')$

in A's coordinate system, B's worldline is:

$$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} 0 \\ t' \end{pmatrix} = \begin{pmatrix} t' \sinh\phi \\ t' \cosh\phi \end{pmatrix}$$

$$\Rightarrow x = t' \sinh\phi = t \tanh\phi$$

SR: rapidity ϕ vs velocity β

What is ϕ ?

observer B has worldline $(x', t') = (0, t')$

in A's coordinate system, B's worldline is:

$$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} 0 \\ t' \end{pmatrix} = \begin{pmatrix} t' \sinh\phi \\ t' \cosh\phi \end{pmatrix}$$

$$\Rightarrow x = t' \sinh\phi = t \tanh\phi = \beta t$$

where velocity $\beta := v/c \equiv v = \tanh\phi$

SR: calibration

Where does $(x', t') = (0, 1)$ lie for observer A?

SR: calibration

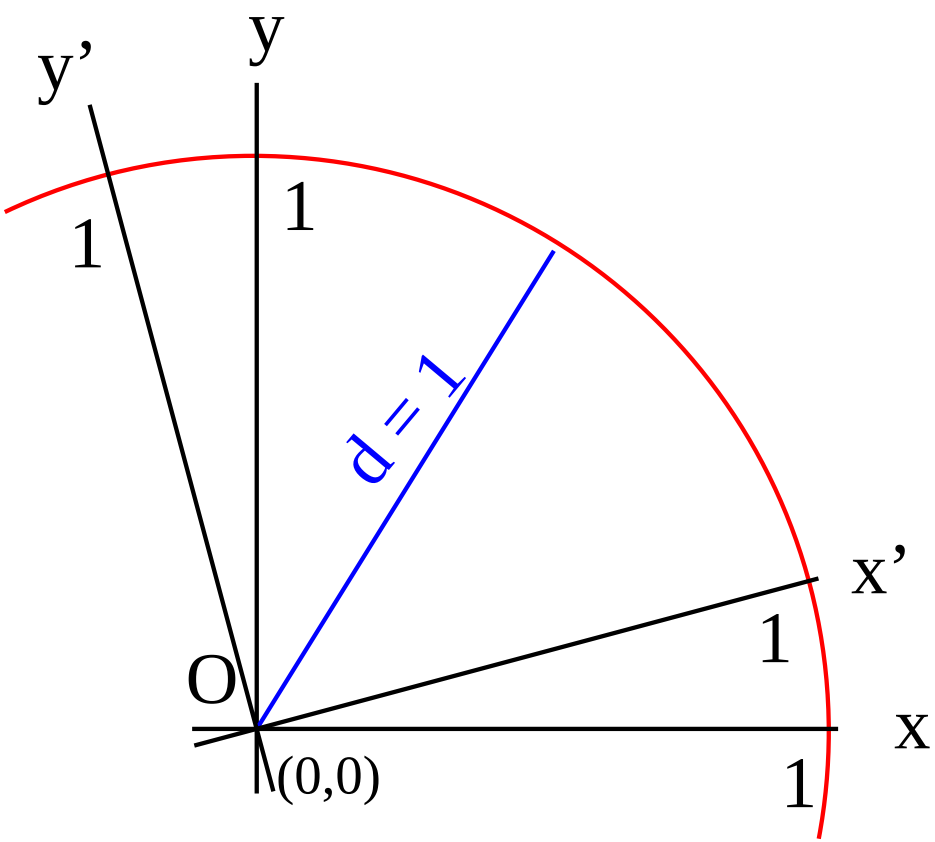
Where does $(x', t') = (0, 1)$ lie for observer A?

Where does $(x', t') = (1, 0)$ lie for observer A?

SR: calibration

Where does $(x', t') = (0, 1)$ lie for observer A?

Where does $(x', t') = (1, 0)$ lie for observer A?

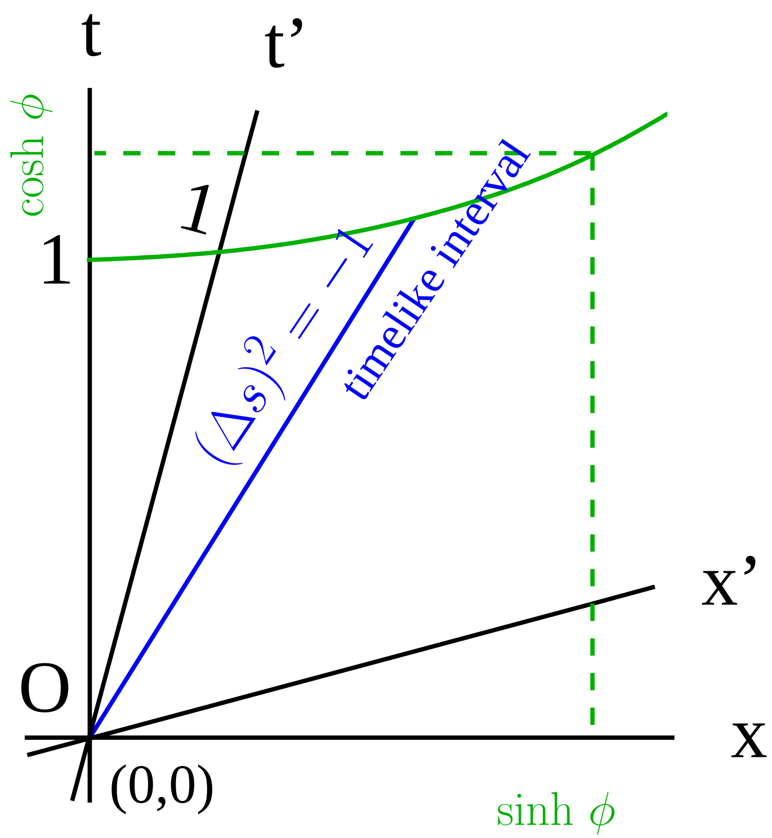


constant distance $d = 1$ from $(0, 0)$

SR: calibration

Where does $(x', t') = (0, 1)$ lie for observer A?

Where does $(x', t') = (1, 0)$ lie for observer A?



constant interval

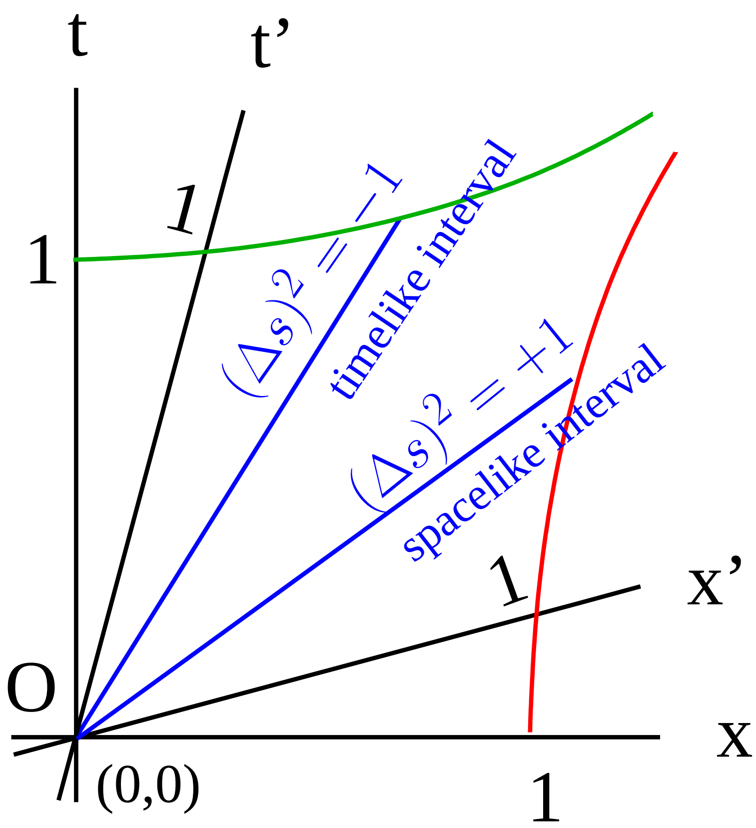
$(\Delta s)^2 = -1$ from $(0, 0)$

$$\Lambda^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sinh\phi \\ \cosh\phi \end{pmatrix}$$

SR: calibration

Where does $(x', t') = (0, 1)$ lie for observer A?

Where does $(x', t') = (1, 0)$ lie for observer A?



constant interval

$(\Delta s)^2 = +1$ from $(0, 0)$

$$\Lambda^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh\phi \\ \sinh\phi \end{pmatrix}$$

SR: calibration

Where does $(x', t') = (0, 1)$ lie for observer A?

Where does $(x', t') = (1, 0)$ lie for observer A?

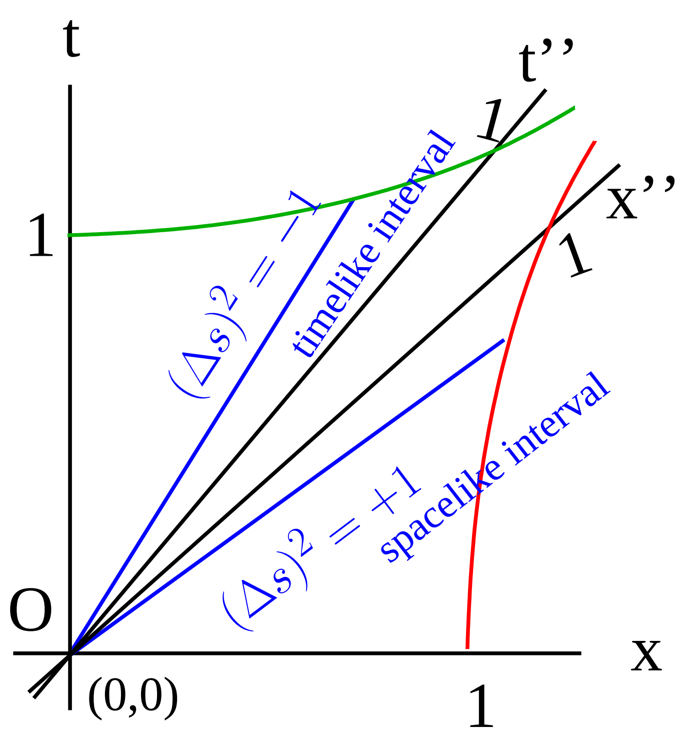
Can high ϕ push the t' axis close to the x axis?

SR: calibration

Where does $(x', t') = (0, 1)$ lie for observer A?

Where does $(x', t') = (1, 0)$ lie for observer A?

Can high ϕ push the t' axis close to the x axis?

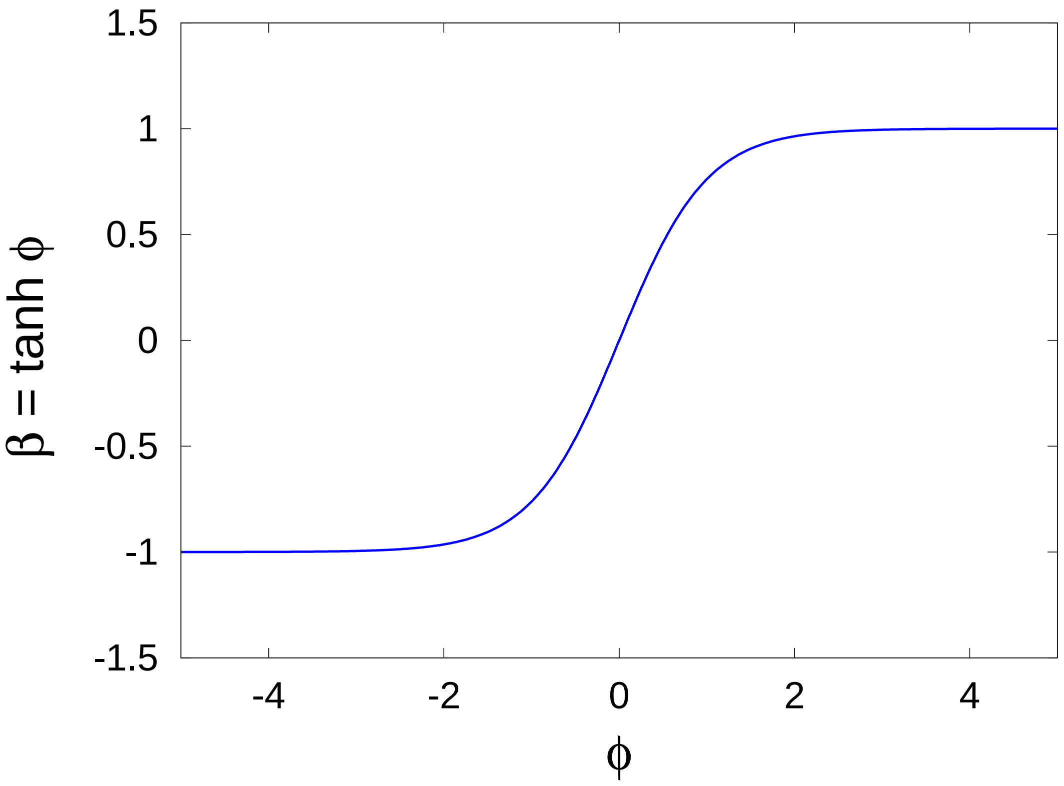


SR: calibration

Where does $(x', t') = (0, 1)$ lie for observer A?

Where does $(x', t') = (1, 0)$ lie for observer A?

Can high ϕ push the t' axis close to the x axis?



SR: effect of Λ on $x = t$ (photons)

What happens to a photon under Lorentz transformation?

SR: effect of Λ on $x = t$ (photons)

What happens to a photon under Lorentz transformation?

SR: effect of Λ on $x = t$ (photons)

What happens to a photon under Lorentz transformation?

Observer A: photon worldline is the set of spacetime events $\{(t,t) \,|\, t_1 < t < t_2\}$ for some t_1, t_2

SR: effect of Λ on $x = t$ (photons)

What happens to a photon under Lorentz transformation?

Observer A: photon worldline is the set of spacetime events $\{(t, t) \,|\, t_1 < t < t_2\}$ for some t_1, t_2

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} t \\ t \end{pmatrix}$$

SR: effect of Λ on $x = t$ (photons)

What happens to a photon under Lorentz transformation?

Observer A: photon worldline is the set of spacetime events $\{(t,t) \,|\, t_1 < t < t_2\}$ for some t_1, t_2

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} (\cosh\phi - \sinh\phi)t \\ (-\sinh\phi + \cosh\phi)t \end{pmatrix}$$

SR: effect of Λ on $x = t$ (photons)

What happens to a photon under Lorentz transformation?

Observer A: photon worldline is the set of spacetime events $\{(t,t) \,|\, t_1 < t < t_2\}$ for some t_1, t_2

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} (\cosh\phi - \sinh\phi)t \\ (-\sinh\phi + \cosh\phi)t \end{pmatrix}$$

$\Rightarrow$ worldline is $x' = (\cosh\phi - \sinh\phi)t = t'$ i.e. the set of spacetime events

$\{(t', t') \,|\, (\cosh\phi - \sinh\phi)t_1 < t' < (\cosh\phi - \sinh\phi)t_2\}$

SR: effect of Λ on $x = t$ (photons)

What happens to a photon under Lorentz transformation?

Observer A: photon worldline is the set of spacetime events $\{(t,t)\,|\,t_1 < t < t_2\}$ for some t_1, t_2

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} (\cosh\phi - \sinh\phi)t \\ (-\sinh\phi + \cosh\phi)t \end{pmatrix}$$

$\Rightarrow$ worldline is $x' = (\cosh\phi - \sinh\phi)t = t'$ i.e. the set of spacetime events

$\{(t',t')\,|\,(\cosh\phi - \sinh\phi)t_1 < t' < (\cosh\phi - \sinh\phi)t_2\}$

- photon speed same in both reference frames

SR: effect of Λ on $x = t$ (photons)

What happens to a photon under Lorentz transformation?

Observer A: photon worldline is the set of spacetime events $\{(t,t)\,|\,t_1 < t < t_2\}$ for some t_1, t_2

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} (\cosh\phi - \sinh\phi)t \\ (-\sinh\phi + \cosh\phi)t \end{pmatrix}$$

$\Rightarrow$ worldline is $x' = (\cosh\phi - \sinh\phi)t = t'$ i.e. the set of spacetime events

$\{(t', t')\,|\,(\cosh\phi - \sinh\phi)t_1 < t' < (\cosh\phi - \sinh\phi)t_2\}$

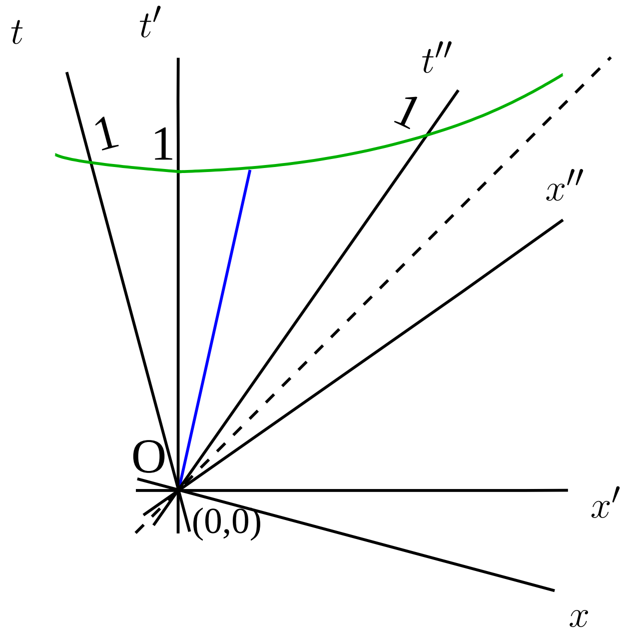
- photon speed same in both reference frames
- w:Michelson-Morley experiment (1887)

SR: adding velocities

interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?

SR: adding velocities

interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?

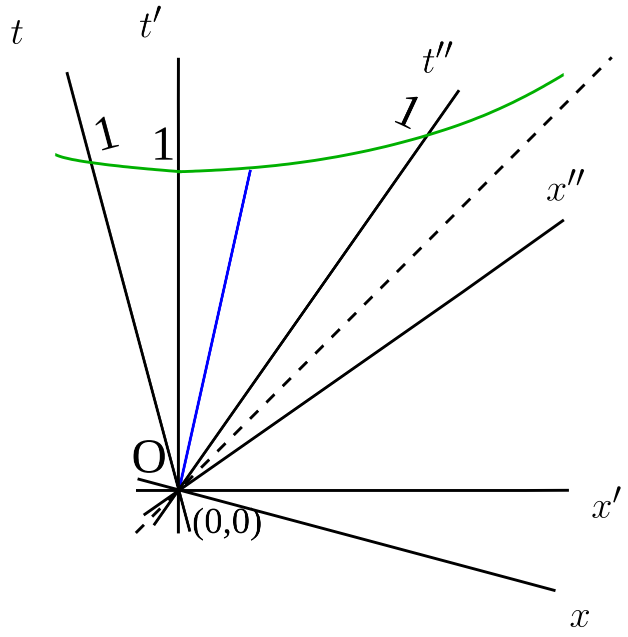


$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\phi_1 & -\sinh\phi_1 \\ -\sinh\phi_1 & \cosh\phi_1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

where $\tanh\phi_1 = \beta_1 = 0.1$

SR: adding velocities

interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?

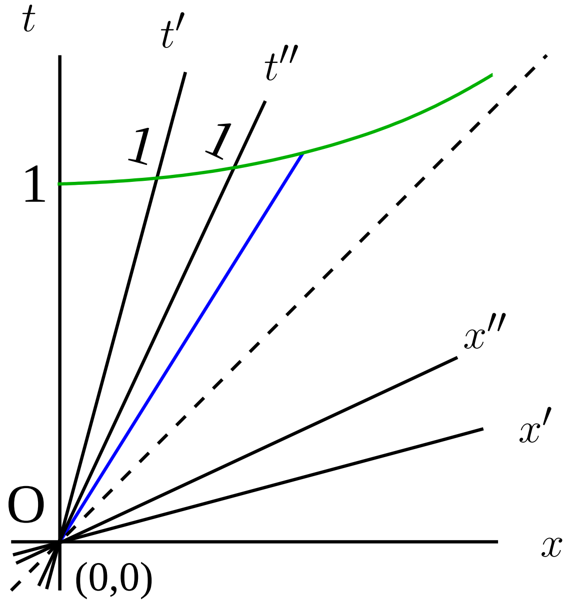


$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \begin{pmatrix} \cosh\phi_2 & -\sinh\phi_2 \\ -\sinh\phi_2 & \cosh\phi_2 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

where $\tanh\phi_2 = \beta_2 = 0.5$

SR: adding velocities

interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?

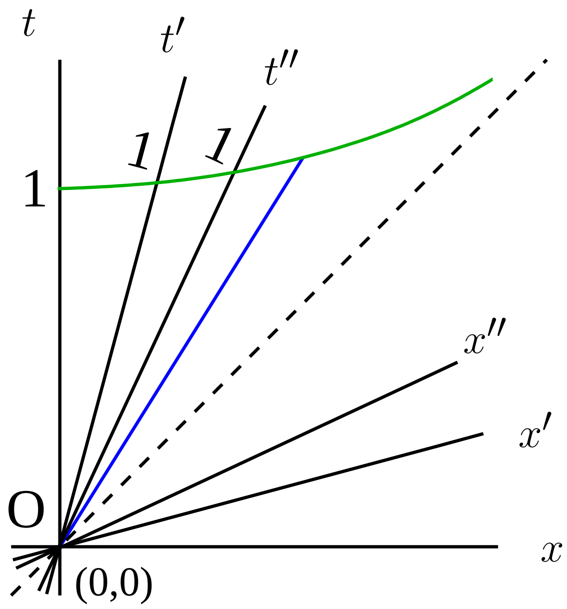


$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \begin{pmatrix} \cosh\phi_2 & -\sinh\phi_2 \\ -\sinh\phi_2 & \cosh\phi_2 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$$

where $\tanh\phi_2 = \beta_2 = 0.5$

SR: adding velocities

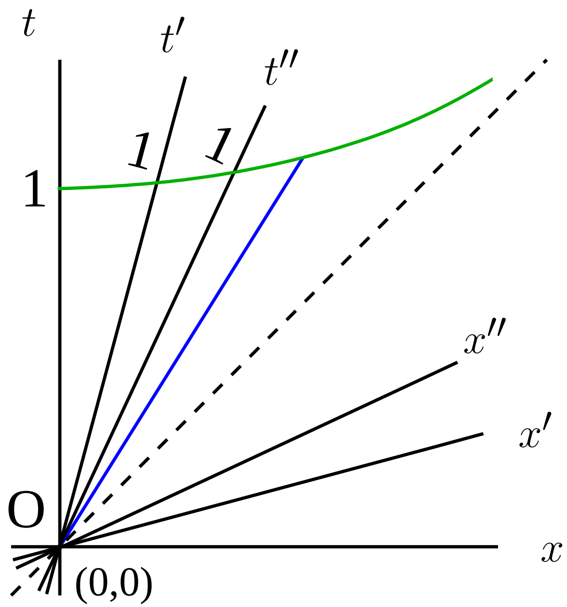
interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?



$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \Lambda(\phi_2)\Lambda(\phi_1) \begin{pmatrix} x \\ t \end{pmatrix}$$

SR: adding velocities

interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?

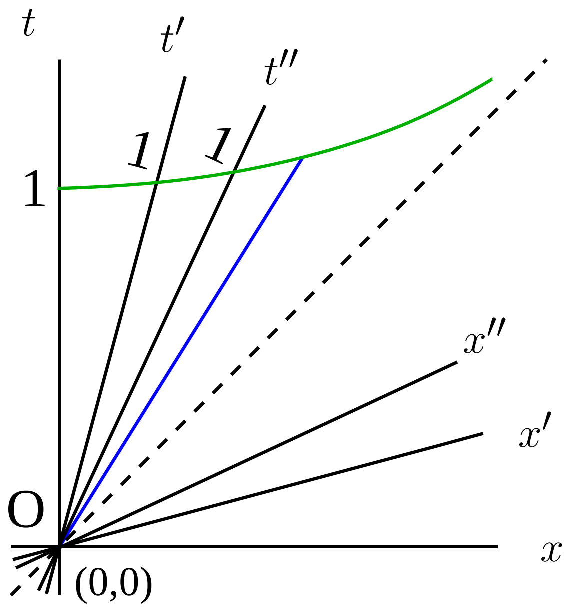


$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \Lambda(\phi_2)\Lambda(\phi_1) \begin{pmatrix} x \\ t \end{pmatrix}$$

but $\Lambda(\phi_2)\Lambda(\phi_1) = \Lambda(\phi_1 + \phi_2)$

SR: adding velocities

interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?

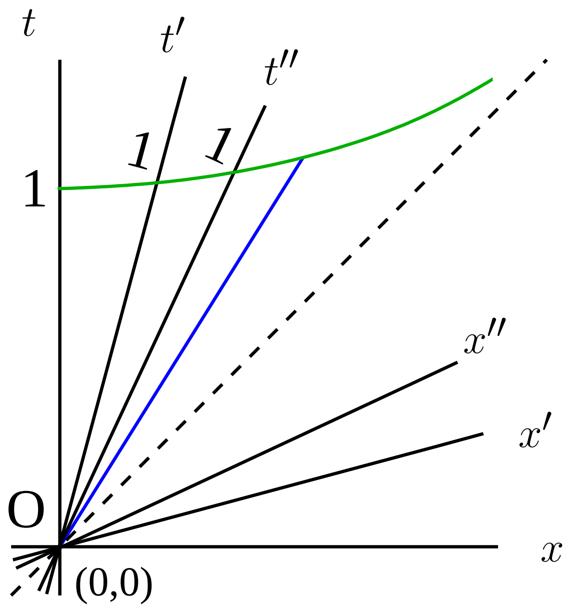


$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \Lambda(\phi_2)\Lambda(\phi_1) \begin{pmatrix} x \\ t \end{pmatrix}$$

but $\Lambda(\phi_2)\Lambda(\phi_1) = \Lambda(\phi_1 + \phi_2)$
cf. rotation θ_1 “plus” rotation θ_2 = rotation $\theta_1 + \theta_2$

SR: adding velocities

interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?



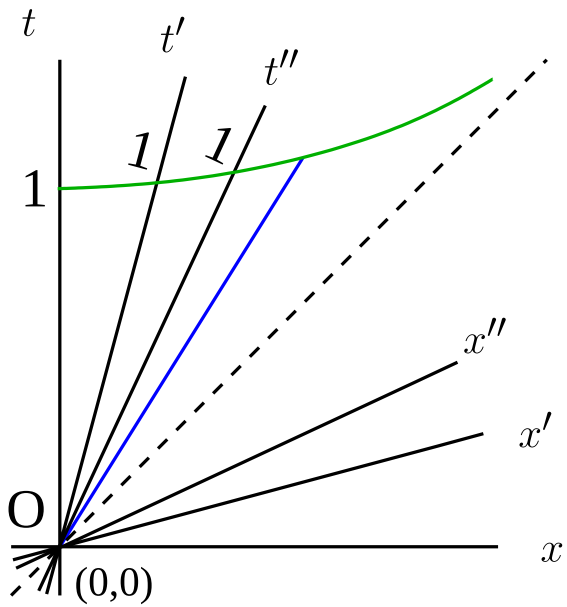
$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \Lambda(\phi_2)\Lambda(\phi_1) \begin{pmatrix} x \\ t \end{pmatrix}$$

but $\Lambda(\phi_2)\Lambda(\phi_1) = \Lambda(\phi_1 + \phi_2)$

so $\beta_3 = \tanh(\phi_1 + \phi_2)$

SR: adding velocities

interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?



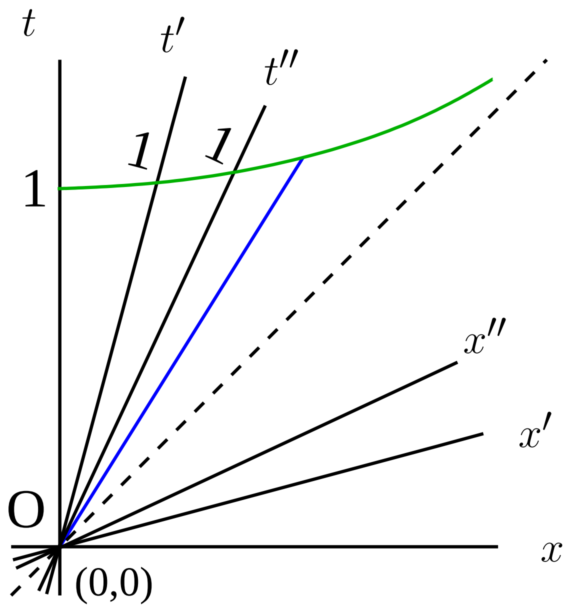
$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \Lambda(\phi_2)\Lambda(\phi_1) \begin{pmatrix} x \\ t \end{pmatrix}$$

but $\Lambda(\phi_2)\Lambda(\phi_1) = \Lambda(\phi_1 + \phi_2)$

so $\beta_3 = \tanh(\phi_1 + \phi_2) = \frac{\tanh\phi_1 + \tanh\phi_2}{1 + \tanh\phi_1 \tanh\phi_2}$

SR: adding velocities

interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?



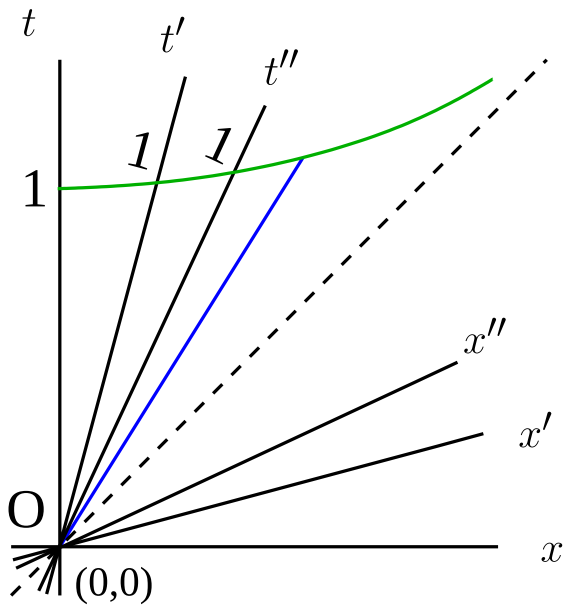
$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \Lambda(\phi_2)\Lambda(\phi_1) \begin{pmatrix} x \\ t \end{pmatrix}$$

but $\Lambda(\phi_2)\Lambda(\phi_1) = \Lambda(\phi_1 + \phi_2)$

so $\beta_3 = \tanh(\phi_1 + \phi_2) \quad = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}$

SR: adding velocities

interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?



$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \Lambda(\phi_2)\Lambda(\phi_1) \begin{pmatrix} x \\ t \end{pmatrix}$$

but $\Lambda(\phi_2)\Lambda(\phi_1) = \Lambda(\phi_1 + \phi_2)$

so $\beta_3 = \tanh(\phi_1 + \phi_2) \quad = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \quad = \frac{0.1+0.5}{1+0.1\times 0.5} \approx 0.57$

SR: Lorentz factor

Λ: alternative to hyperbolic trig functions

SR: Lorentz factor

Λ: alternative to hyperbolic trig functions

$$\Lambda(\phi) := \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

$$\cosh\phi := \frac{\mathrm{e}^{\phi}+\mathrm{e}^{-\phi}}{2}$$

$$\sinh\phi := \frac{\mathrm{e}^{\phi}-\mathrm{e}^{-\phi}}{2}$$

w:hyperbolic function

SR: Lorentz factor

Λ: alternative to hyperbolic trig functions

$$\Lambda(\beta) := \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

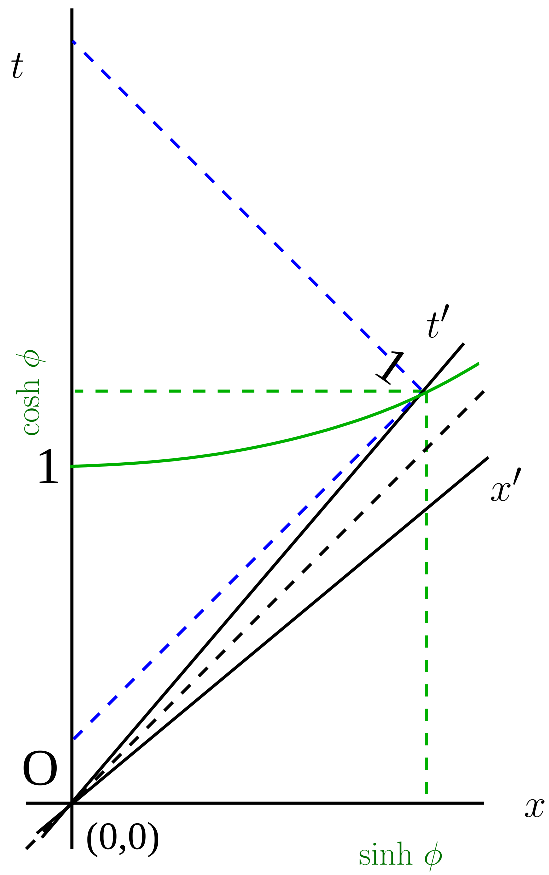
$\beta = \tanh\phi$

$\gamma := (1 - \beta^2)^{-1/2}$ = Lorentz factor

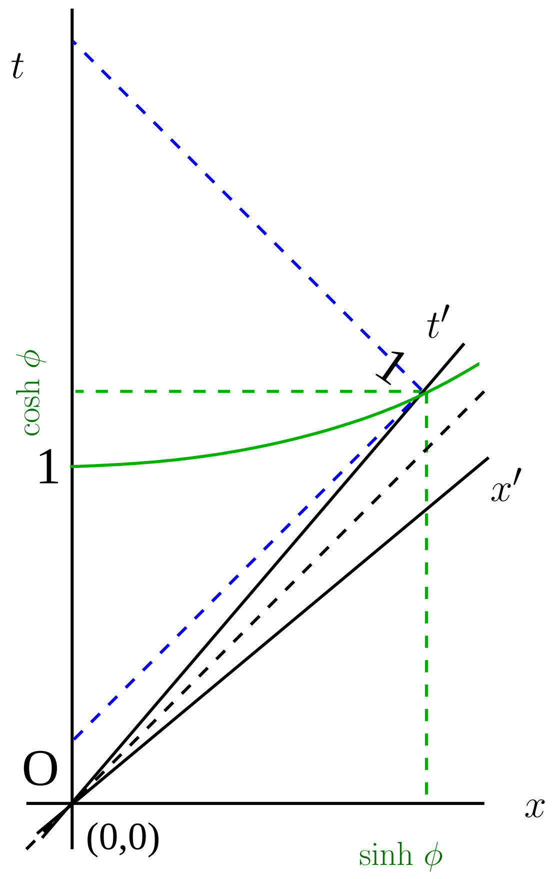
$\gamma = \cosh\phi$

$\beta\gamma = \sinh\phi$

SR: worldline time dilation

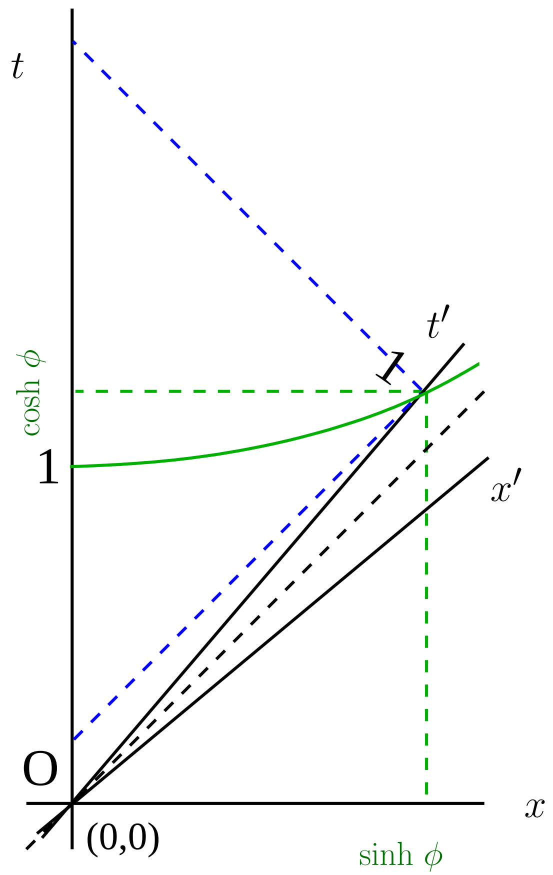


SR: worldline time dilation



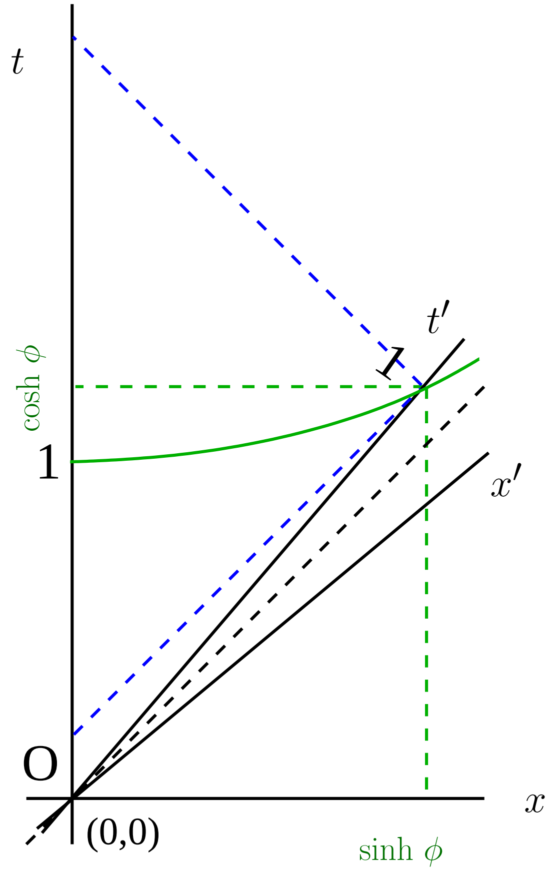
$$\cosh\phi \equiv \gamma$$

SR: worldline time dilation



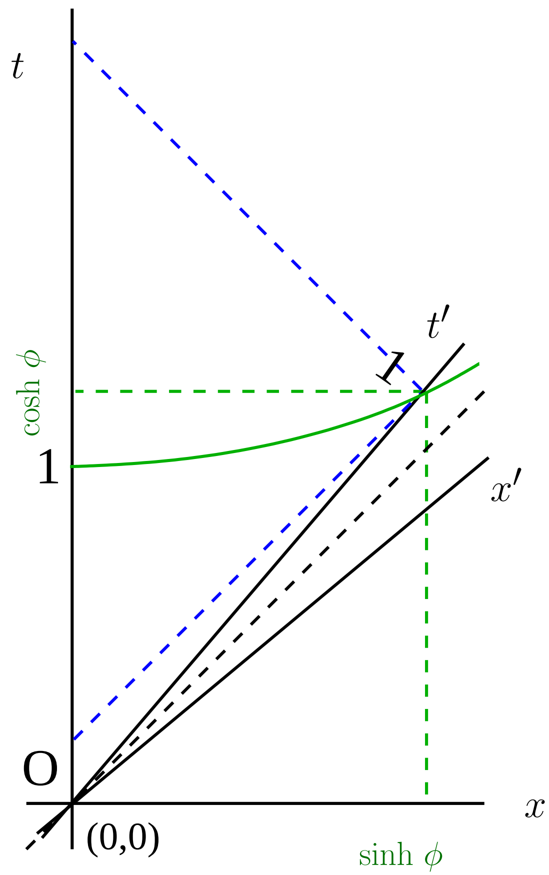
$$\cosh\phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

SR: worldline time dilation



$$\cosh\phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} > 1$$

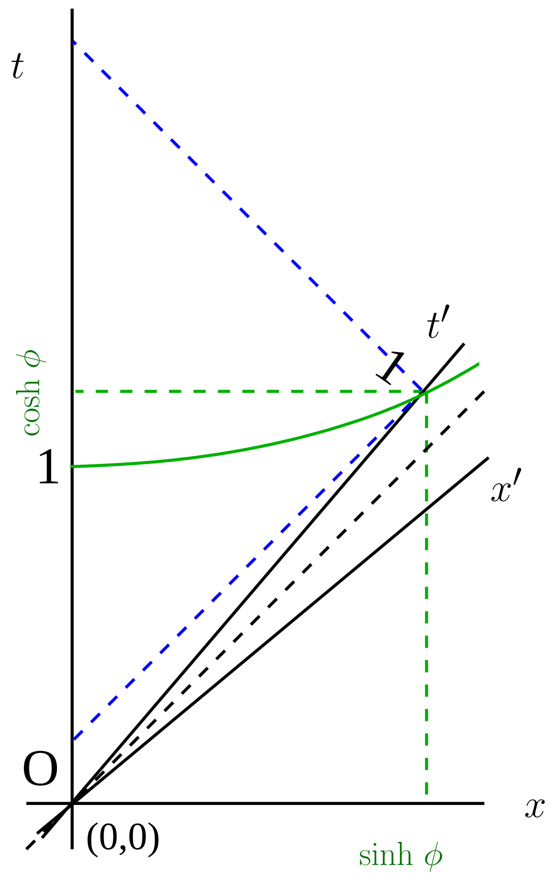
SR: worldline time dilation



worldline “time dilation”

$$\cosh\phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} > 1$$

SR: worldline time dilation

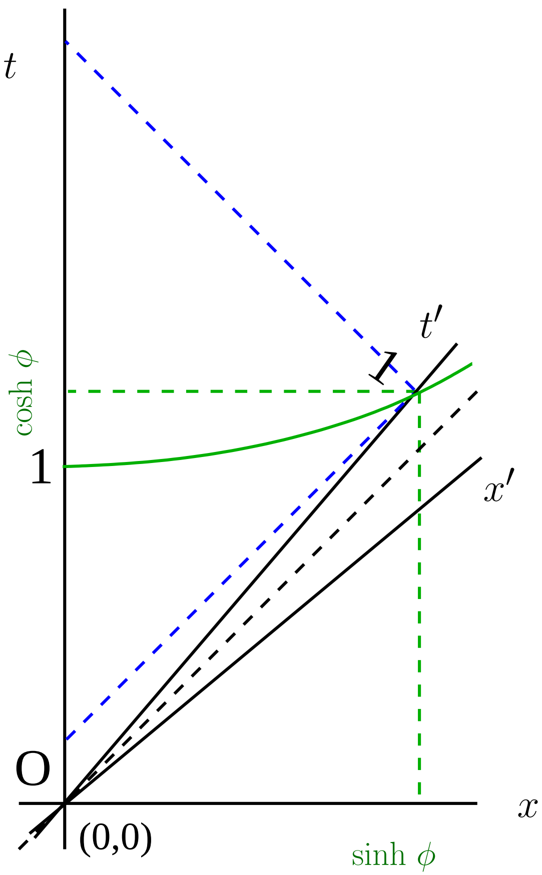


worldline "time dilation"

muons: mean lifetime 2197 ns $\ll 15$ km

$$\cosh\phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} > 1$$

SR: worldline time dilation



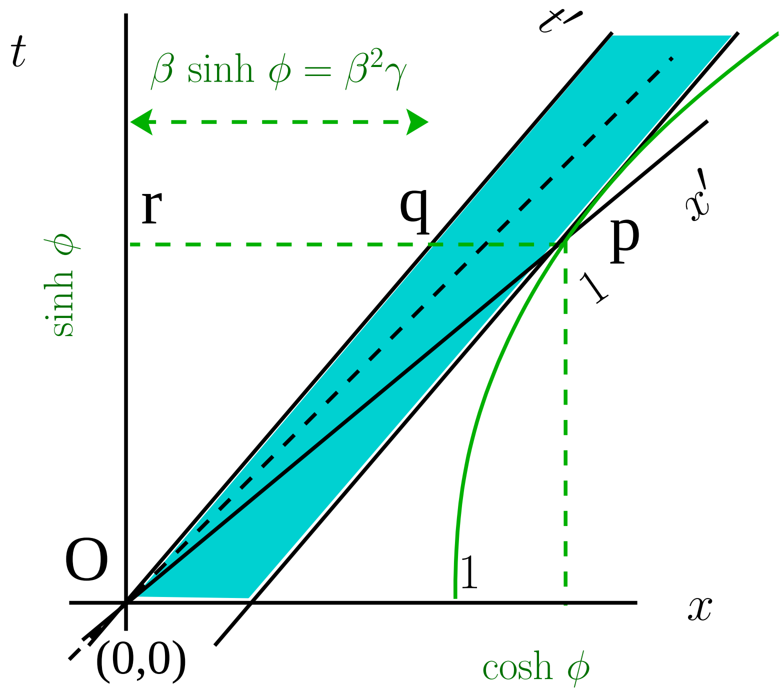
$$\cosh\phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} > 1$$

worldline "time dilation"

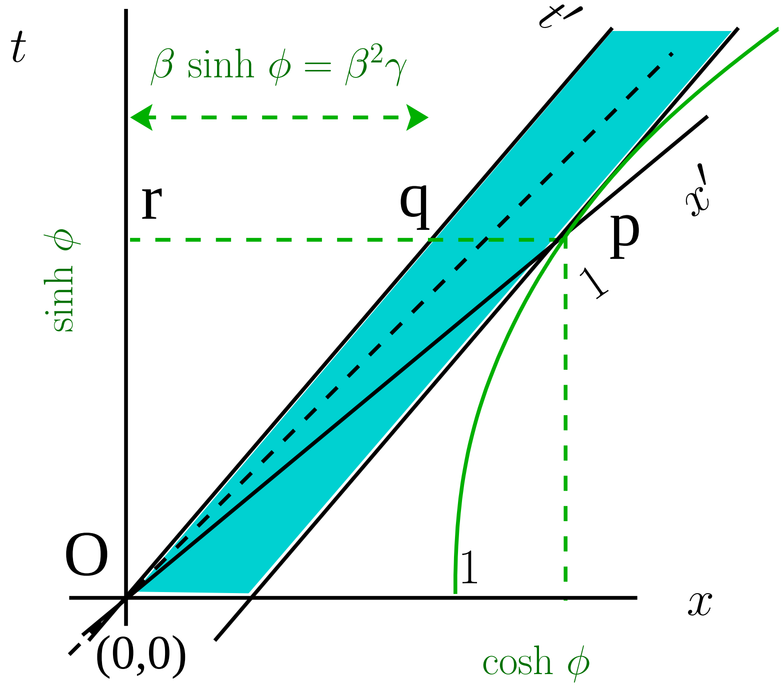
muons: mean lifetime 2197 ns $\ll 15$ km

time dilation $\Rightarrow$ muons can hit the ground

SR: worldsheet space contraction

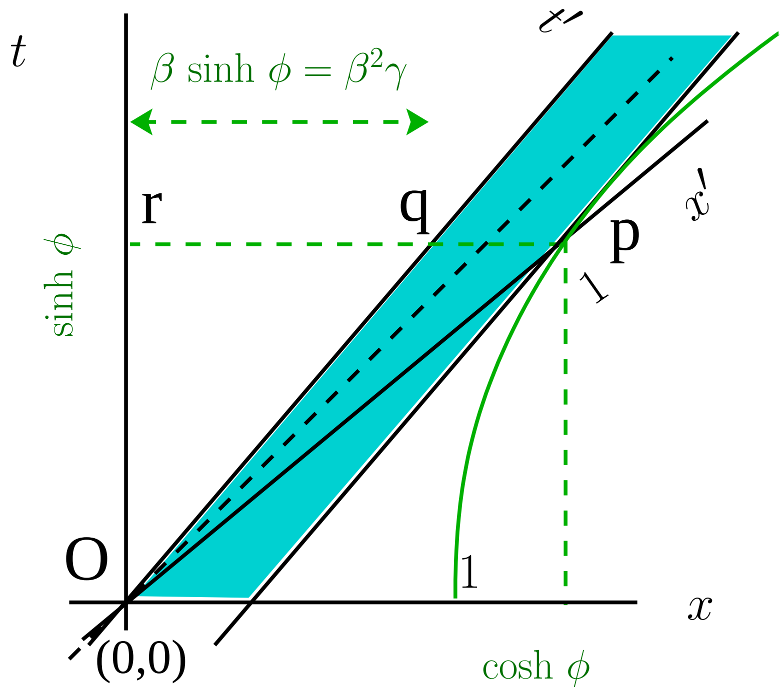


SR: worldsheet space contraction



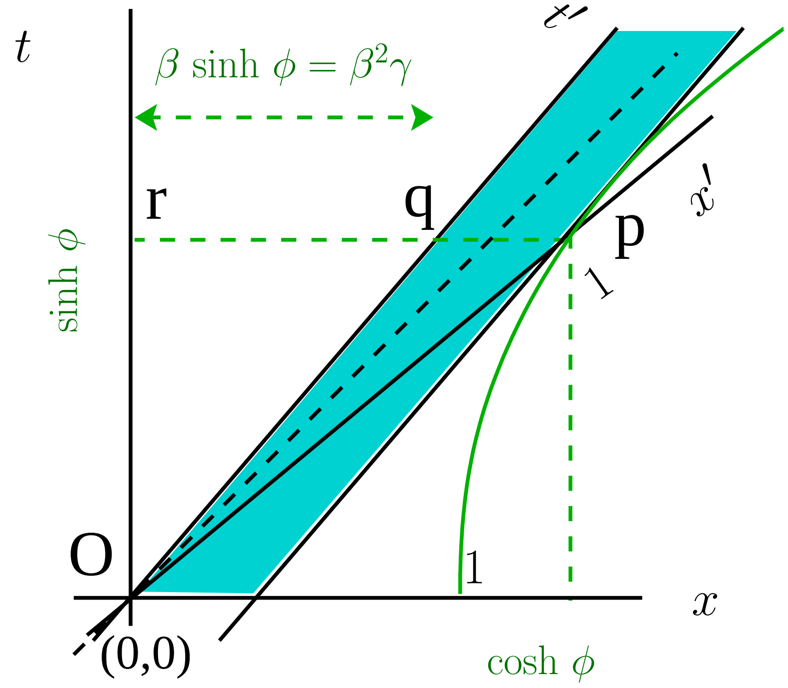
$$\sqrt{\Delta s^2(\mathrm{q},\mathrm{p})} =$$

SR: worldsheet space contraction



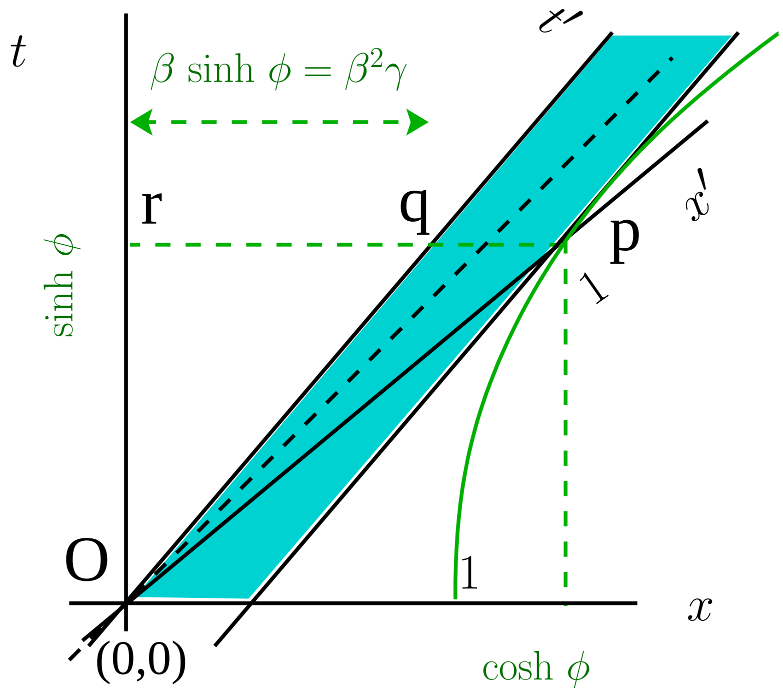
$$\sqrt{\Delta s^2(\mathrm{q},\mathrm{p})} = \cosh\phi - \beta\sinh\phi$$

SR: worldsheet space contraction



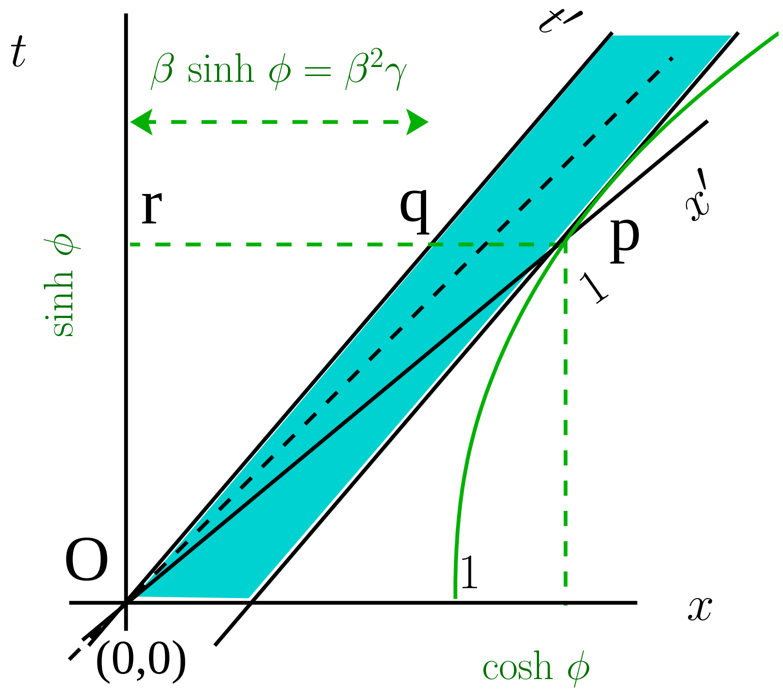
$$\sqrt{\Delta s^2(\mathrm{q},\mathrm{p})} = \gamma - \beta\beta\gamma$$

SR: worldsheet space contraction



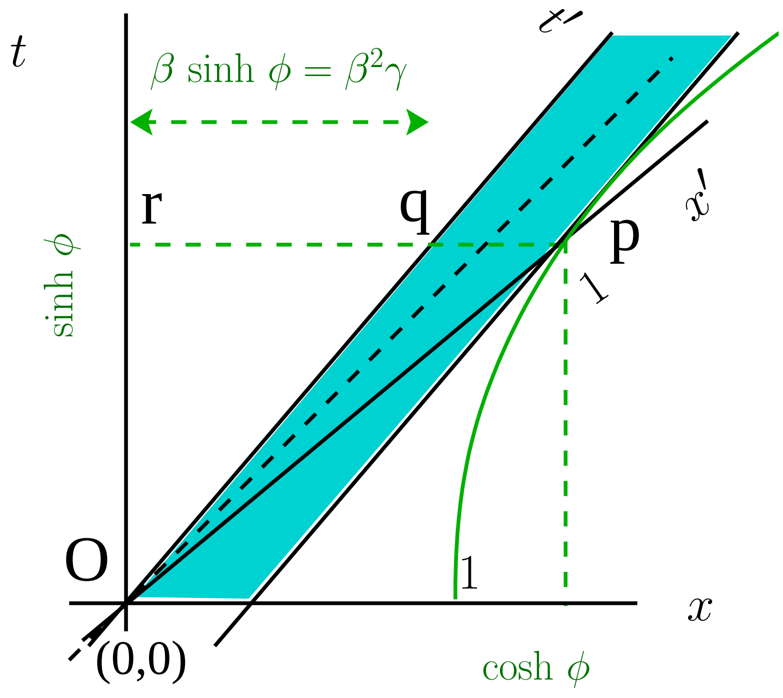
$$\sqrt{\Delta s^2(\mathrm{q},\mathrm{p})} = (1-\beta^2)\gamma$$

SR: worldsheet space contraction



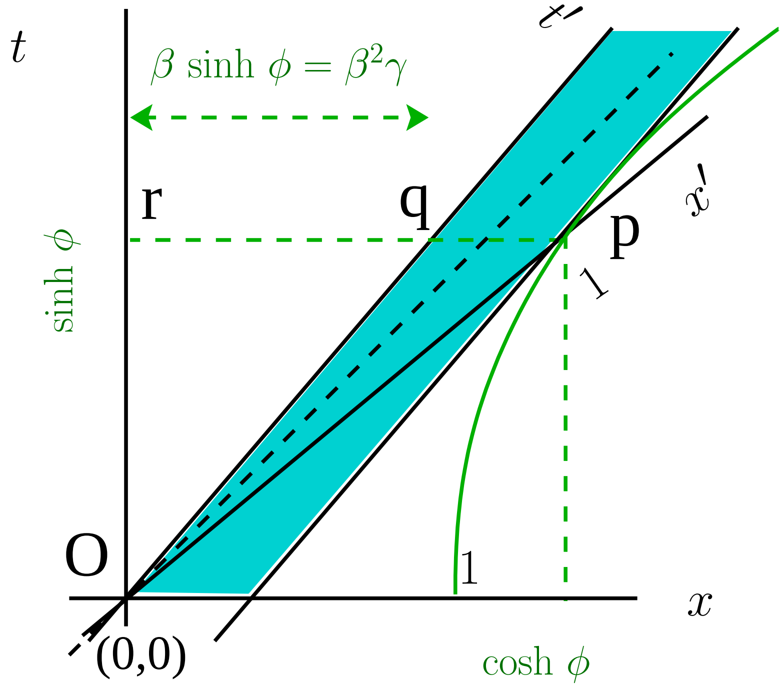
$$\sqrt{\Delta s^2(\mathrm{q},\mathrm{p})} = (1-\beta^2)^{1+(-1/2)}$$

SR: worldsheet space contraction



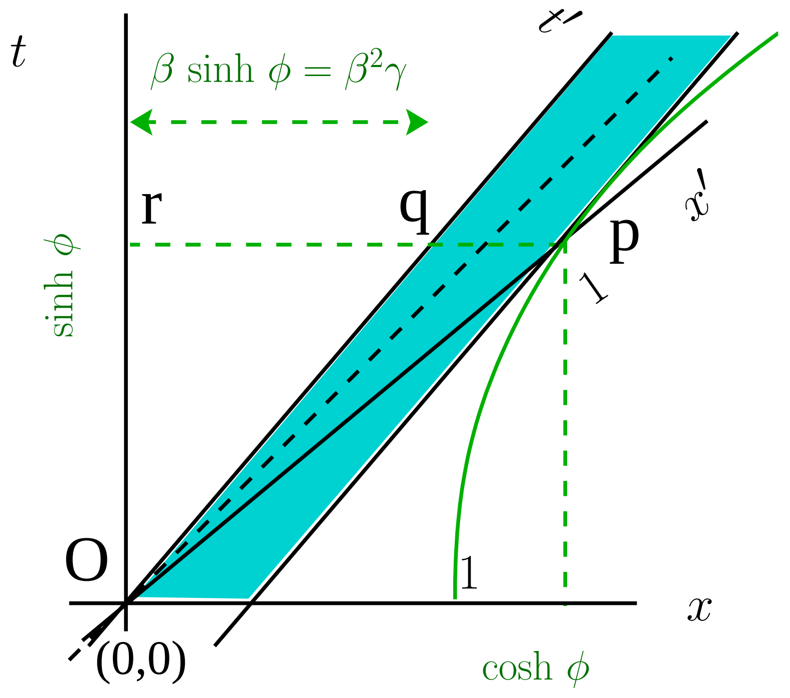
$$\sqrt{\Delta s^2(\mathrm{q},\mathrm{p})} = (1-\beta^2)^{1/2}$$

SR: worldsheet space contraction



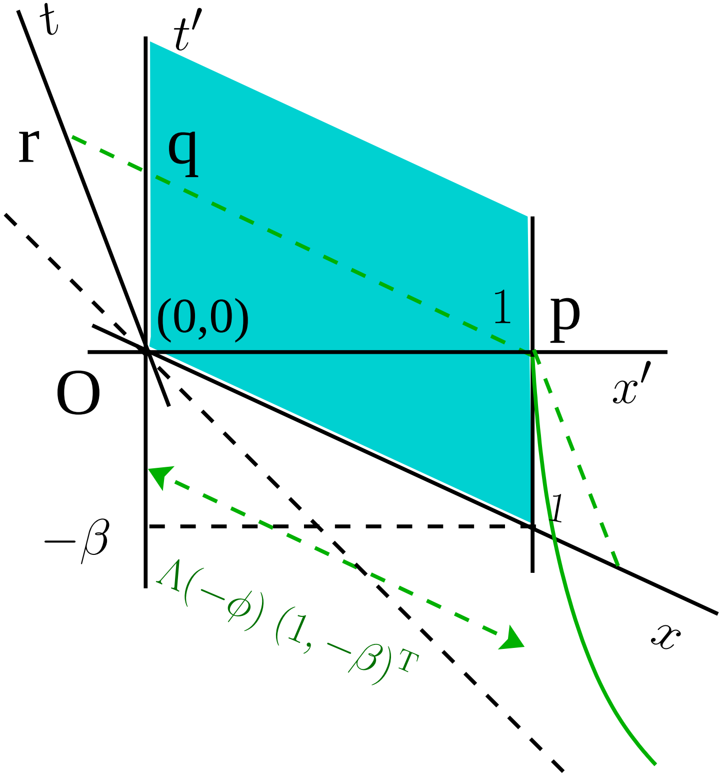
$$\sqrt{\Delta s^2(\mathrm{q},\mathrm{p})} = \gamma^{-1} < 1$$

SR: worldsheet space contraction

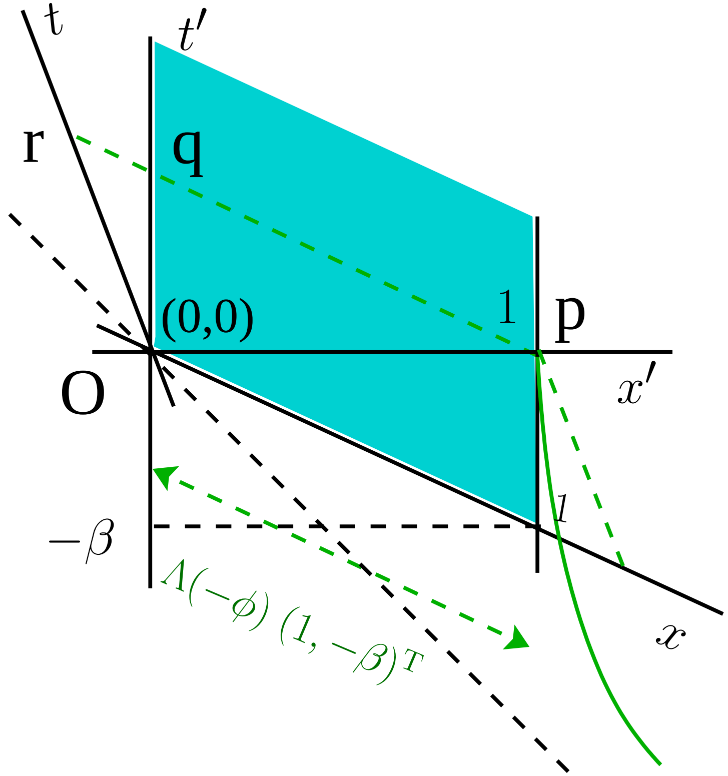


$$\sqrt{\Delta s^2(\mathrm{q},\mathrm{p})} = \gamma^{-1} < 1$$ worldsheet "space contraction"

SR: worldsheet space contraction

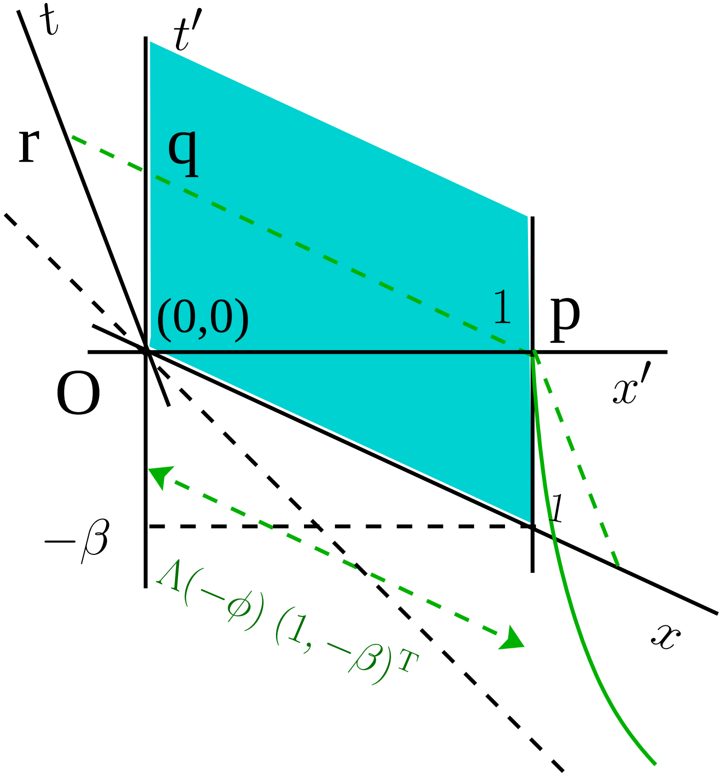


SR: worldsheet space contraction



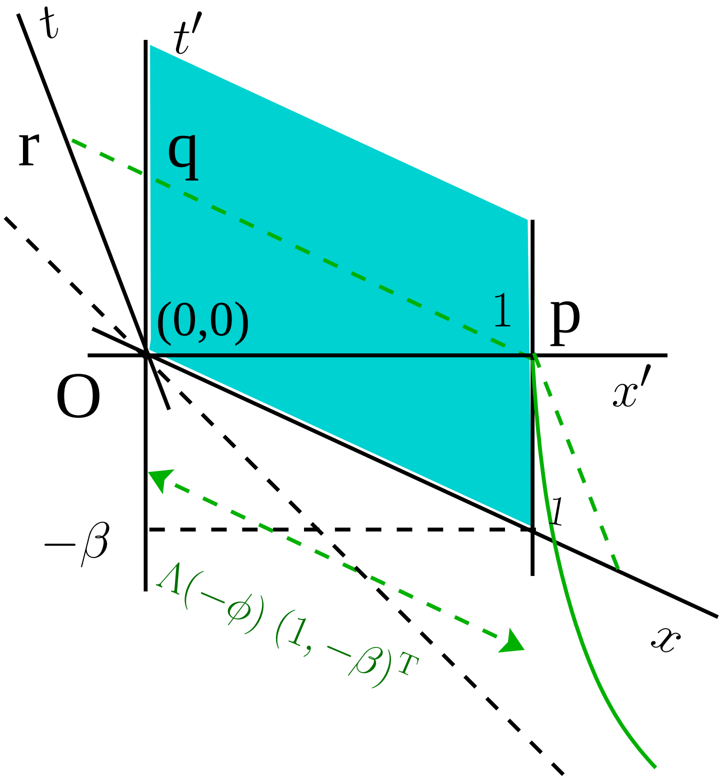
$$\Lambda^{-1}\begin{pmatrix}1\\-\beta\end{pmatrix}=\begin{pmatrix}\cosh\phi-\beta\sinh\phi\\ \sinh\phi-\beta\cosh\phi\end{pmatrix}$$

SR: worldsheet space contraction



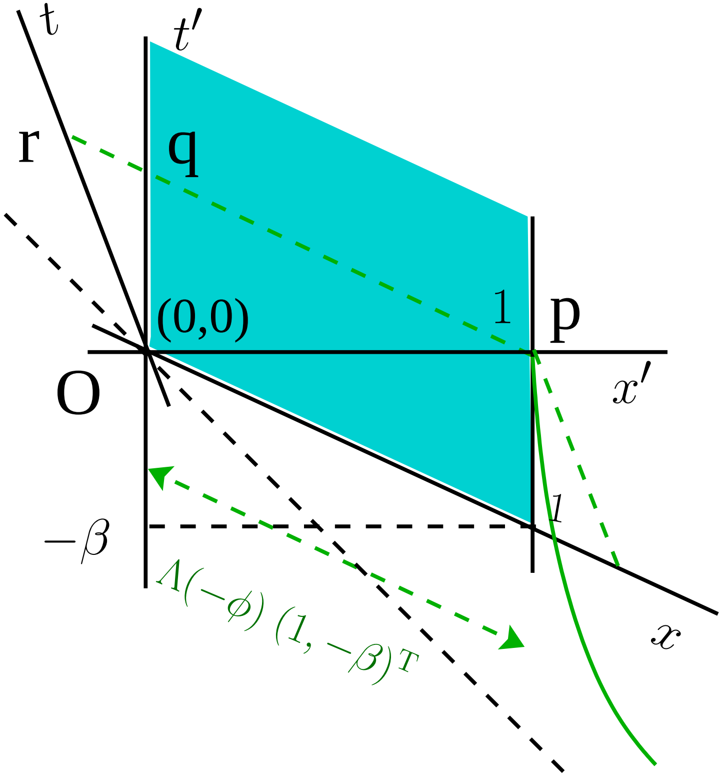
$$\Lambda^{-1}\begin{pmatrix} 1 \\ -\beta \end{pmatrix} = \begin{pmatrix} \cosh\phi(1-\beta^2) \\ \cosh\phi(\tanh\phi - \tanh\phi) \end{pmatrix}$$

SR: worldsheet space contraction



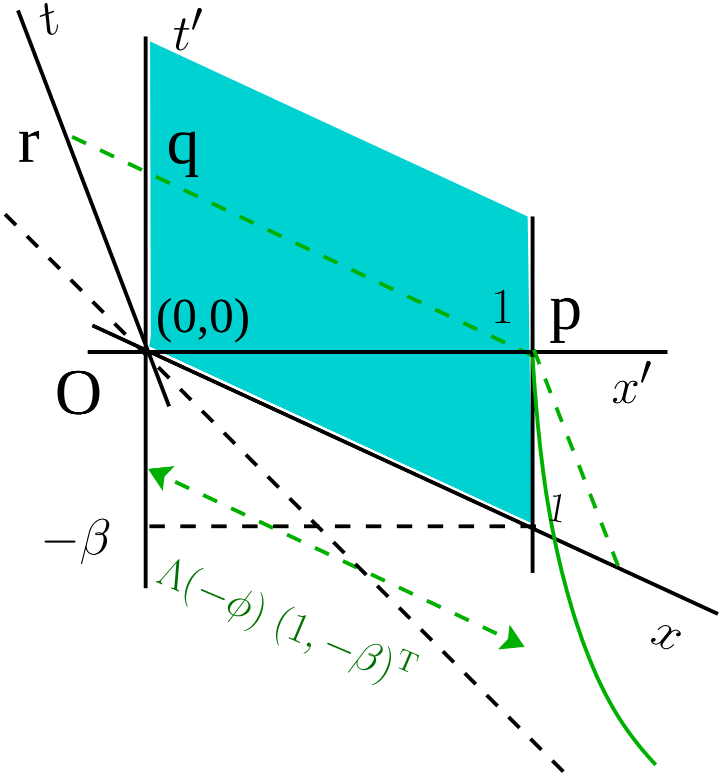
$$\Lambda^{-1}\begin{pmatrix}1\\-\beta\end{pmatrix}=\begin{pmatrix}\gamma(1-\beta^2)\\0\end{pmatrix}$$

SR: worldsheet space contraction



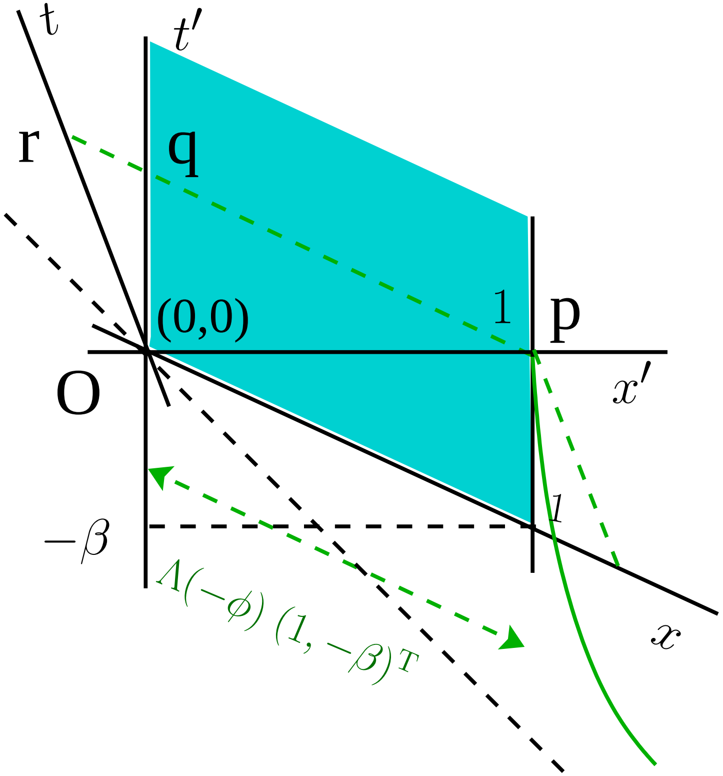
$$\Lambda^{-1}\begin{pmatrix}1\\-\beta\end{pmatrix}=\begin{pmatrix}\gamma^{-1}\\0\end{pmatrix}$$

SR: worldsheet space contraction



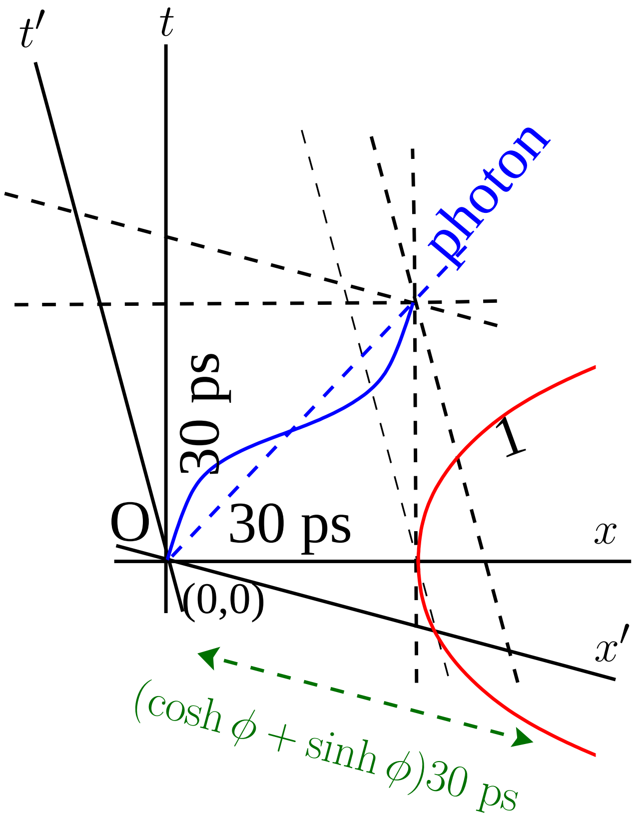
$$\sqrt{\Delta s^2(\mathrm{q},\mathrm{p})} = \gamma^{-1} < 1$$

SR: worldsheet space contraction

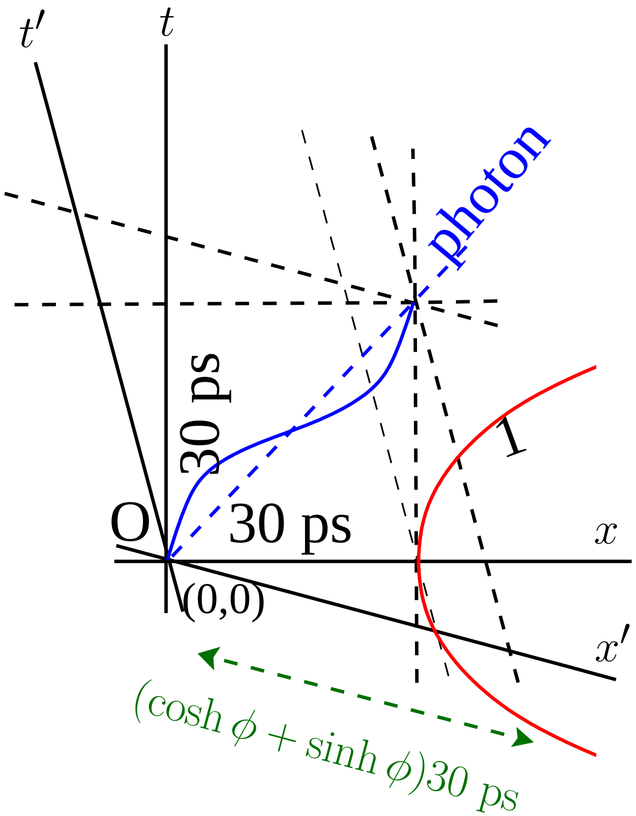


$$\sqrt{\Delta s^2(\mathrm{q},\mathrm{p})} = \gamma^{-1} < 1$$ worldsheet "space contraction"

SR: Doppler shift

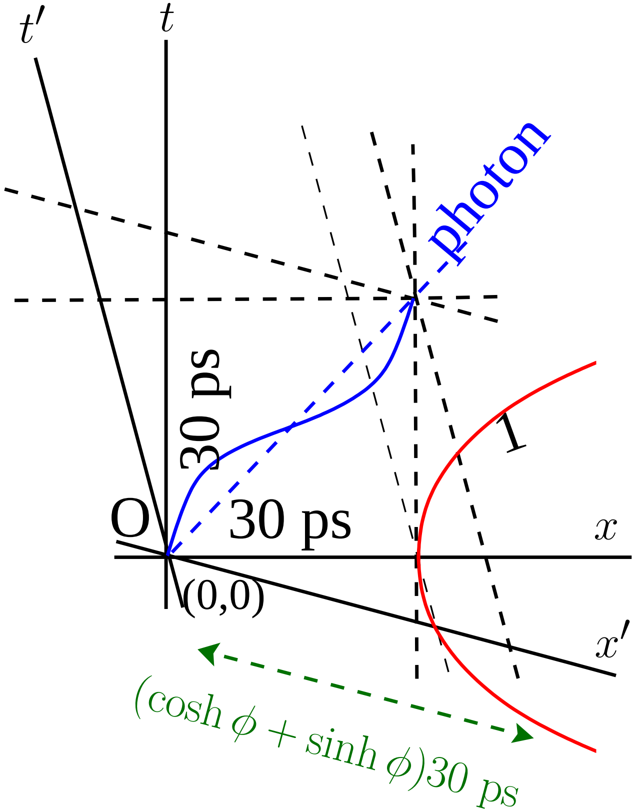


SR: Doppler shift



see photon worldline calculation

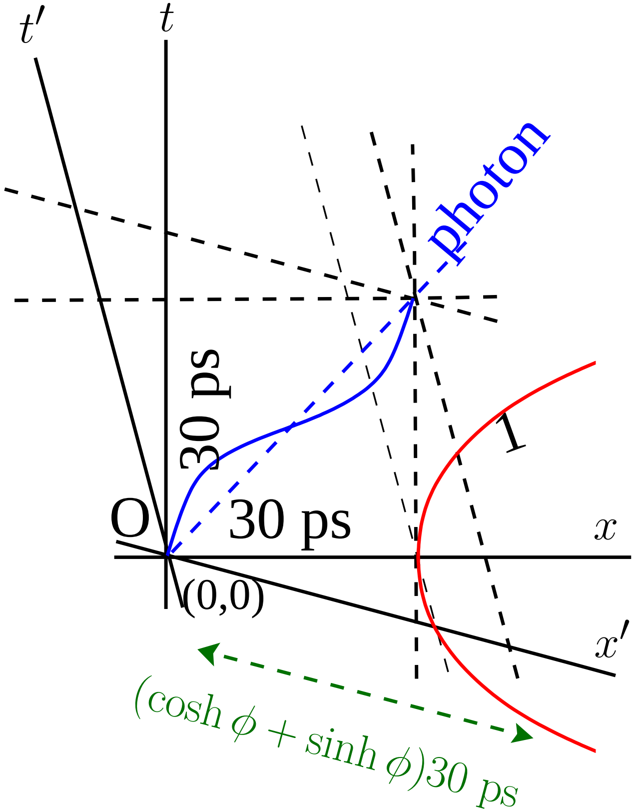
SR: Doppler shift



see photon worldline calculation

$$x' = (\cosh\phi + \sinh\phi)t$$

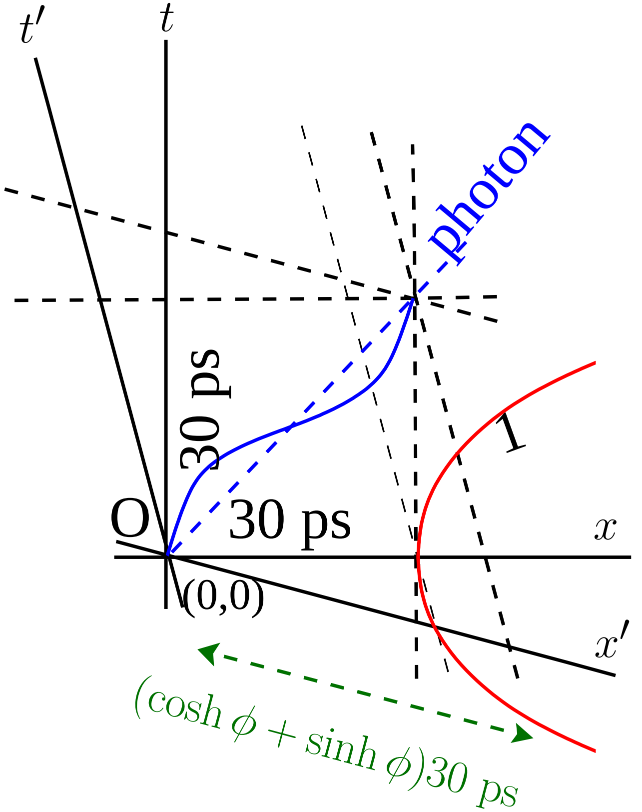
SR: Doppler shift



see photon worldline calculation

$$x' = (\cosh\phi + \sinh\phi)x$$

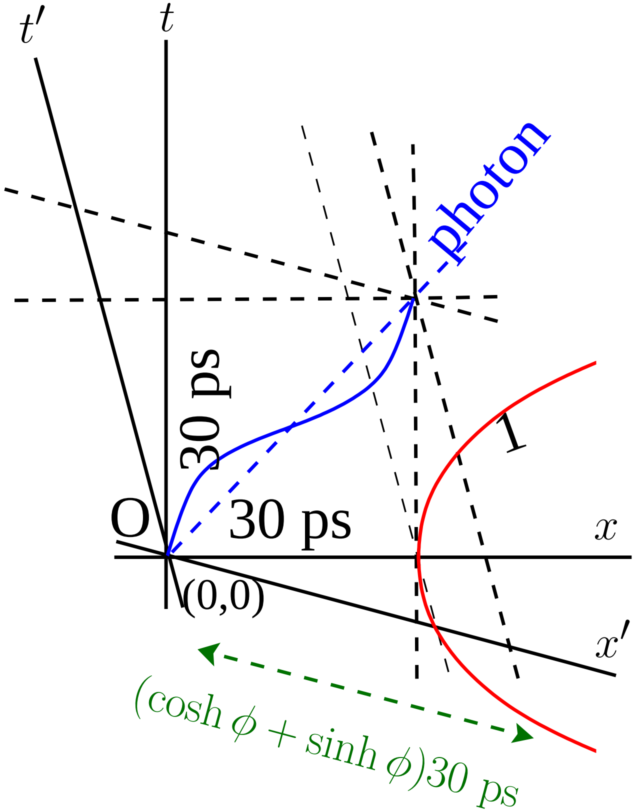
SR: Doppler shift



see photon worldline calculation

$$x'/x = \cosh\phi + \sinh\phi$$

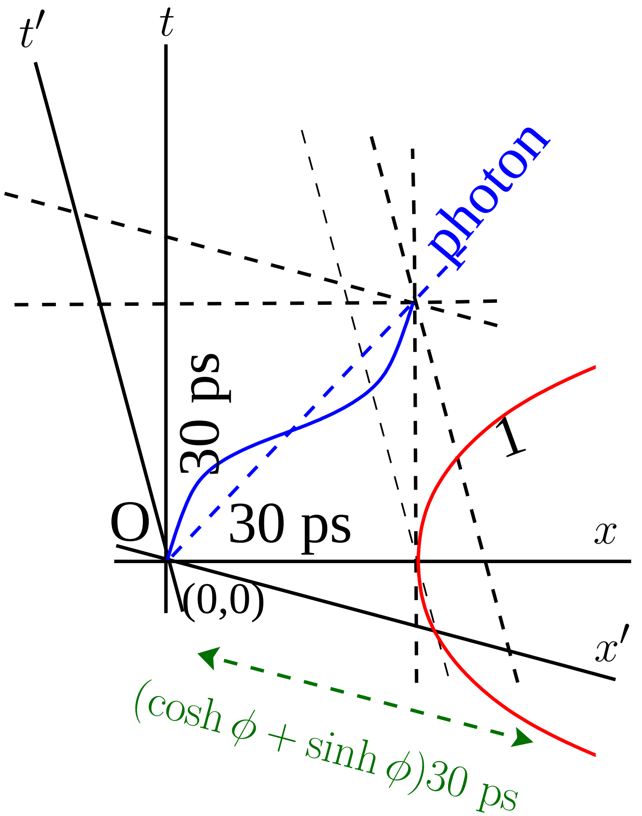
SR: Doppler shift



see photon worldline calculation

$x'/x = \gamma + \beta\gamma$

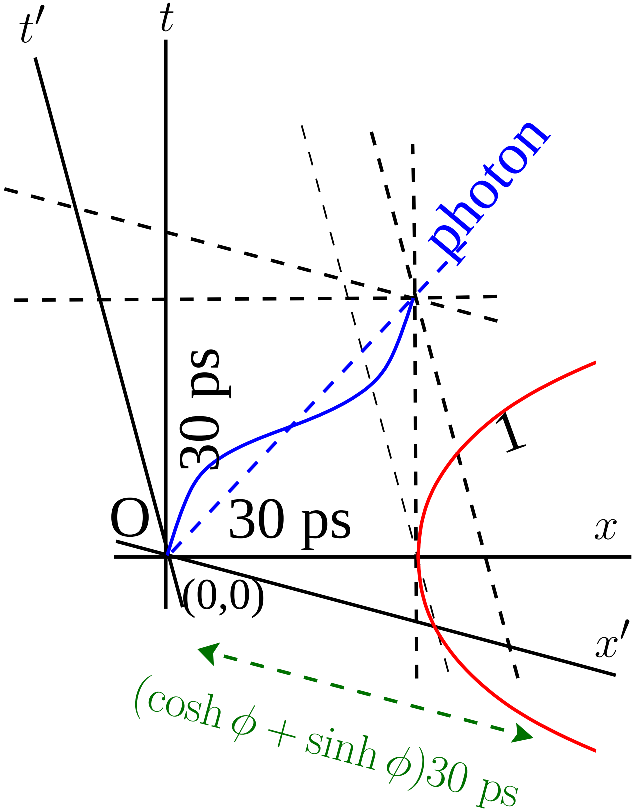
SR: Doppler shift



see photon worldline calculation

$x'/x = \gamma(1+\beta)$

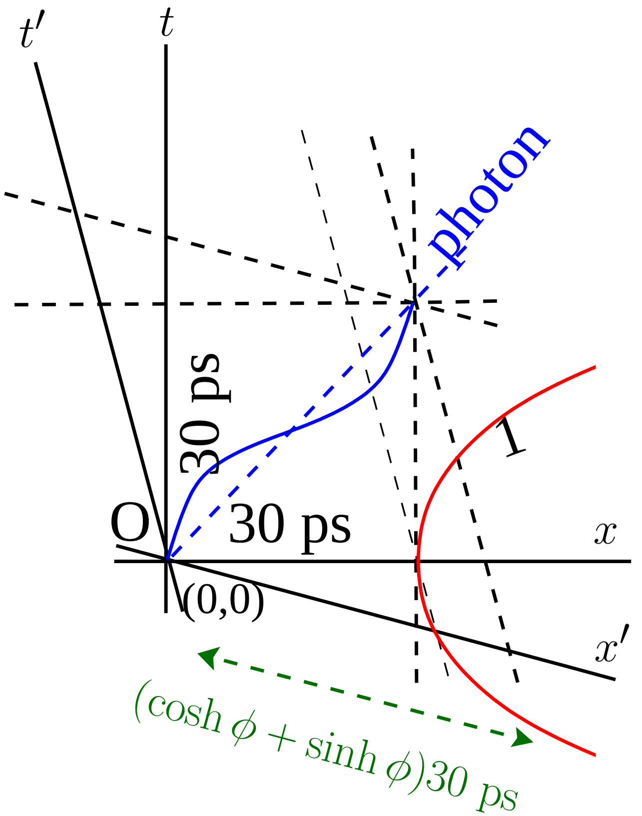
SR: Doppler shift



see photon worldline calculation

$$x'/x = \gamma(1+\beta) = \frac{1+\beta}{\sqrt{1-\beta^2}}$$

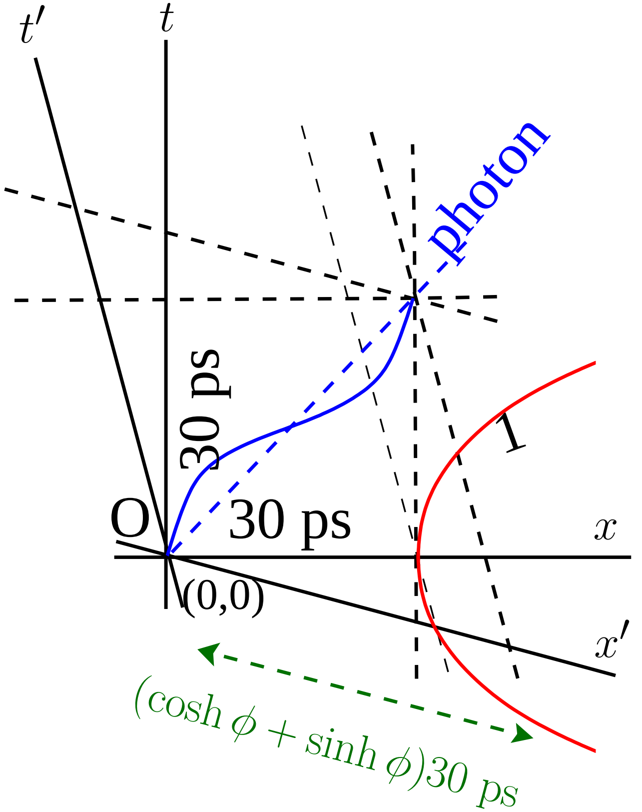
SR: Doppler shift



see photon worldline calculation

$$x'/x = \gamma(1+\beta) = \frac{\sqrt{(1+\beta)^2}}{\sqrt{(1-\beta)(1+\beta)}}$$

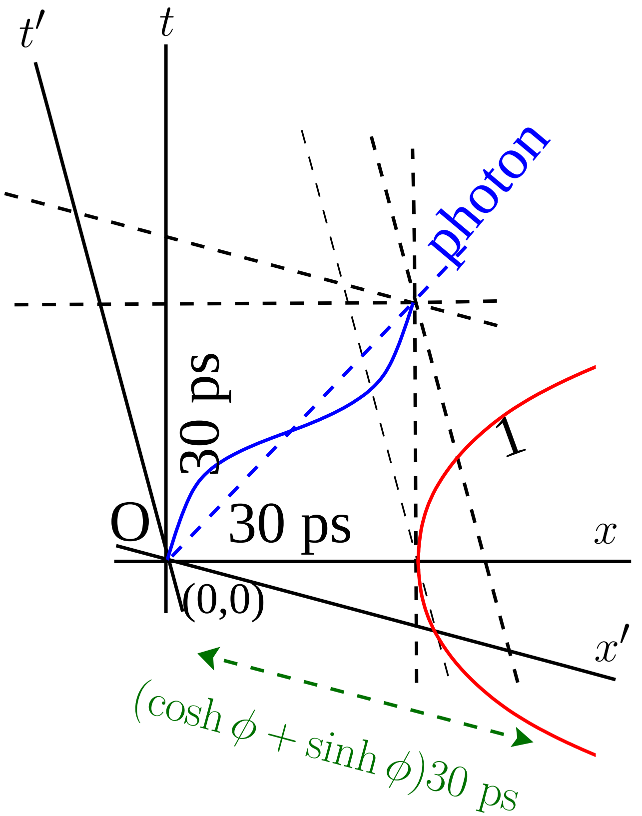
SR: Doppler shift



see photon worldline calculation

$$x'/x = \gamma(1+\beta) = \sqrt{\frac{1+\beta}{1-\beta}}$$

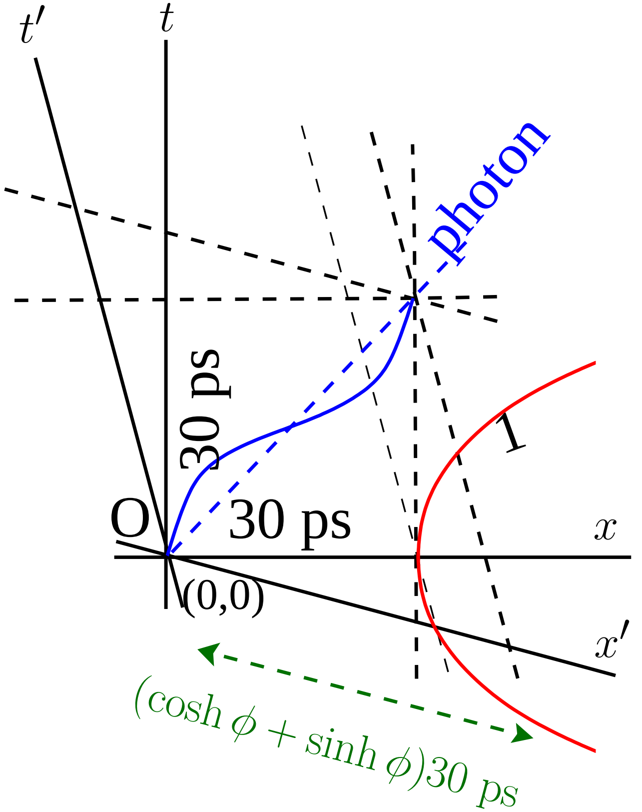
SR: Doppler shift



see photon worldline calculation

$$1 + z := \lambda'/\lambda = \gamma(1+\beta) \equiv \sqrt{\frac{1+\beta}{1-\beta}}$$ redshift

SR: Doppler shift

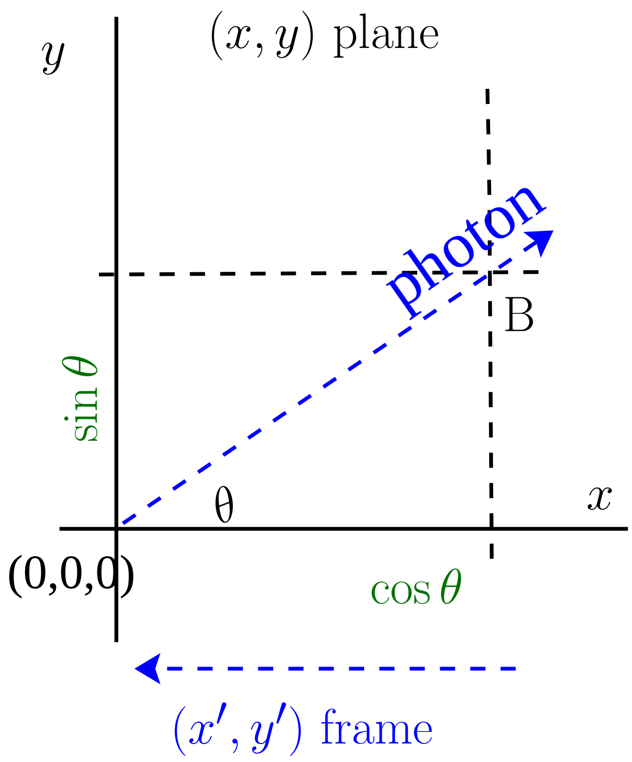


see photon worldline calculation

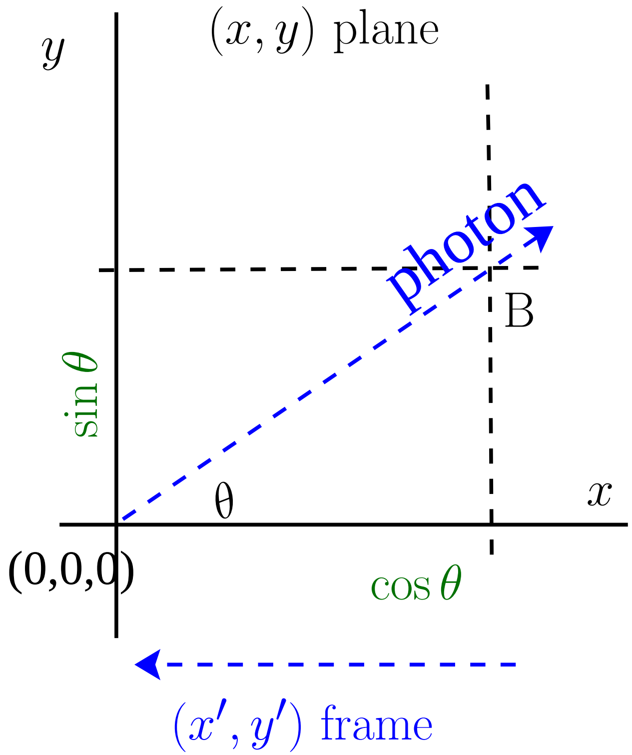
$$1 + z := \lambda'/\lambda = \gamma(1+\beta) \equiv \sqrt{\frac{1+\beta}{1-\beta}}$$ redshift

$\Rightarrow$ when $\beta \ll 1$, $z \approx \beta$

SR: relativistic aberration



SR: relativistic aberration



event B: $(x, y, t) = (\cos\theta, \sin\theta, 1)$

SR: relativistic aberration

event B: $(x, y, t) = (\cos\theta, \sin\theta, 1)$

SR: relativistic aberration

event B: $(x, y, t) = (\cos\theta, \sin\theta, 1)$

$$\Lambda^{-1} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix}$$

SR: relativistic aberration

event B: $(x, y, t) = (\cos\theta, \sin\theta, 1)$

$$\Lambda^{-1} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & \beta\gamma \\ 0 & 1 & 0 \\ \beta\gamma & 0 & \gamma \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix}$$

SR: relativistic aberration

event B: $(x, y, t) = (\cos\theta, \sin\theta, 1)$

$$\Lambda^{-1} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma\cos\theta + \beta\gamma \\ \sin\theta \\ \beta\gamma\cos\theta + \gamma \end{pmatrix}$$

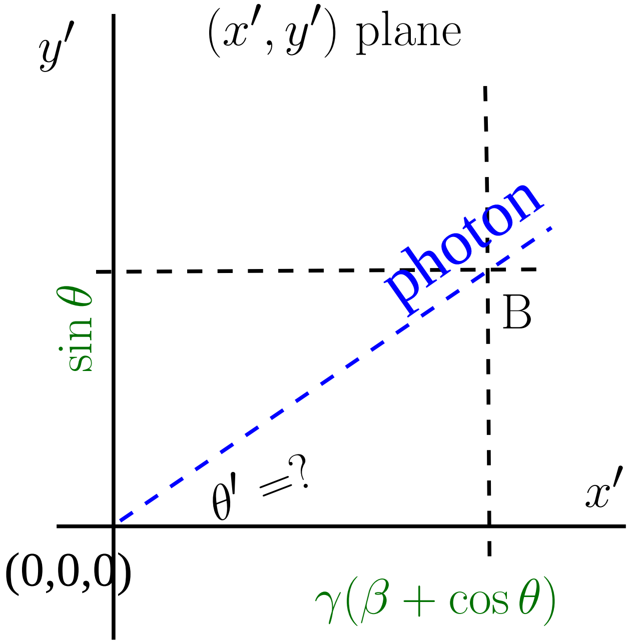
SR: relativistic aberration

event B: $(x, y, t) = (\cos\theta, \sin\theta, 1)$

$$\Lambda^{-1} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma(\cos\theta + \beta) \\ \sin\theta \\ \gamma(1 + \beta\cos\theta) \end{pmatrix}$$

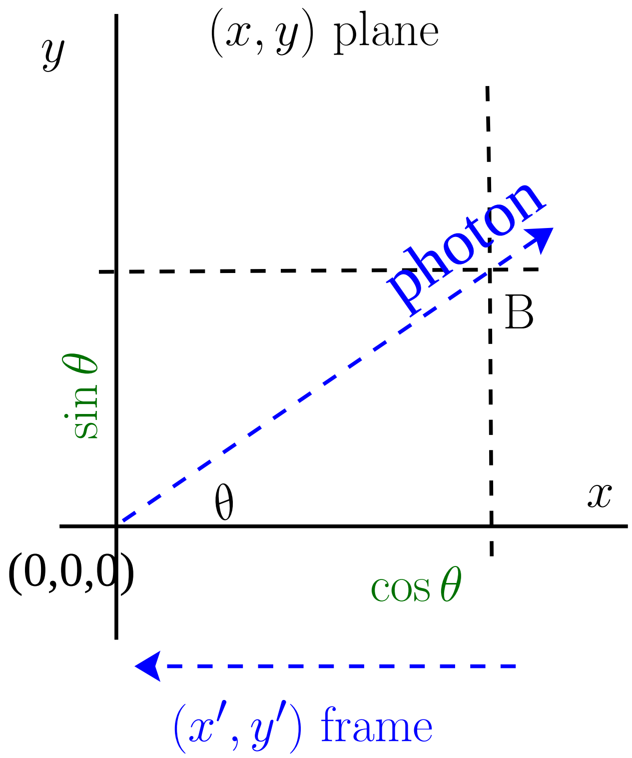
SR: relativistic aberration

event B: $(x, y, t) = (\cos\theta, \sin\theta, 1)$



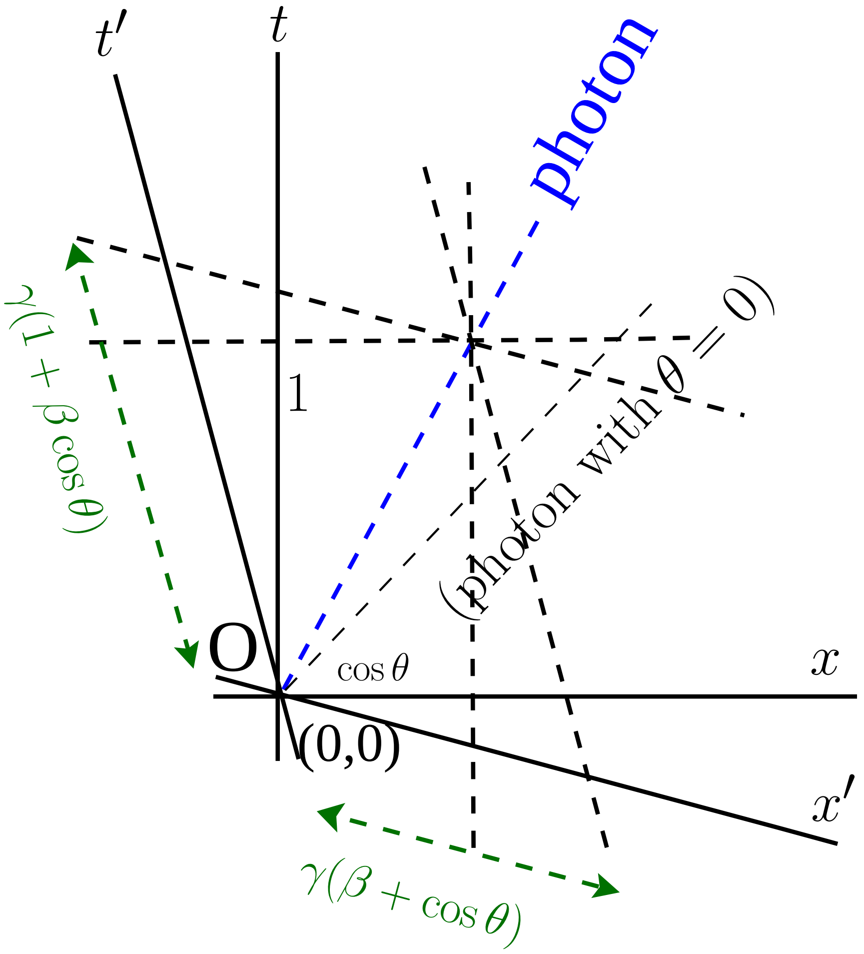
SR: relativistic aberration

event B: $(x, y, t) = (\cos\theta, \sin\theta, 1)$



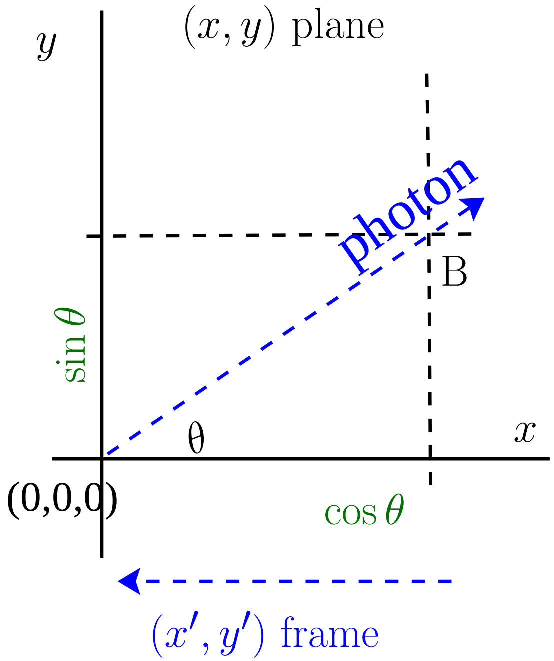
SR: relativistic aberration

event B: $(x, y, t) = (\cos\theta, \sin\theta, 1)$



SR: relativistic aberration

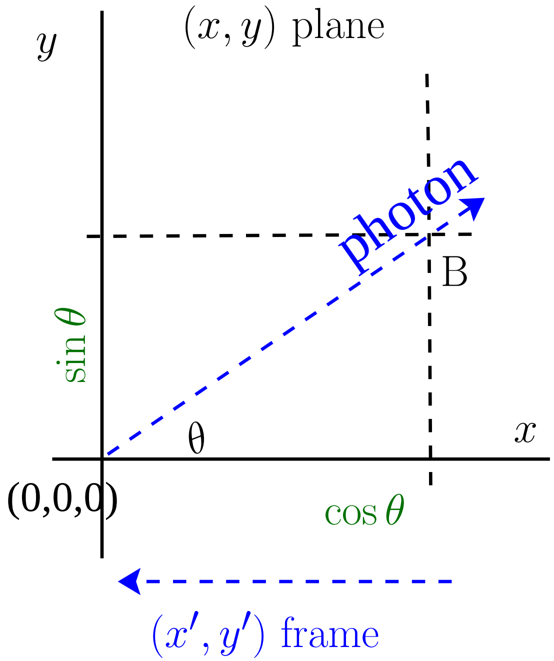
event B: $(x, y, t) = (\cos\theta, \sin\theta, 1)$



$$\tan\theta' = \frac{\sin\theta}{\gamma(\beta + \cos\theta)}$$

SR: relativistic aberration

event B: $(x, y, t) = (\cos\theta, \sin\theta, 1)$

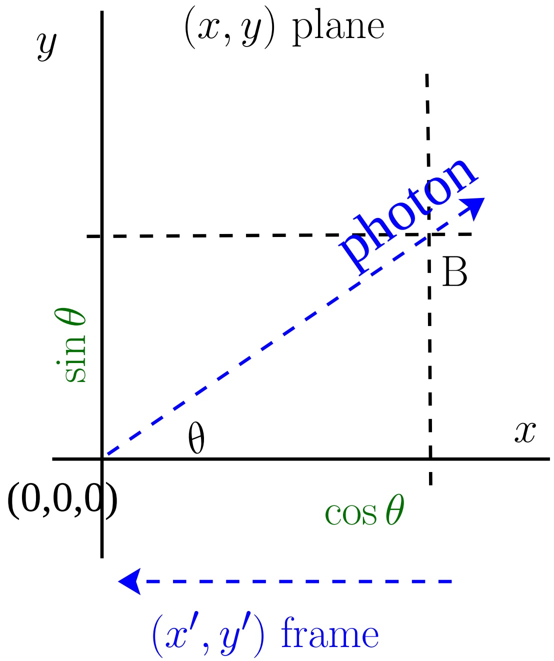


$\tan\theta' = \frac{\sin\theta}{\gamma(\beta+\cos\theta)} < \tan\theta$ if $0 < \beta < 1$

w:Relativistic aberration (2011-02-22: quality=weak)

SR: relativistic aberration

event B: $(x, y, t) = (\cos\theta, \sin\theta, 1)$

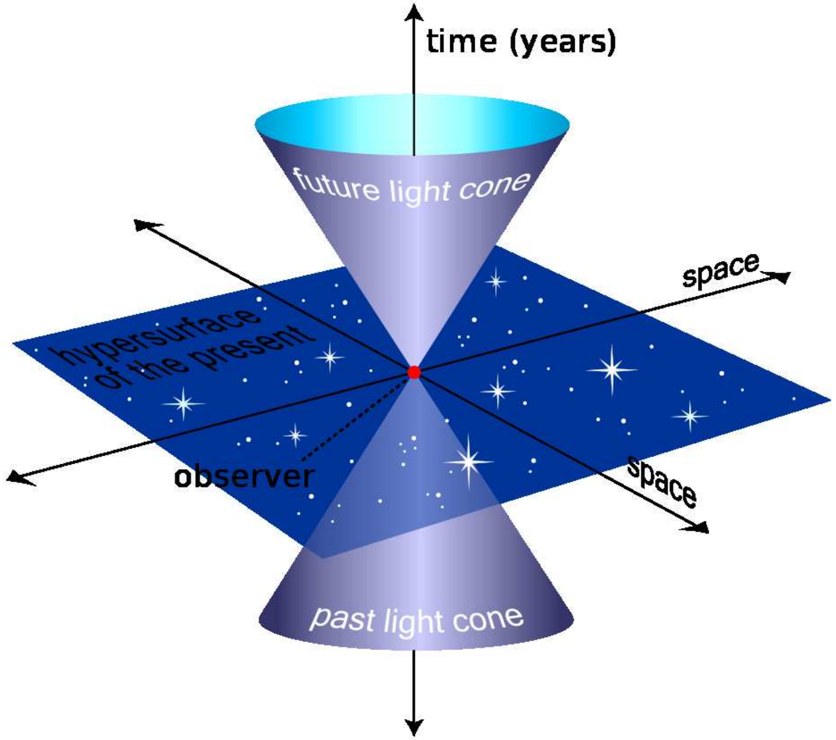


$\tan\theta' = \frac{\sin\theta}{\gamma(\beta+\cos\theta)} < \tan\theta$ if $0 < \beta < 1$

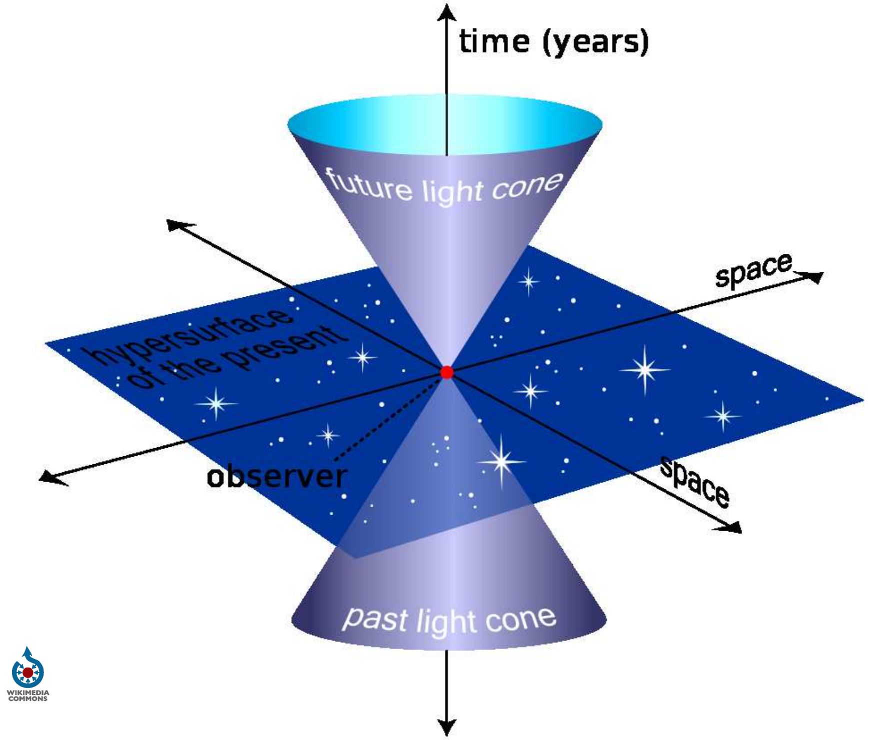
w:Relativistic aberration (2011-02-22: quality=weak)

$\Rightarrow$ relativistic beaming, e.g. AGN jets

SR: world line



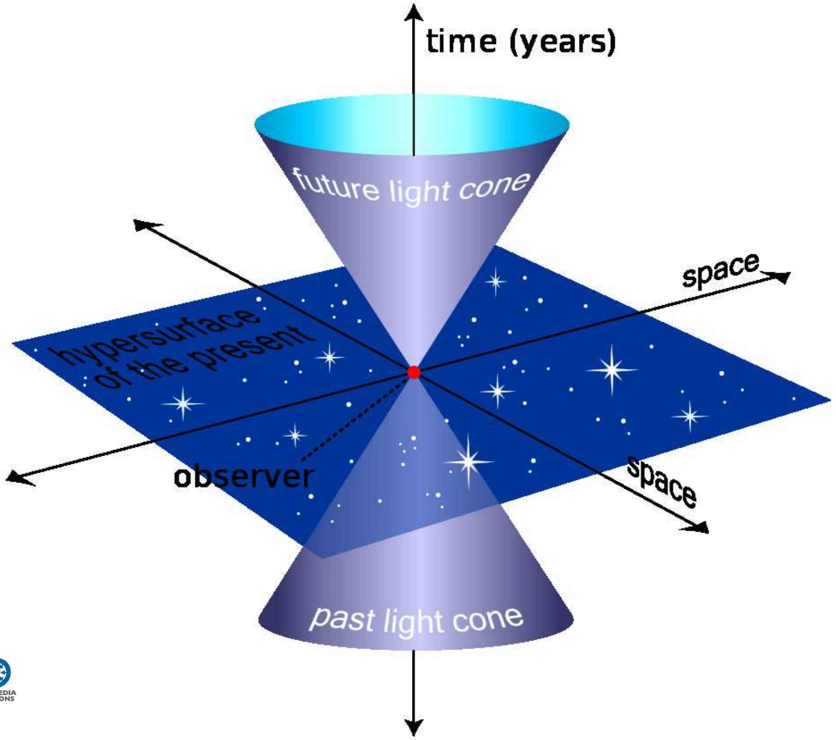
SR: world line



lightlike interval = null interval: $(\Delta s)^2 = 0$

spacetime =

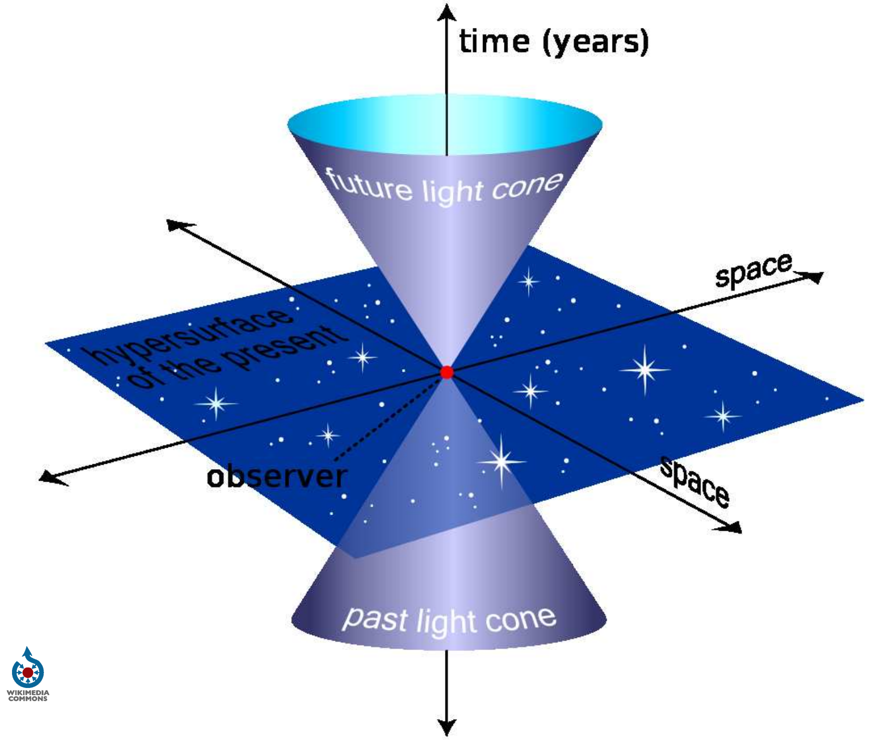
SR: world line



lightlike interval = null interval: $(\Delta s)^2 = 0$

spacetime = on past w:light cone + inside past light cone

SR: world line

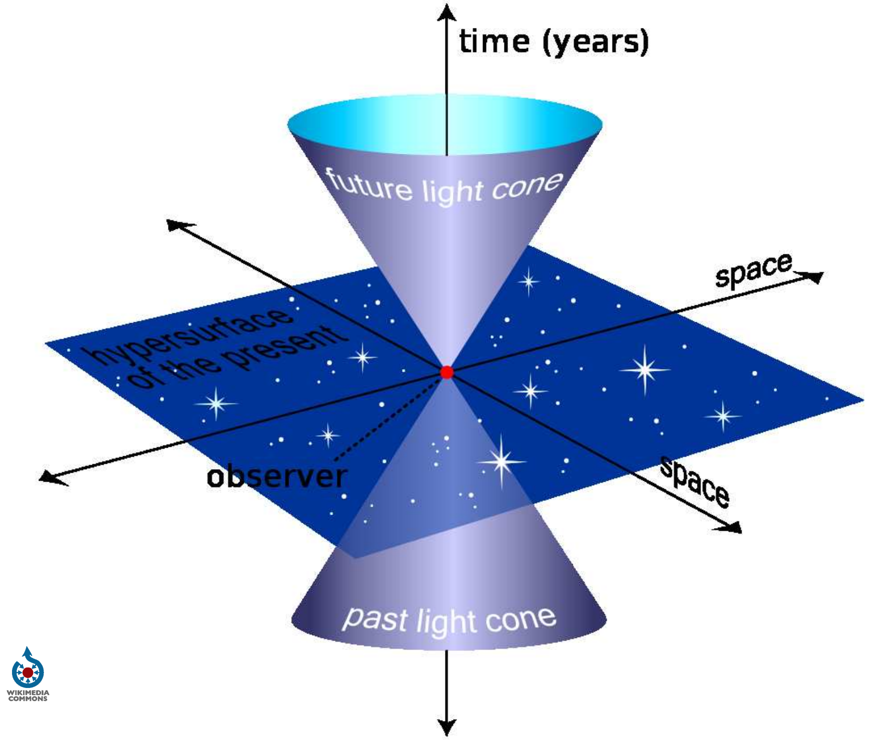


lightlike interval = null interval: $(\Delta s)^2 = 0$

spacetime = on past w:light cone + inside past light cone

+ on future light cone + inside future light cone

SR: world line



lightlike interval = null interval: $(\Delta s)^2 = 0$

spacetime = on past w:light cone + inside past light cone

+ on future light cone + inside future light cone

+ elsewhere

SR: world line

Lorentz transform of world line

SR: world line

Lorentz transform of world line

SR: world line

Lorentz transform of world line

• coordinate time in spacetime model $\neq$ time in your brain (thinking)

SR: world line

Lorentz transform of world line

- coordinate time in spacetime model $\neq$ time in your brain (thinking)
- $\frac{\mathrm{d}t}{\mathrm{d}t_{\mathrm{thinking}}}$ can be positive or negative

SR: world line

Lorentz transform of world line

- coordinate time in spacetime model $\neq$ time in your brain (thinking)
- $\frac{\mathrm{d}t}{\mathrm{d}\lambda}$ can be positive or negative, λ arbitrary real parameter

SR: world line

Lorentz transform of world line

- coordinate time in spacetime model $\neq$ time in your brain (thinking)
- $\frac{\mathrm{d}t}{\mathrm{d}\lambda}$ can be positive or negative, λ arbitrary real parameter
- "elsewhere" spacetime events can change from past to future even though $\frac{\mathrm{d}t}{\mathrm{d}\lambda} > 0$

SR: world line

Lorentz transform of world line

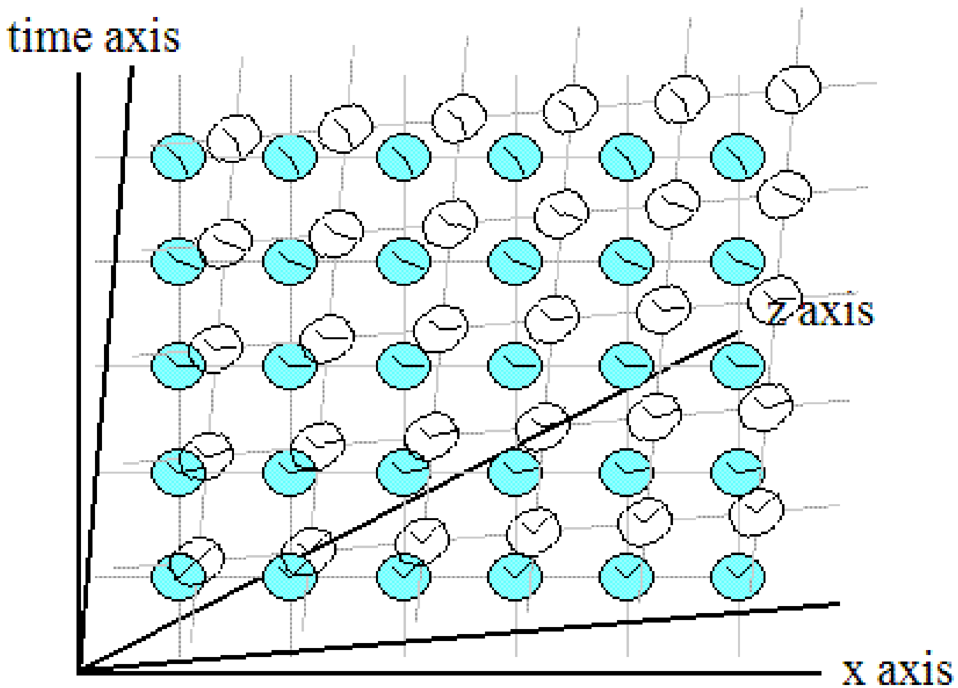
- coordinate time in spacetime model $\neq$ time in your brain (thinking)
- $\frac{\mathrm{d}t}{\mathrm{d}\lambda}$ can be positive or negative, λ arbitrary real parameter
- “elsewhere” spacetime events can change from past to future even though $\frac{\mathrm{d}t}{\mathrm{d}\lambda} > 0$
- w:proper time $\tau :=$ time along a worldline measured by clock following that worldline

SR: world line

Lorentz transform of world line

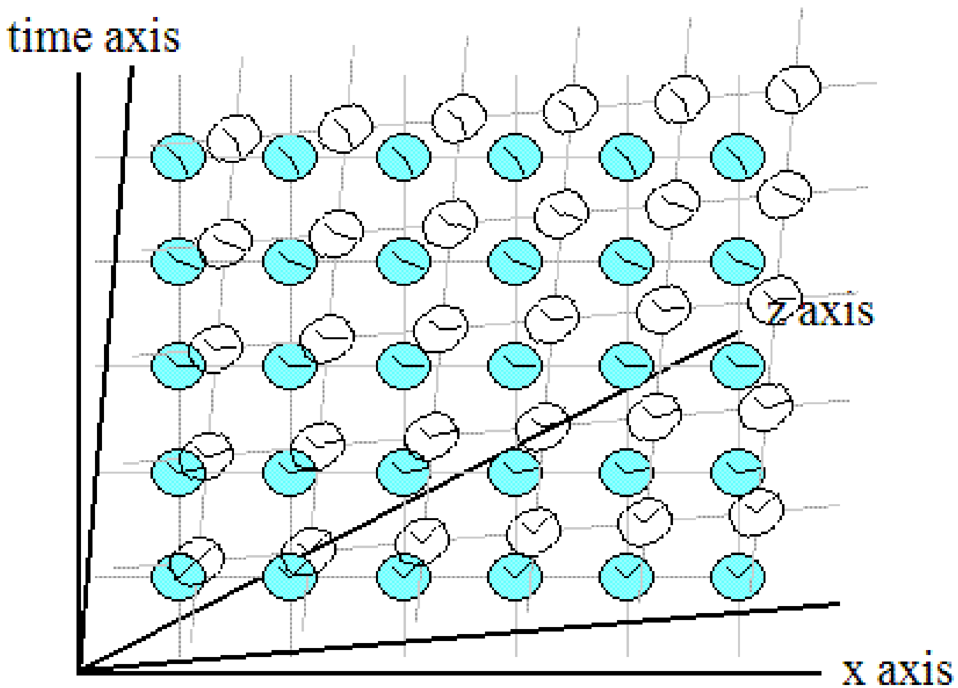
- coordinate time in spacetime model $\neq$ time in your brain (thinking)
- $\frac{\mathrm{d}t}{\mathrm{d}\lambda}$ can be positive or negative, λ arbitrary real parameter
- “elsewhere” spacetime events can change from past to future even though $\frac{\mathrm{d}t}{\mathrm{d}\lambda} > 0$
- w:proper time $\tau :=$ time along a worldline measured by clock following that worldline
- often $\mathrm{d}\tau$ is useful for integrating

SR: Rietdijk–Putnam–Penrose p.



Relativity shows that the inertial frames of reference of relatively moving objects do not overlie each other.

SR: Rietdijk–Putnam–Penrose p.

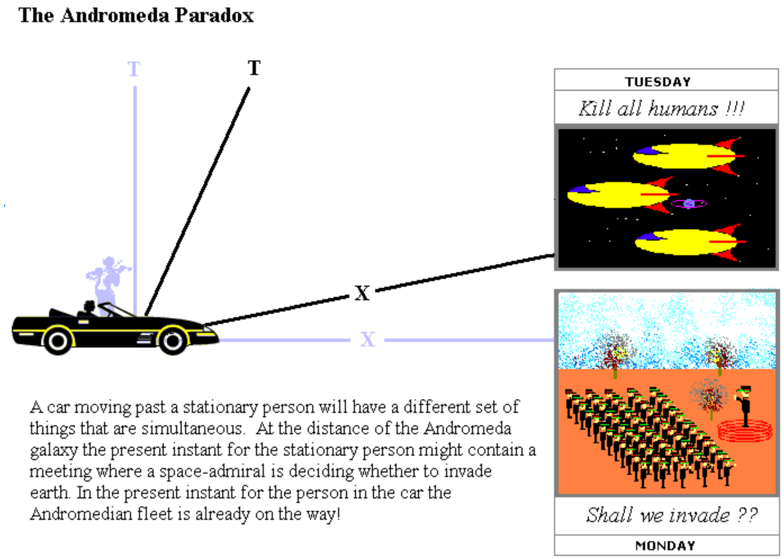


Relativity shows that the inertial frames of reference of relatively moving objects do not overlie each other.

b:Inertialoverlay.GIF

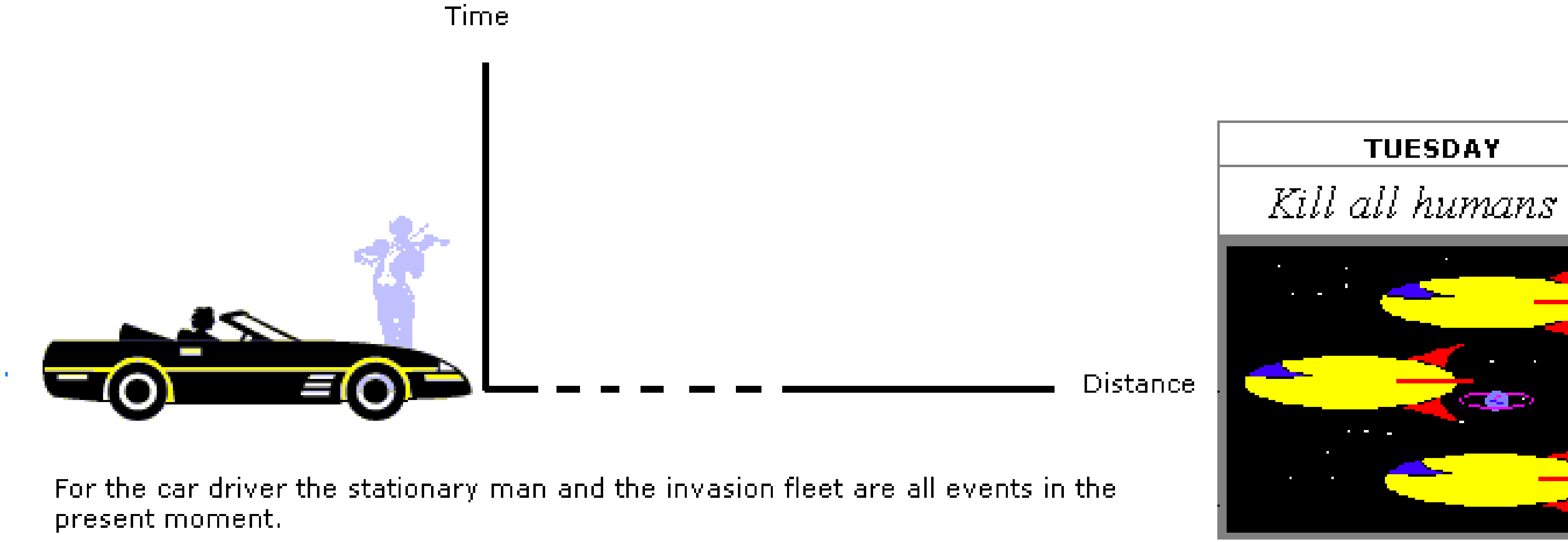
- each observer can synchronise clocks + rods

SR: Rietdijk–Putnam–Penrose p.



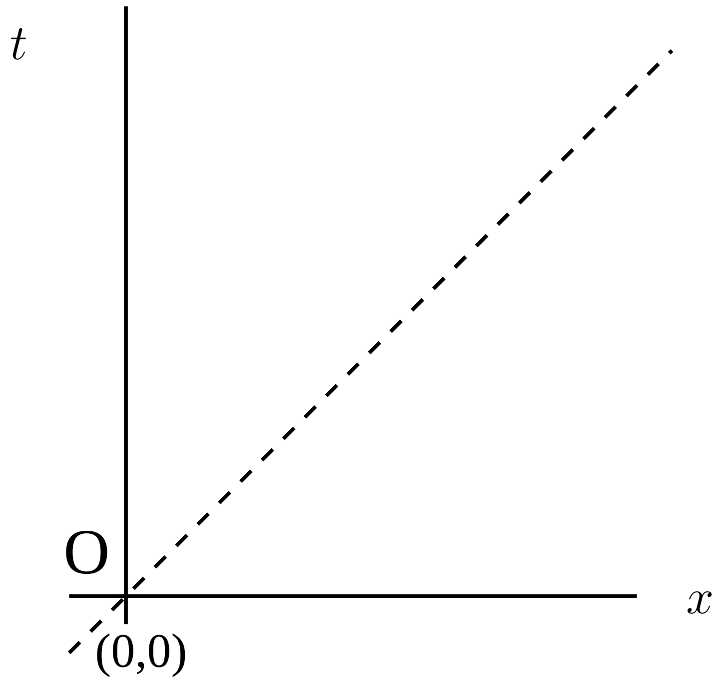
w:Rietdijk-Putnam argument b:Rel2.gif

SR: Rietdijk–Putnam–Penrose p.



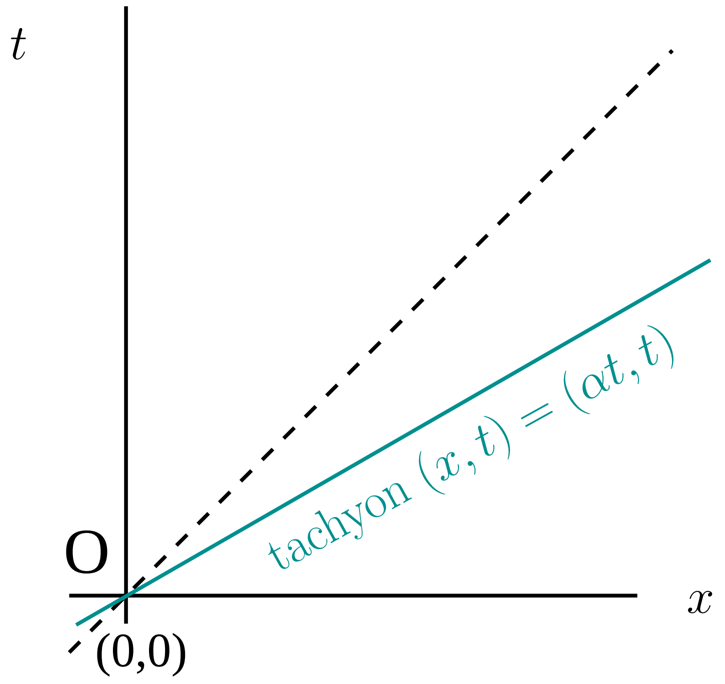
w:Rietdijk-Putnam argument b:Rel3.gif

SR: tachyons and causality



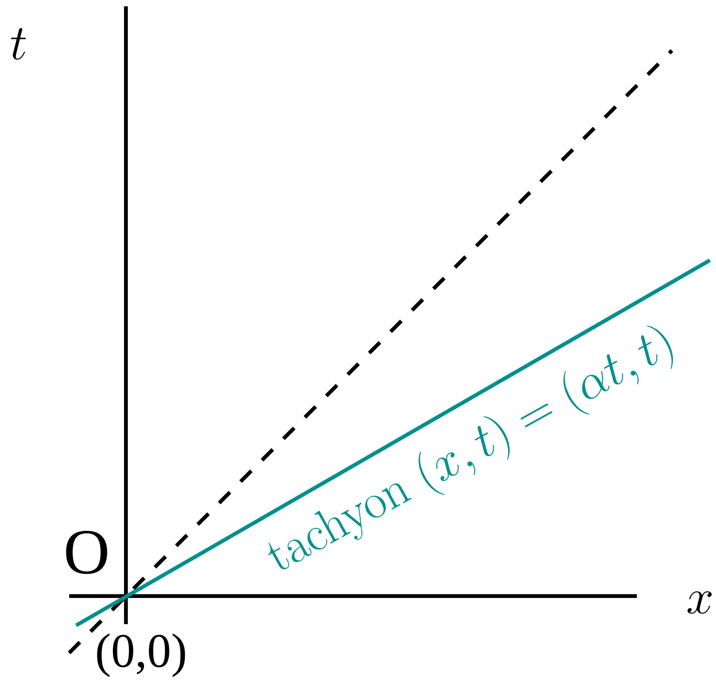
observer “at rest”

SR: tachyons and causality



add a tachyon with speed $\alpha > 1$

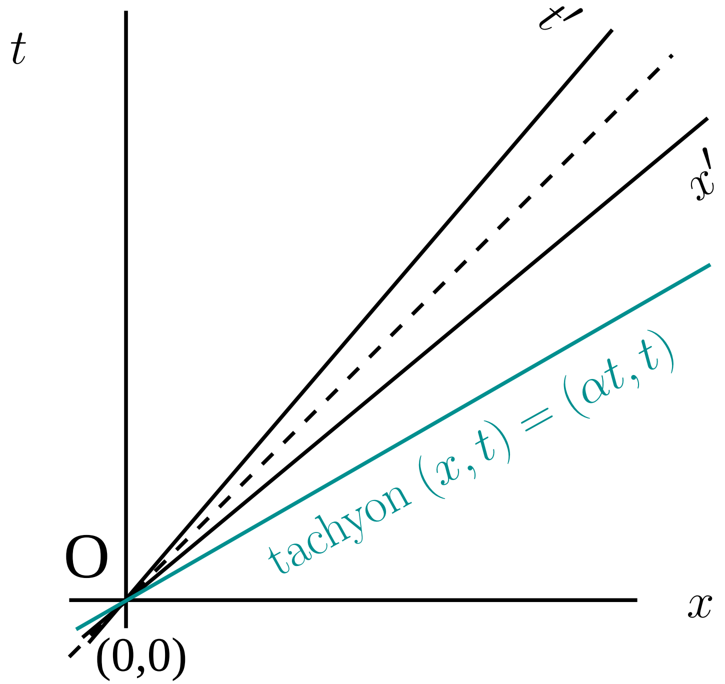
SR: tachyons and causality



add a tachyon with speed $\alpha > 1$

choose rocket at speed β with $1/\alpha < \beta < 1$

SR: tachyons and causality

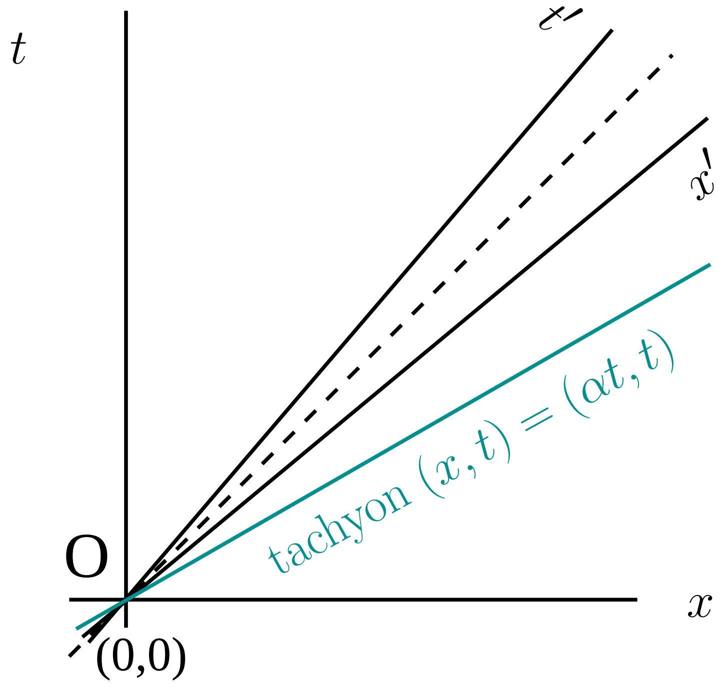


add a tachyon with speed $\alpha > 1$

choose rocket at speed β with $1/\alpha < \beta < 1$

add axes x', t' for the rocket

SR: tachyons and causality



add a tachyon with speed $\alpha > 1$

choose rocket at speed β with $1/\alpha < \beta < 1$

add axes x', t' for the rocket

rocket frame: $(\alpha t, t)$ becomes $\Lambda\ (\alpha t, t)^{\mathrm{T}}$

SR: tachyons and causality

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t \\ t \end{pmatrix} =$$

SR: tachyons and causality

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t \\ t \end{pmatrix} = \begin{pmatrix} \gamma\alpha t - \beta\gamma t \\ -\alpha\beta\gamma t + \gamma t \end{pmatrix}$$

SR: tachyons and causality

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t \\ t \end{pmatrix} = \gamma t \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

SR: tachyons and causality

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t \\ t \end{pmatrix} = \gamma t \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$x' = \gamma t(\alpha - \beta) > 0$ since $\alpha > 1 > \beta$

SR: tachyons and causality

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t \\ t \end{pmatrix} = \gamma t \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$x' = \gamma t(\alpha - \beta) > 0$ since $\alpha > 1 > \beta$

$t' = \gamma t(-\alpha\beta + 1) < 0$ since we chose $\beta > 1/\alpha$

SR: tachyons and causality

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t \\ t \end{pmatrix} = \gamma t \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$x' = \gamma t(\alpha - \beta) > 0$ since $\alpha > 1 > \beta$

$t' = \gamma t(-\alpha\beta + 1) < 0$ since we chose $\beta > 1/\alpha$

$\mathrm{d}t'/\mathrm{d}t < 0$

same sequence of spacetime events = tachyon worldline:

t increases for observer “at rest”,

t' decreases for rocket observer (with $\beta > 1/\alpha$)

SR: tachyons and causality

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t \\ t \end{pmatrix} = \gamma t \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$x' = \gamma t(\alpha - \beta) > 0$ since $\alpha > 1 > \beta$

$t' = \gamma t(-\alpha\beta + 1) < 0$ since we chose $\beta > 1/\alpha$

$\mathrm{d}t'/\mathrm{d}t < 0$

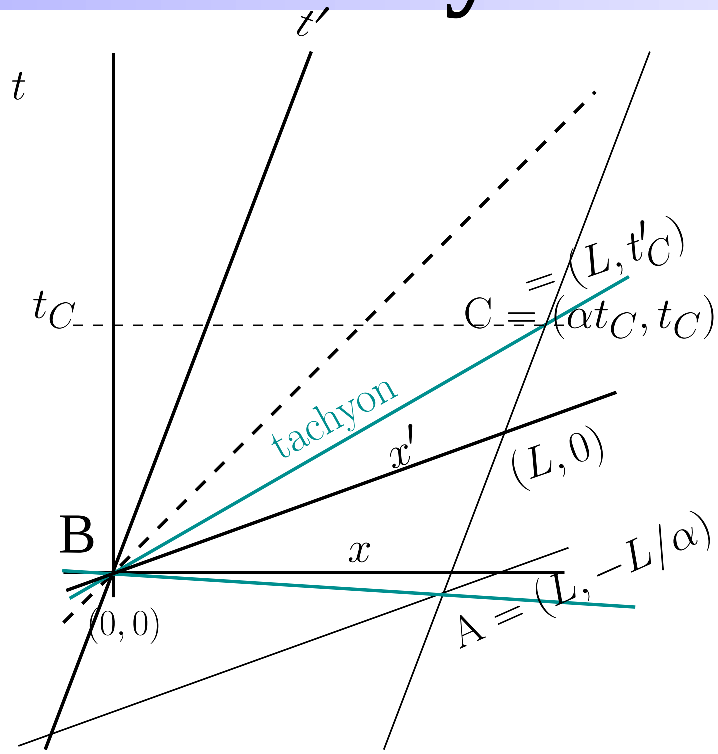
same sequence of spacetime events = tachyon worldline:

t increases for observer “at rest”,

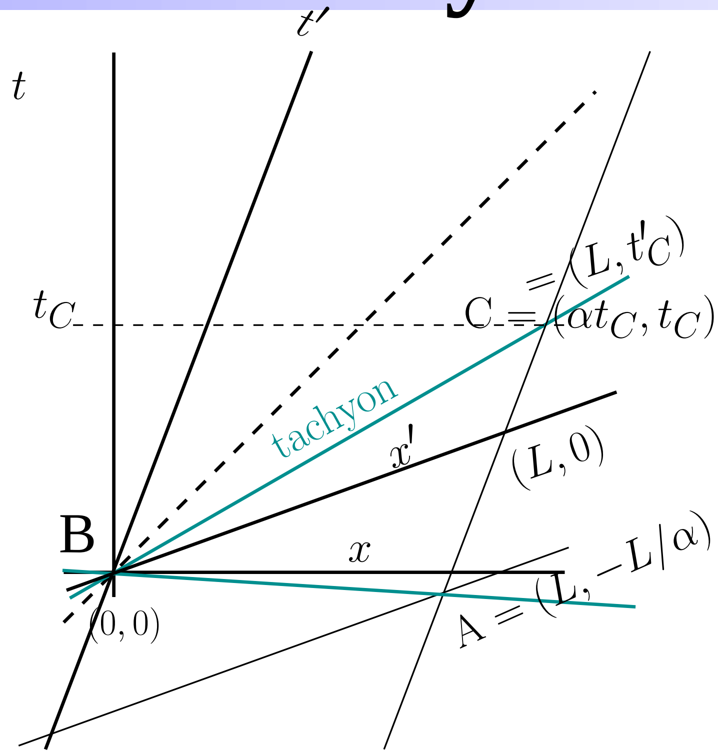
t' decreases for rocket observer (with $\beta > 1/\alpha$)

- observer at rest: tachyon emitted at origin
- rocket: tachyon absorbed at origin

SR: tachyonic antitelephone

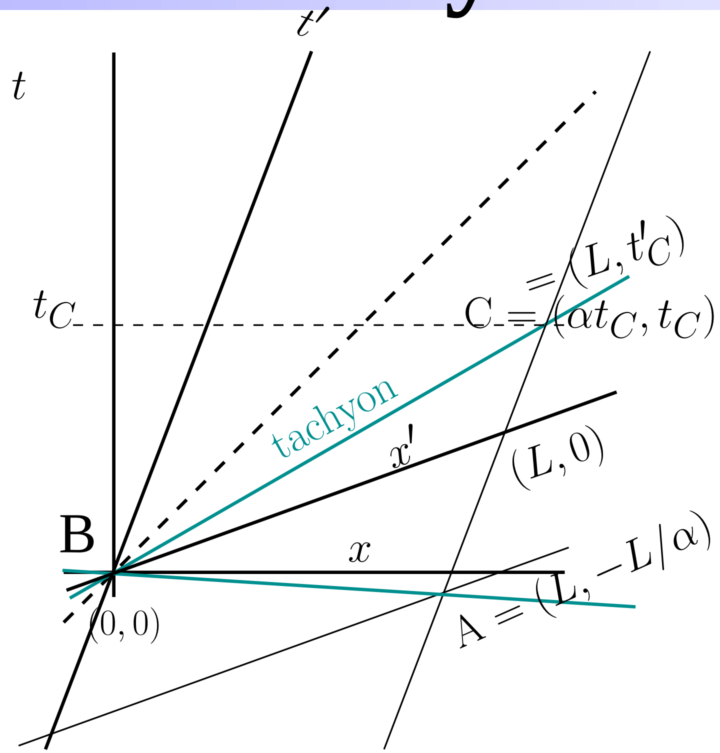


SR: tachyonic antitelephone



B stationary: (x, t) frame

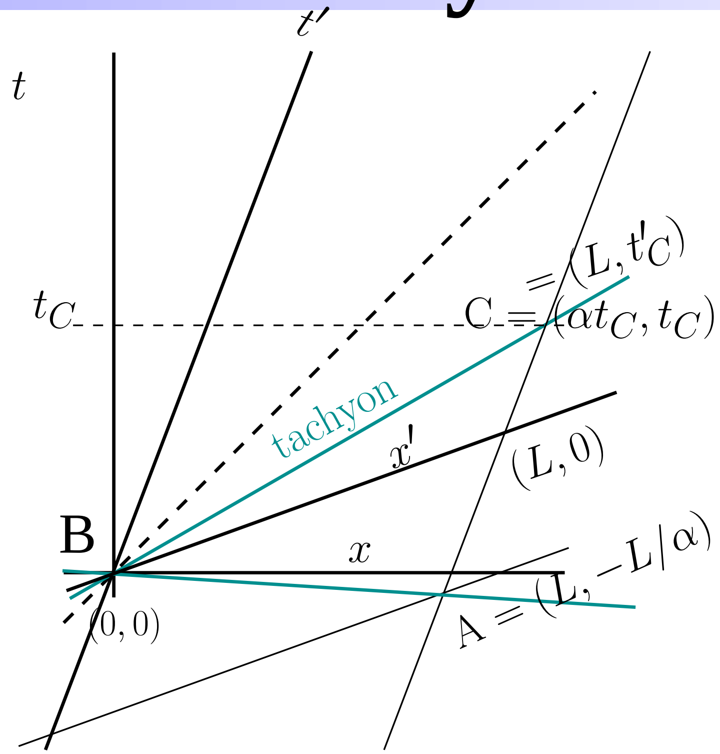
SR: tachyonic antitelephone



B stationary: (x, t) frame

A moving at speed β: (x', t') frame

SR: tachyonic antitelephone

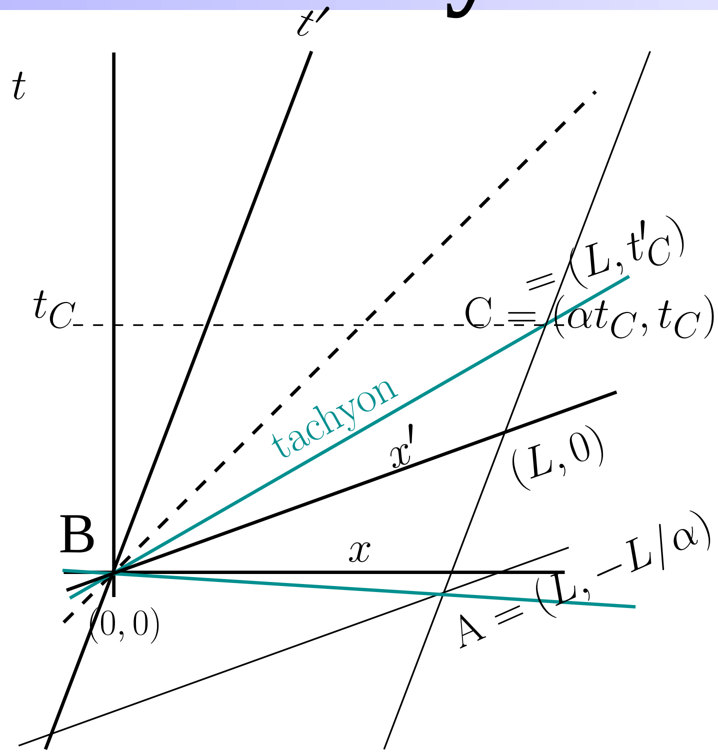


B stationary: (x, t) frame

A moving at speed β: (x', t') frame

A: tachyon at $\alpha > 1$ to B

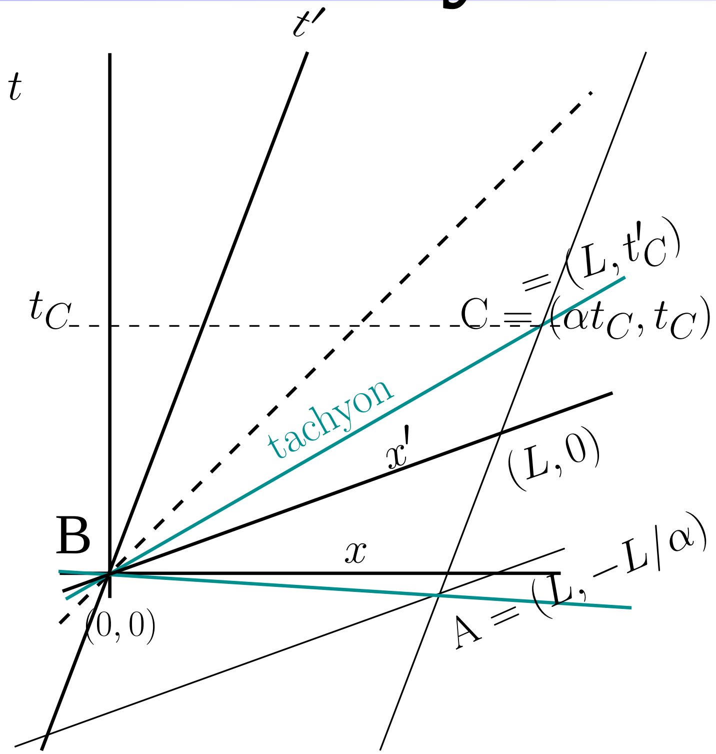
SR: tachyonic antitelephone



B stationary: (x, t) frame
A moving at speed β: (x', t') frame

B: tachyon at $\alpha > 1$ to C

SR: tachyonic antitelephone

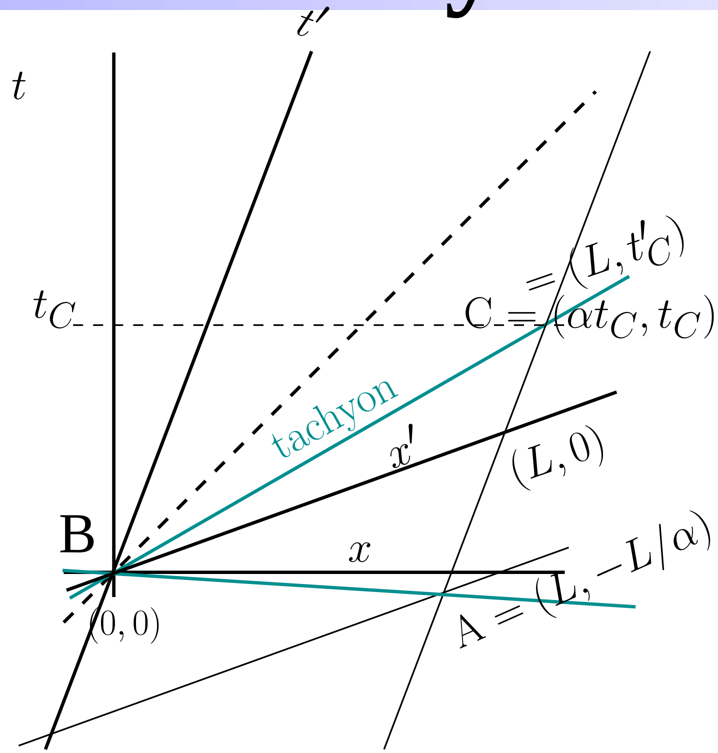


C: $$\begin{pmatrix} L \\ t'_{\rm C} \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t_{\rm C} \\ t_{\rm C} \end{pmatrix}$$

B stationary: (x, t) frame

A moving at speed β: (x', t') frame

SR: tachyonic antitelephone

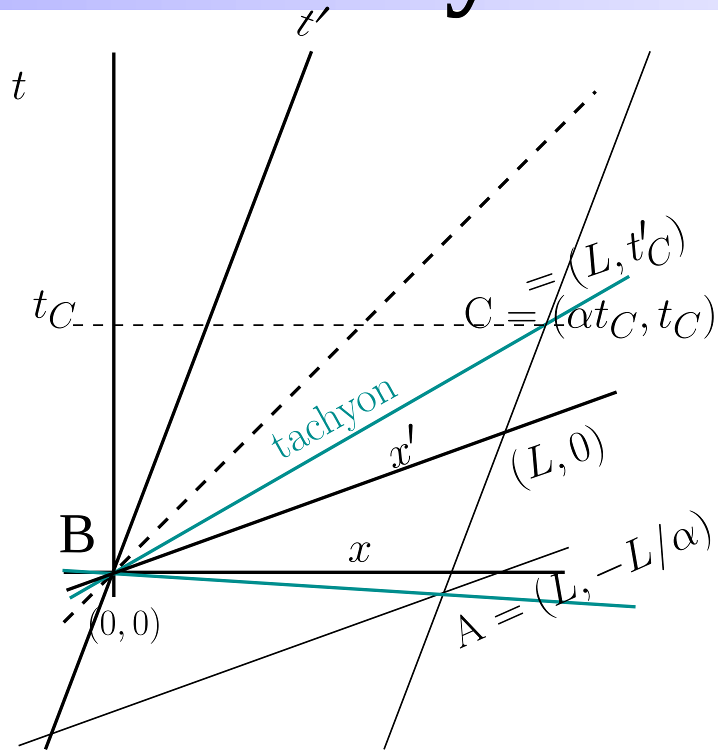


$$\begin{pmatrix} L \\ t'_{\mathrm{C}} \end{pmatrix} = \gamma t_{\mathrm{C}} \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

B stationary: (x, t) frame

A moving at speed β: (x', t') frame

SR: tachyonic antitelephone



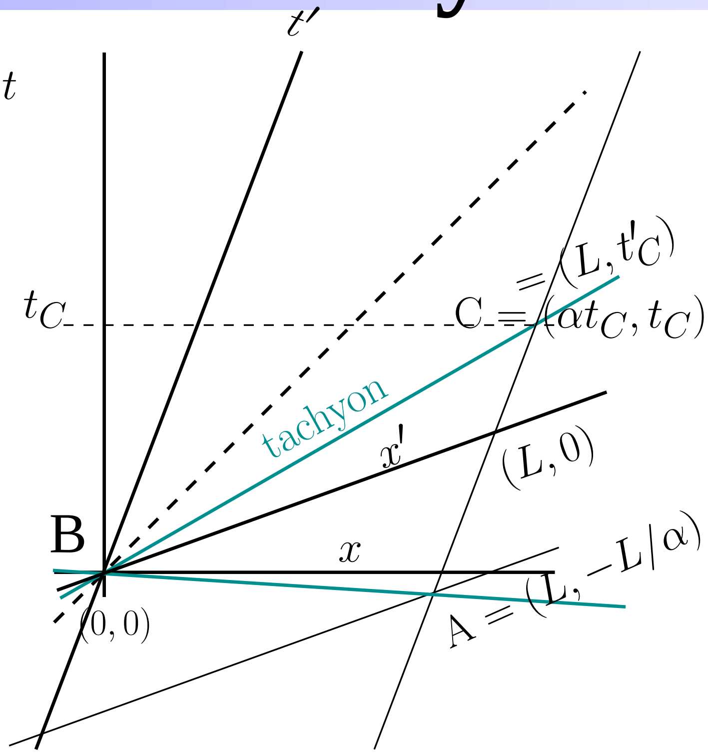
$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$$t'_C = \gamma t_C (1 - \alpha\beta)$$

B stationary: (x, t) frame

A moving at speed β: (x', t') frame

SR: tachyonic antitelephone



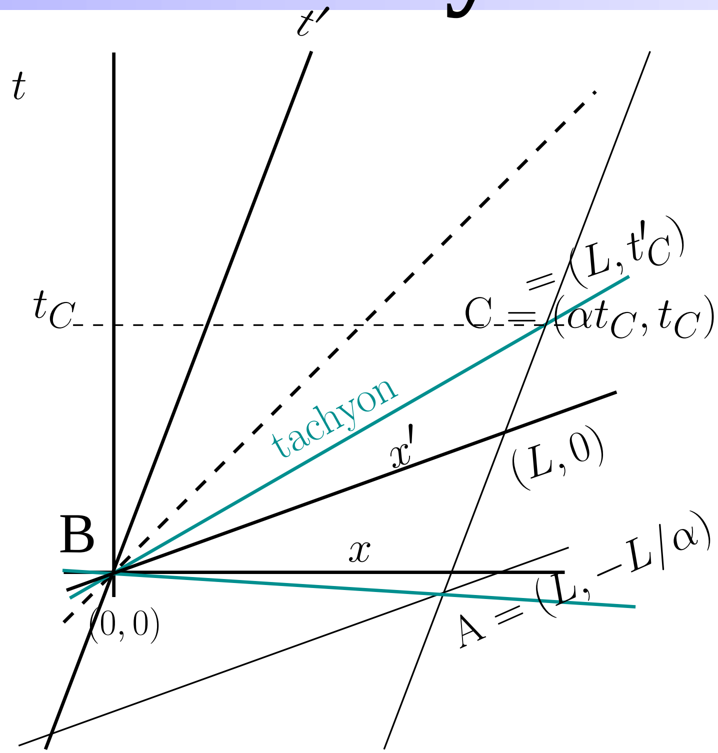
$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$$t'_C = \gamma \frac{L}{\gamma(\alpha-\beta)}(1 - \alpha\beta)$$

B stationary: (x, t) frame

A moving at speed β: (x', t') frame

SR: tachyonic antitelephone



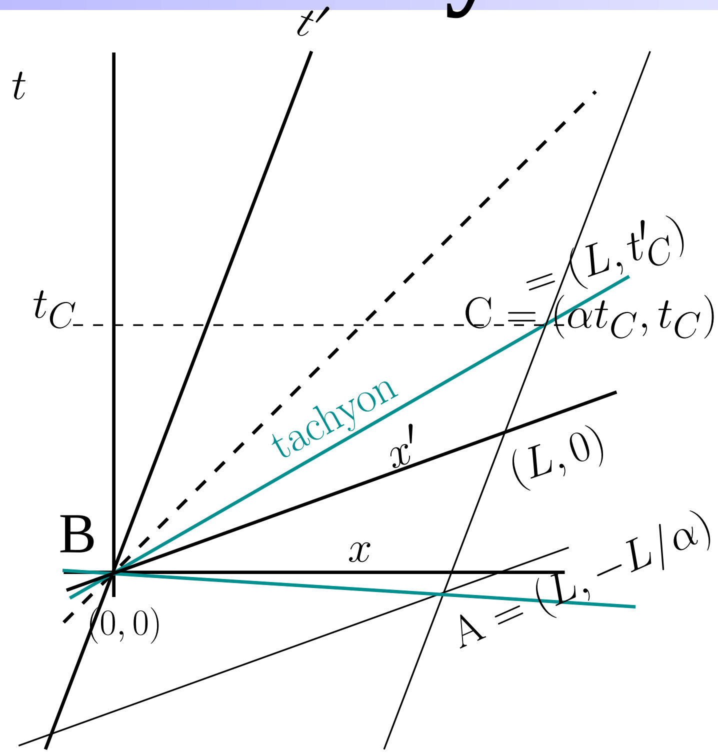
$$\begin{pmatrix} L \\ t'_{\mathrm{C}} \end{pmatrix} = \gamma t_{\mathrm{C}} \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$$t'_{\mathrm{C}} = L\frac{1-\alpha\beta}{\alpha-\beta}$$

B stationary: (x, t) frame

A moving at speed β: (x', t') frame

SR: tachyonic antitelephone



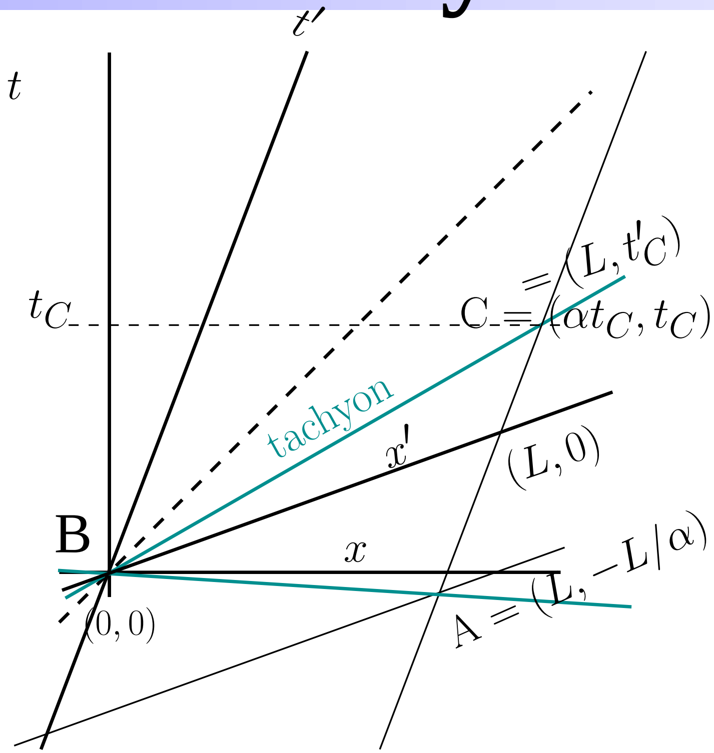
$$\begin{pmatrix} L \\ t'_{\mathrm{C}} \end{pmatrix} = \gamma t_{\mathrm{C}} \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$$t'_{\mathrm{C}} = L \tfrac{1-\alpha\beta}{\alpha-\beta}$$

$$t'_{\mathrm{C}} - t'_{\mathrm{A}} = L \left(\tfrac{1-\alpha\beta}{\alpha-\beta} + \tfrac{1}{\alpha} \right)$$

B stationary: (x, t) frame
A moving at speed β: (x', t') frame

SR: tachyonic antitelephone



$$\begin{pmatrix} L \\ t'_{\mathrm{C}} \end{pmatrix} = \gamma t_{\mathrm{C}} \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

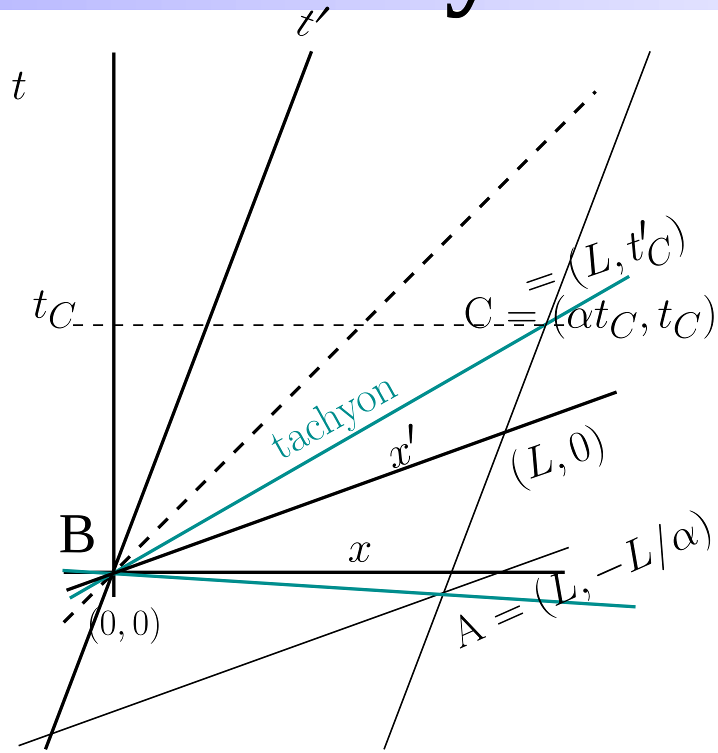
$$t'_{\mathrm{C}} = L\frac{1-\alpha\beta}{\alpha-\beta}$$

$$t'_{\mathrm{C}} - t'_{\mathrm{A}} = L\frac{\alpha-\alpha^2\beta+\alpha-\beta}{\alpha(\alpha-\beta)}$$

B stationary: (x, t) frame

A moving at speed β: (x', t') frame

SR: tachyonic antitelephone



$$\begin{pmatrix} L \\ t'_\mathrm{C} \end{pmatrix} = \gamma t_\mathrm{C} \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

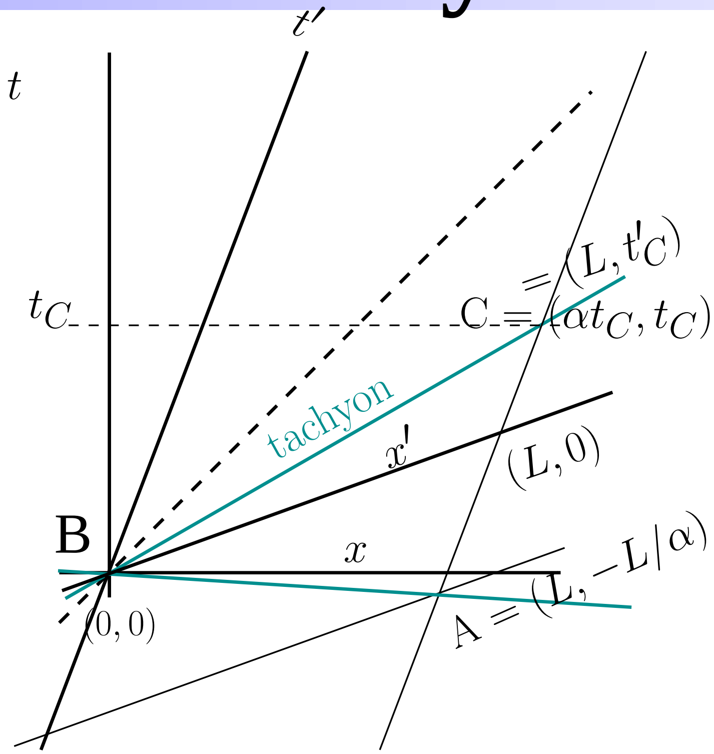
$$t'_\mathrm{C} = L\frac{1-\alpha\beta}{\alpha-\beta}$$

$$t'_\mathrm{C} - t'_\mathrm{A} = L\frac{2\alpha-(\alpha^2+1)\beta}{\alpha(\alpha-\beta)}$$

B stationary: (x, t) frame

A moving at speed β: (x', t') frame

SR: tachyonic antitelephone



$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$t'_C = L \frac{1-\alpha\beta}{\alpha-\beta}$

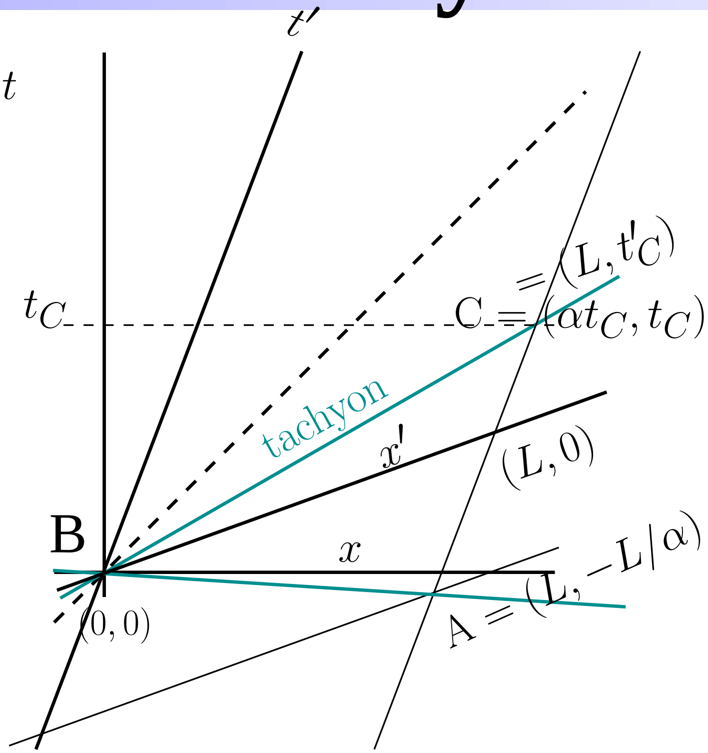
$t'_C - t'_A = L \frac{2\alpha-(\alpha^2+1)\beta}{\alpha(\alpha-\beta)}$

< 0 if $\beta > \frac{2\alpha}{\alpha^2+1}$

B stationary: (x, t) frame

A moving at speed β: (x', t') frame

SR: tachyonic antitelephone



B stationary: (x, t) frame

A moving at speed β: (x', t') frame

$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

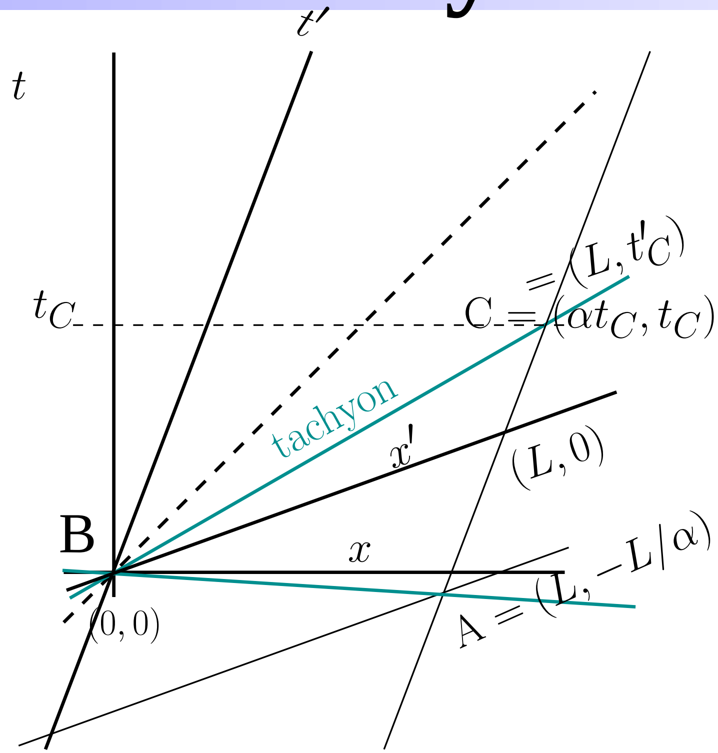
$t'_C = L\frac{1-\alpha\beta}{\alpha-\beta}$

$t'_C - t'_A = L\frac{2\alpha-(\alpha^2+1)\beta}{\alpha(\alpha-\beta)}$

< 0 if $\beta > \frac{2\alpha}{\alpha^2+1}$

A receives tachyonic response at C before sending it

SR: tachyonic antitelephone



B stationary: (x, t) frame

A moving at speed β: (x', t') frame

$$\begin{pmatrix} L \\ t'_{\mathrm{C}} \end{pmatrix} = \gamma t_{\mathrm{C}} \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$$t'_{\mathrm{C}} = L\frac{1-\alpha\beta}{\alpha-\beta}$$

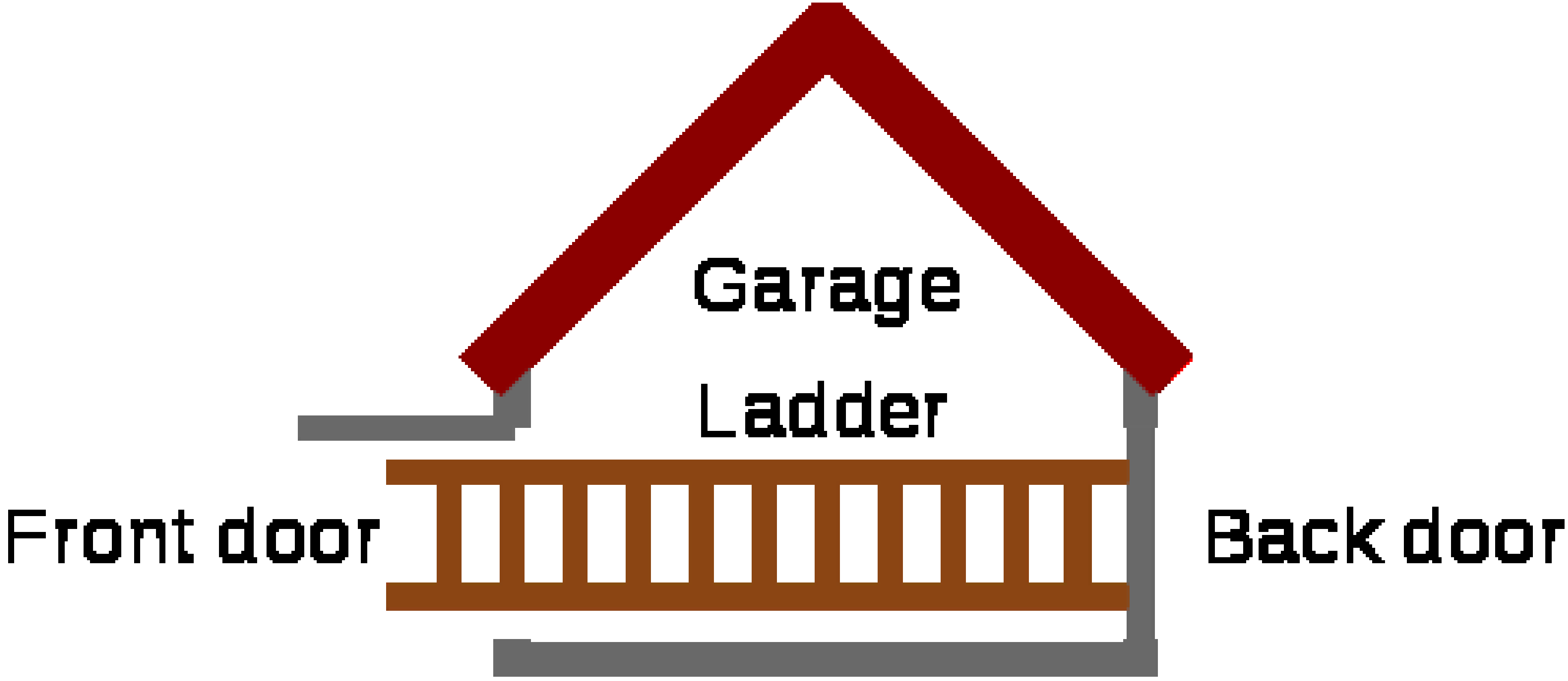
$$t'_{\mathrm{C}} - t'_{\mathrm{A}} = L\frac{2\alpha-(\alpha^2+1)\beta}{\alpha(\alpha-\beta)}$$

< 0 if $\beta > \frac{2\alpha}{\alpha^2+1}$

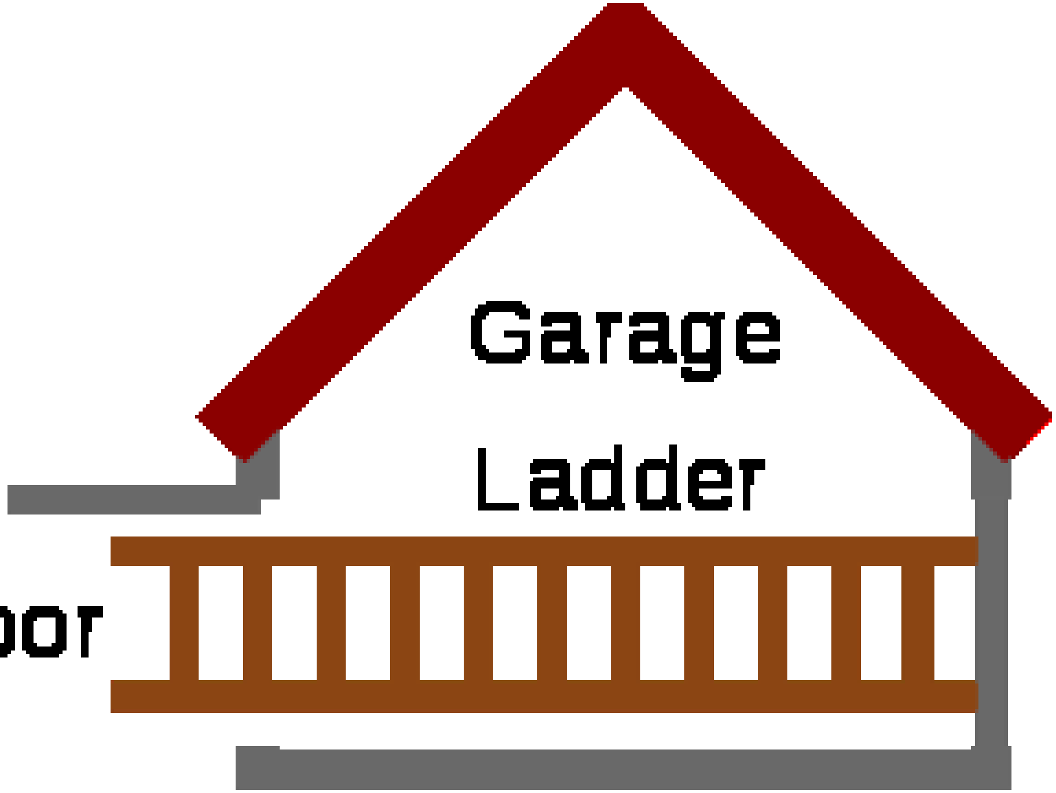
A receives tachyonic response at C before sending it

w:tachyonic antitelephone

SR: pole-barn/ladder paradox

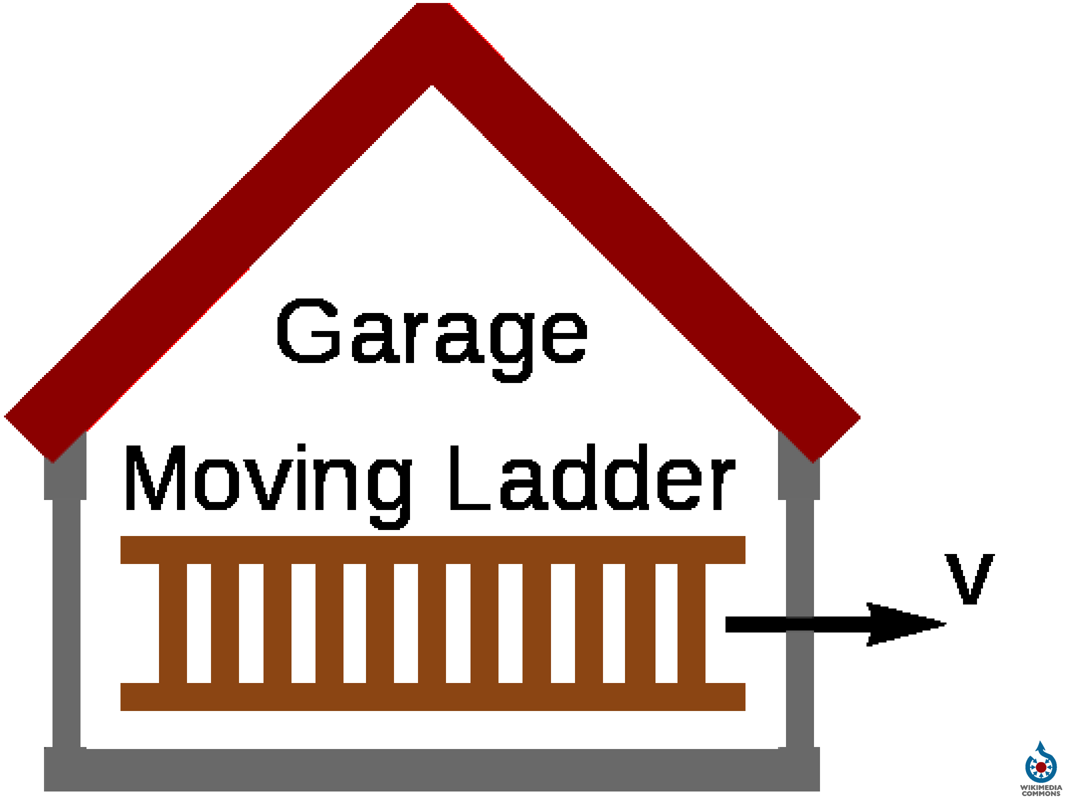


SR: pole-barn/ladder paradox



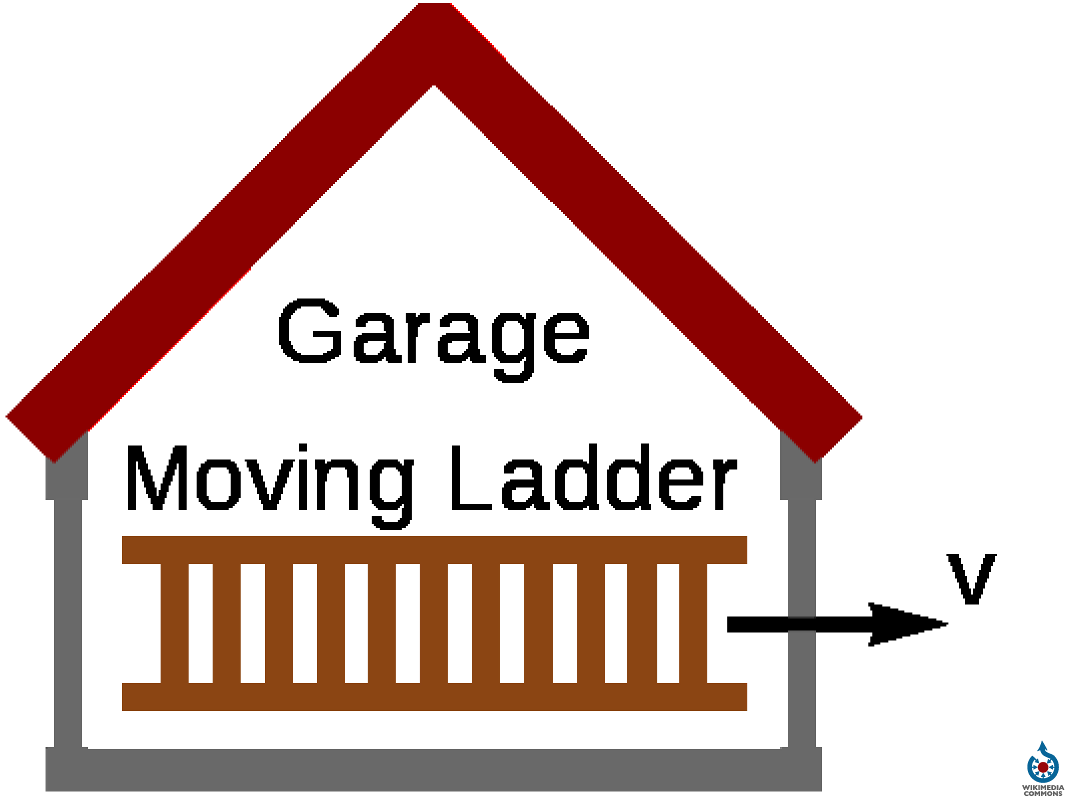
- ladder of length 29.9γ ns, garage length 30 ns

SR: pole-barn/ladder paradox



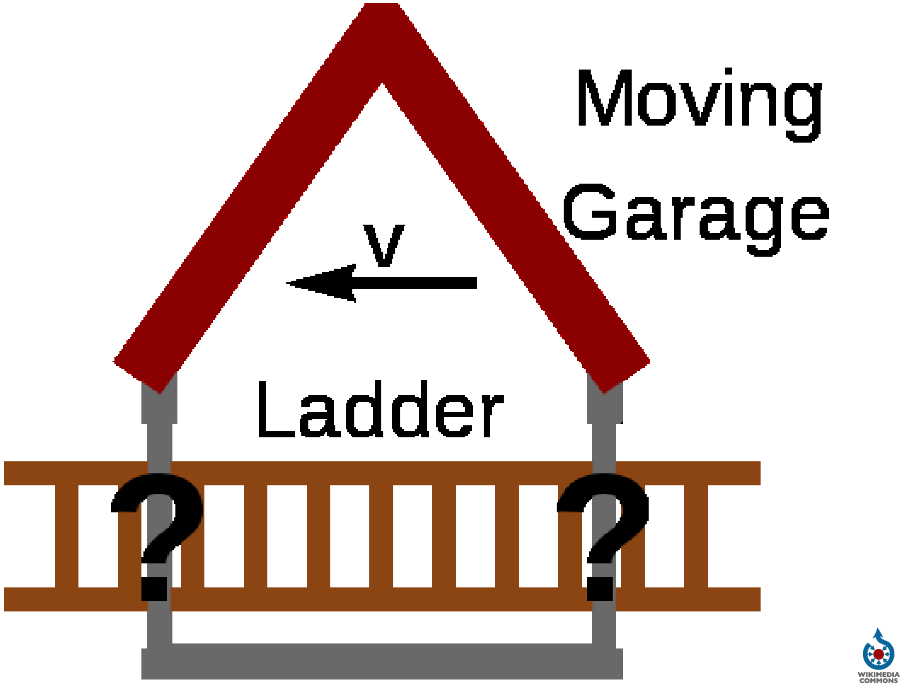
- ladder of length 29.9γ ns, garage length 30 ns
- instantaneously close front + back doors

SR: pole-barn/ladder paradox



- ladder of length 29.9γ ns, garage length 30 ns
- instantaneously close front + back doors
- 29.9γ ns $/\gamma < 30$ ns $\Rightarrow$ OK

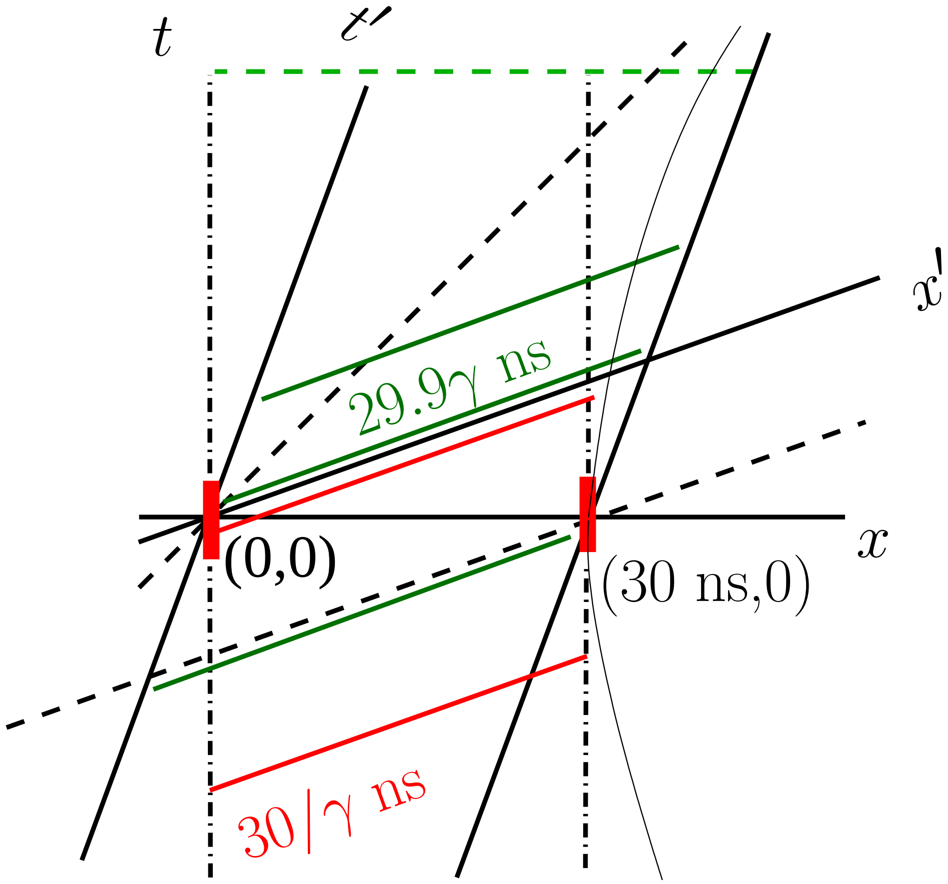
SR: pole-barn/ladder paradox



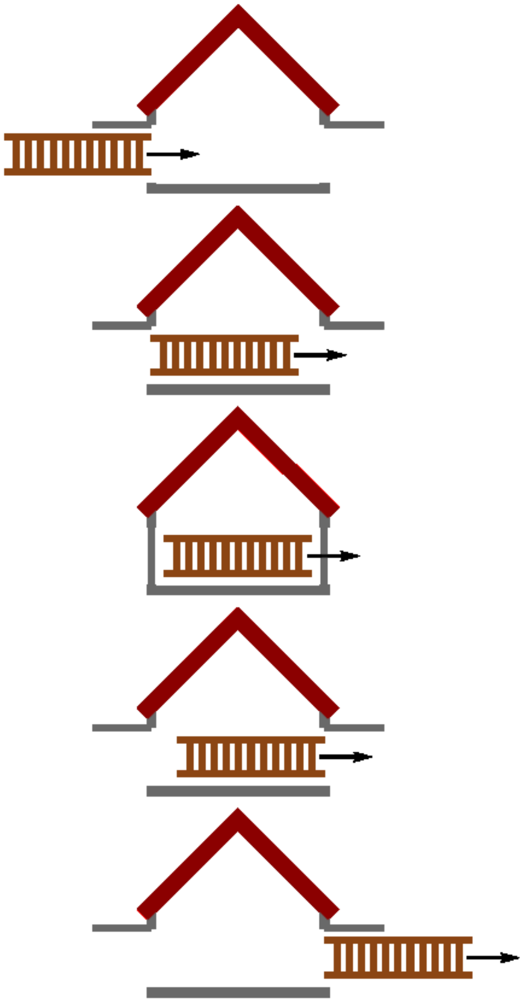
- ladder of length 29.9γ ns, garage length 30 ns
- instantaneously close front + back doors
- ladder frame: garage $30/\gamma$ ns long $\ll 29.9\gamma$ ns!!

Is this possible or not? Make a spacetime diagram.

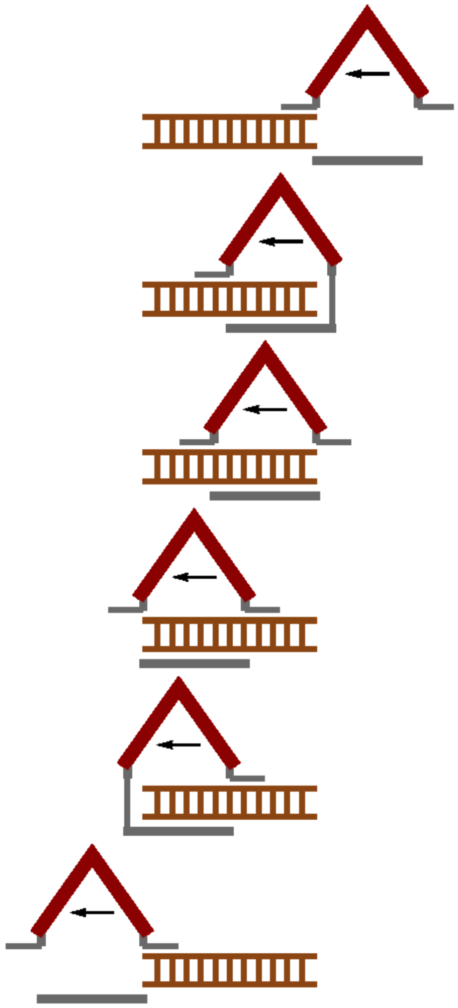
SR: pole-barn/ladder paradox



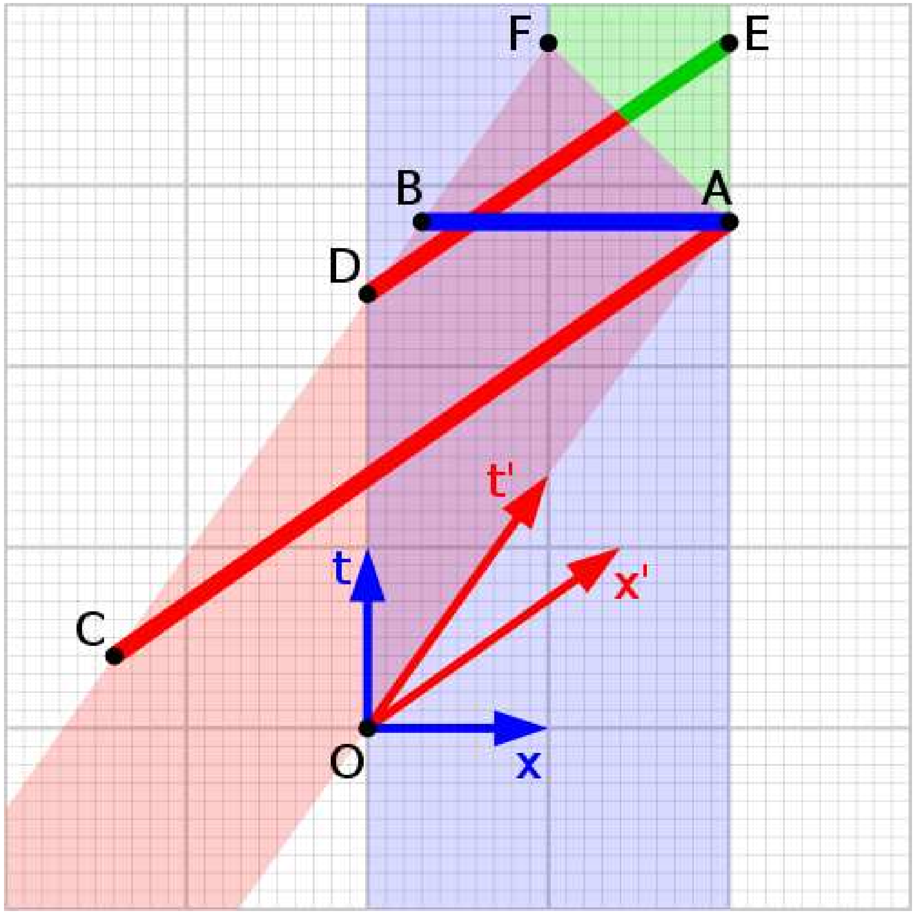
SR: pole-barn/ladder paradox



SR: pole-barn/ladder paradox

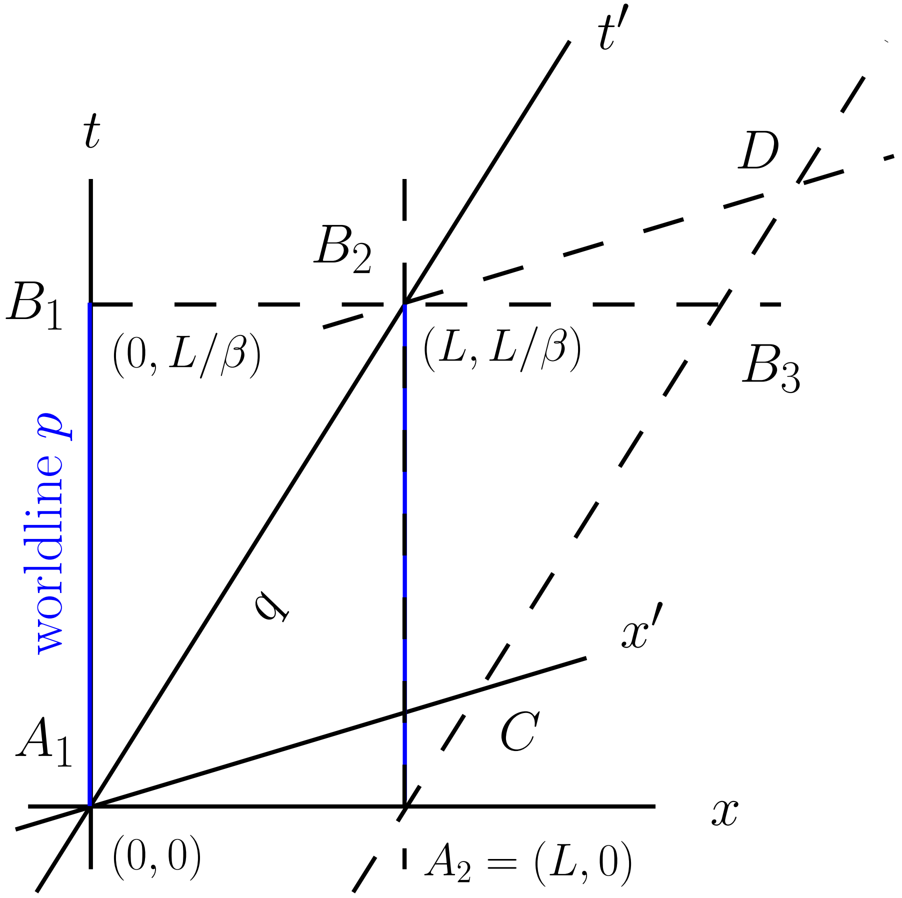


SR: pole-barn/ladder paradox



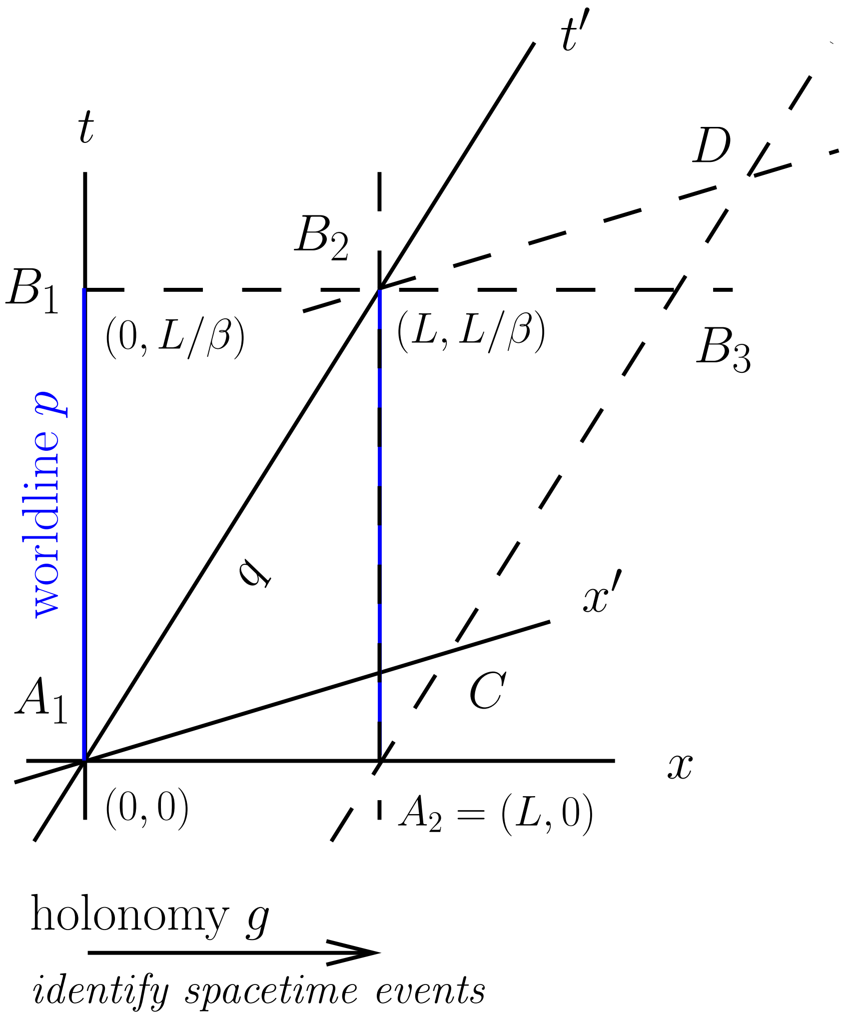
w:Ladder paradox

SR: twins paradox

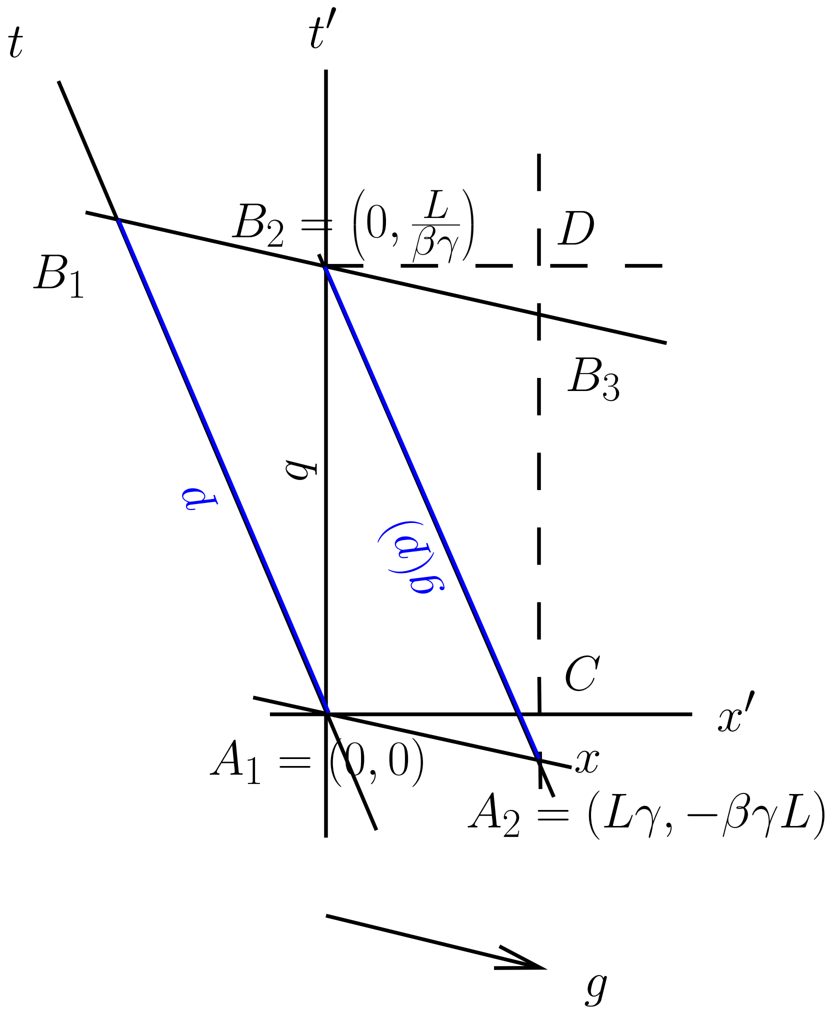


simply connected Minkowski

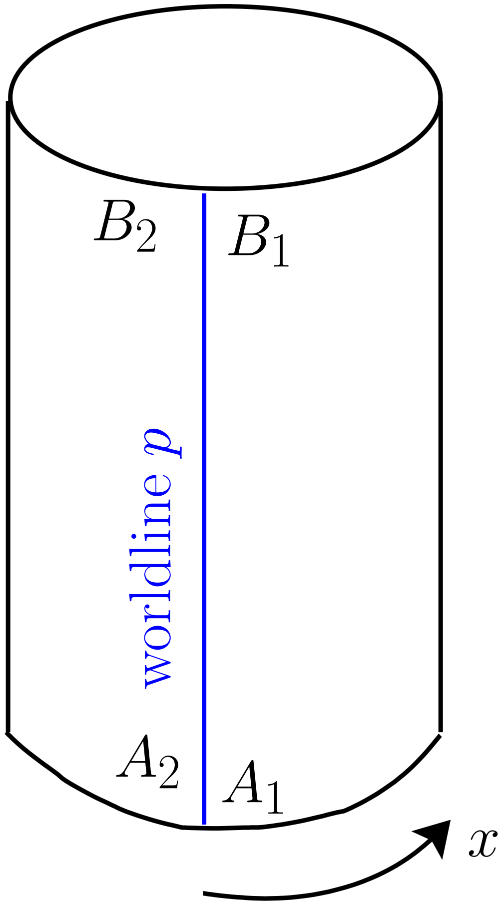
SR: twins paradox



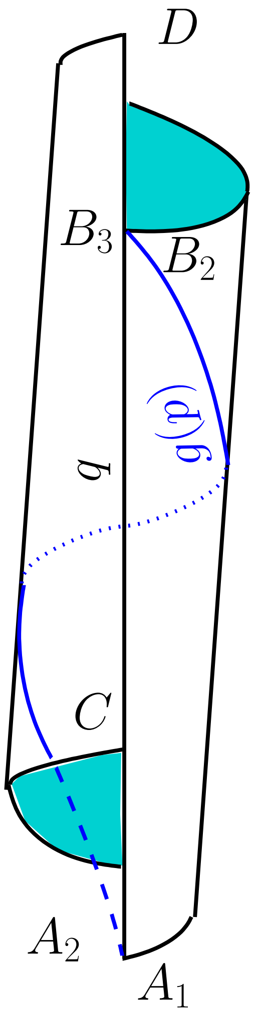
SR: twins paradox



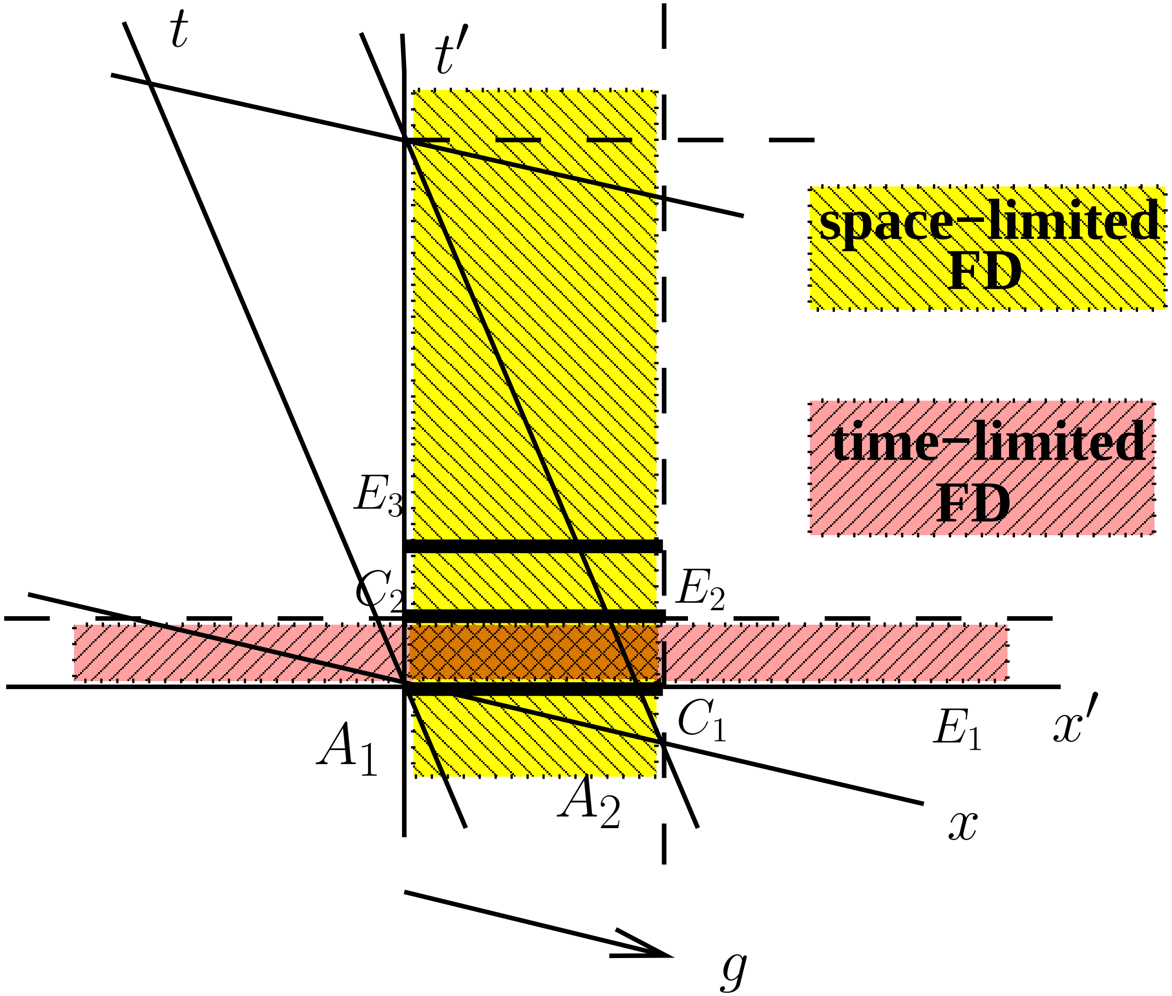
SR: twins paradox



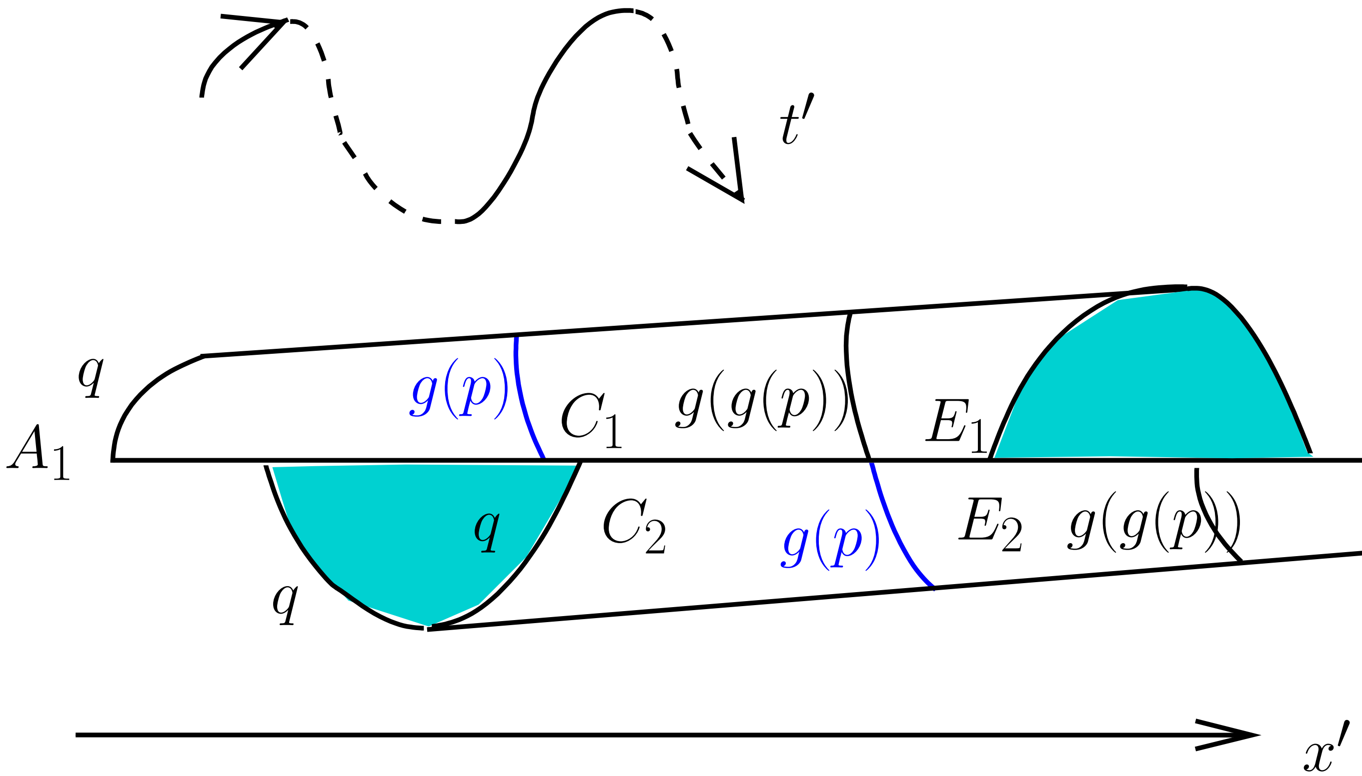
SR: twins paradox



SR: twins paradox



SR: twins paradox



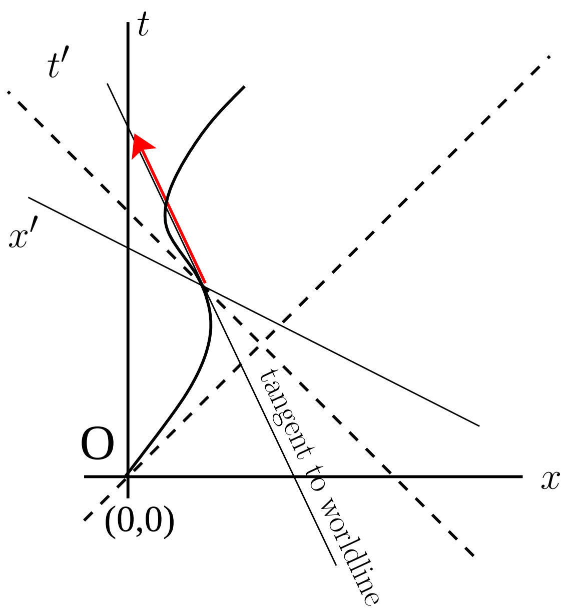
Roukema & Bajtlik 2008, MNRAS, 390, 655
arXiv:astro-ph/0612155

SR: four-velocity, four-momentum

choose x axis so that 3-velocity $u_{\mathrm{Galilean}} = (\beta, 0, 0)^{\mathrm{T}}$ for observer with $(t, x, y, z)^{\mathrm{T}}$ coord system

SR: four-velocity, four-momentum

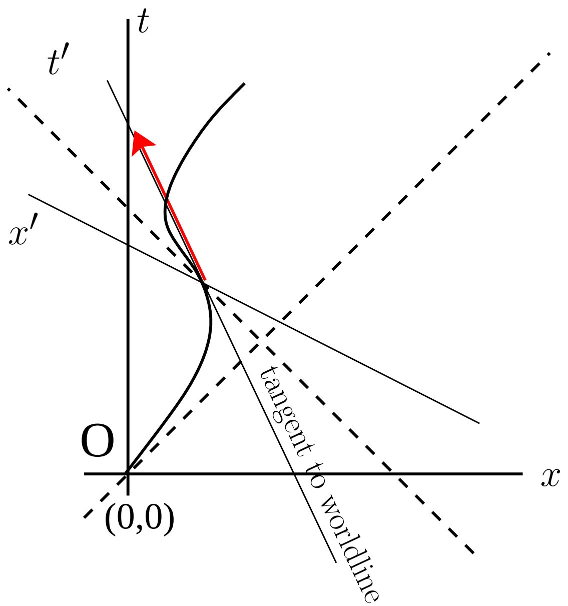
choose x axis so that 3-velocity $u_{\mathrm{Galilean}} = (\beta, 0, 0)^{\mathrm{T}}$ for observer with (t, x, y, z) coord system



- in (t, x) spacetime 2-plane, extend from scalar speed β to spacetime vector = tangent to worldline

SR: four-velocity, four-momentum

choose x axis so that 3-velocity $u_{\mathrm{Galilean}} = (\beta, 0, 0)^{\mathrm{T}}$ for observer with (t, x, y, z) coord system



$$(u^t, u^x) \quad := \quad (\frac{\mathrm{d}}{\mathrm{d}\tau} t(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau))$$

w:four-velocity

- in (t, x) spacetime 2-plane, extend from scalar speed β to spacetime vector = tangent to worldline

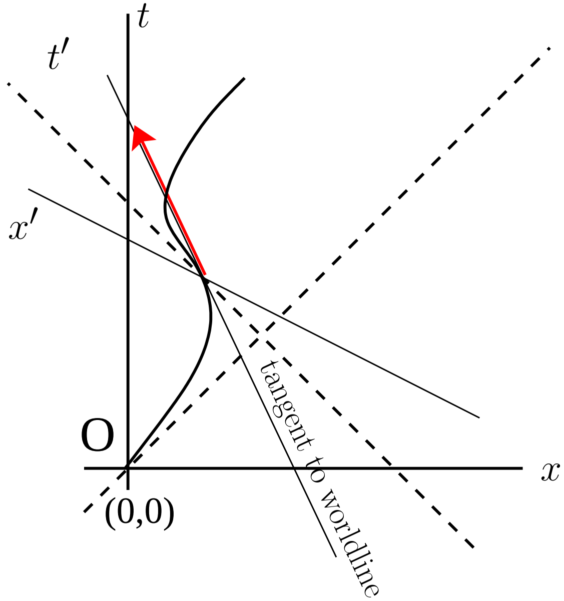
SR: four-velocity, four-momentum

choose x axis so that 3-velocity $u_{\mathrm{Galilean}} = (\beta, 0, 0)^{\mathrm{T}}$ for observer with (t, x, y, z) coord system

$$(u^t, u^x) \quad := \quad (\frac{\mathrm{d}}{\mathrm{d}\tau} t(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau))$$

w:four-velocity

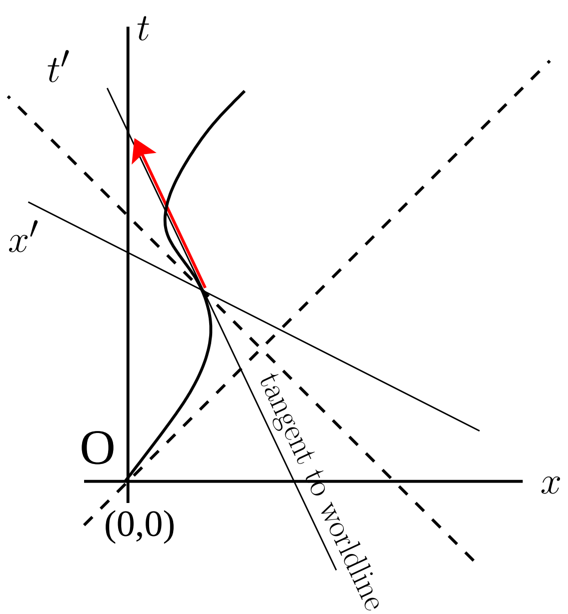
similarly $(u^{t'}, u^{x'}) = (\frac{\mathrm{d}}{\mathrm{d}\tau} t'(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau} x'(\tau))$



- in (t, x) spacetime 2-plane, extend from

SR: four-velocity, four-momentum

choose x axis so that 3-velocity $u_{\mathrm{Galilean}} = (\beta, 0, 0)^{\mathrm{T}}$ for observer with (t, x, y, z) coord system



$$(u^t, u^x) \quad := \quad \left(\frac{\mathrm{d}}{\mathrm{d}\tau} t(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)\right)$$

w:four-velocity

similarly $(u^{t'}, u^{x'}) = \left(\frac{\mathrm{d}}{\mathrm{d}\tau} t'(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau} x'(\tau)\right) = (1, 0)$

- in (t, x) spacetime 2-plane, extend from scalar speed β to spacetime vector = tangent to worldline

SR: four-velocity, four-momentum

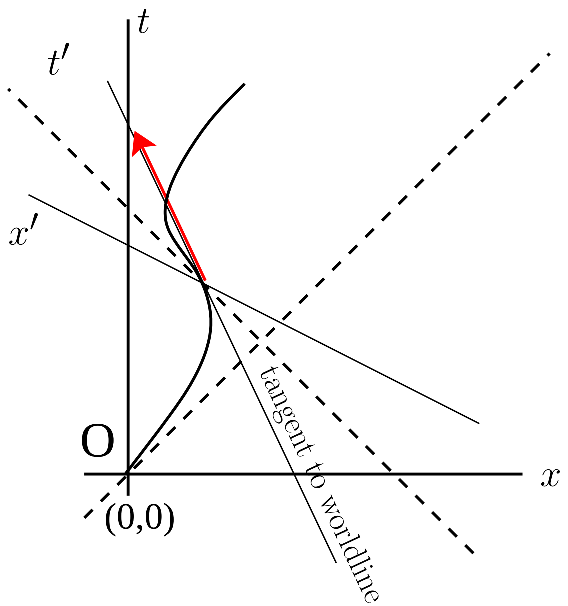
choose x axis so that 3-velocity $u_{\mathrm{Galilean}} = (\beta, 0, 0)^{\mathrm{T}}$ for observer with (t, x, y, z) coord system

$$(u^t, u^x) := \left(\frac{\mathrm{d}}{\mathrm{d}\tau} t(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)\right)$$

w:four-velocity

similarly $(u^{t'}, u^{x'}) = \left(\frac{\mathrm{d}}{\mathrm{d}\tau} t'(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau} x'(\tau)\right) = (1, 0)$

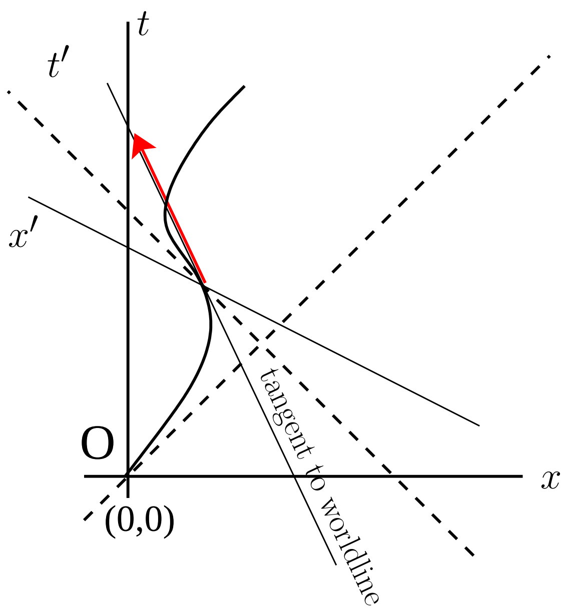
want $\vec{u}$ Lorentz invariant $\Rightarrow$ $(u^t, u^x)^{\mathrm{T}} = \Lambda^{-1} (1, 0)^{\mathrm{T}}$



- in (t, x) spacetime 2-plane, extend from scalar speed β to spacetime vector =

SR: four-velocity, four-momentum

choose x axis so that 3-velocity $u_{\text{Galilean}} = (\beta, 0, 0)^{\text{T}}$ for observer with (t, x, y, z) coord system



$(u^t, u^x) := (\frac{\mathrm{d}}{\mathrm{d}\tau} t(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau))$ w:four-velocity

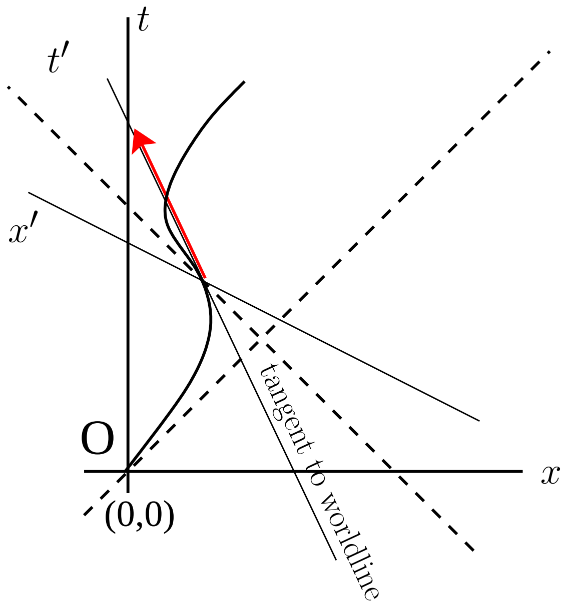
similarly $(u^{t'}, u^{x'}) = (\frac{\mathrm{d}}{\mathrm{d}\tau} t'(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau} x'(\tau)) = (1, 0)$

want $\vec{u}$ Lorentz invariant $\Rightarrow (u^t, u^x)^{\text{T}} = \Lambda^{-1}(1, 0)^{\text{T}} = \gamma(1, \beta)^{\text{T}}$

- in (t, x) spacetime 2-plane, extend from scalar speed β to spacetime vector = tangent to worldline

SR: four-velocity, four-momentum

choose x axis so that 3-velocity $u_{\text{Galilean}} = (\beta, 0, 0)^{\text{T}}$ for observer with (t, x, y, z) coord system



• in (t, x) spacetime 2-plane, extend from scalar speed β to spacetime vector = tangent to worldline

$(u^t, u^x) := (\frac{\mathrm{d}}{\mathrm{d}\tau}t(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau))$ w:four-velocity

similarly $(u^{t'}, u^{x'}) = (\frac{\mathrm{d}}{\mathrm{d}\tau}t'(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau}x'(\tau)) = (1, 0)$

want $\vec{u}$ Lorentz invariant $\Rightarrow (u^t, u^x)^{\text{T}} = \Lambda^{-1}(1, 0)^{\text{T}} = \gamma(1, \beta)^{\text{T}}$

4D: $\vec{u} = \gamma(1, \beta^x, \beta^y, \beta^z)^{\text{T}}$

notation in this pdf:

$\vec{u}$ = 4-vector, ${}^{(3)}\vec{u}$ = spatial component

SR: four-velocity, four-momentum

Is the 3-component (spatial component) of $\vec{u}$ the same as the non-relativistic velocity?

SR: four-velocity, four-momentum

Is the 3-component (spatial component) of $\vec{u}$ the same as the non-relativistic velocity?

$$^{(3)}\vec{u} = \frac{\mathrm{d}}{\mathrm{d}\tau}(x, y, z)^{\mathrm{T}}$$

SR: four-velocity, four-momentum

Is the 3-component (spatial component) of $\vec{u}$ the same as the non-relativistic velocity?

$$^{(3)}\vec{u} = \frac{\mathrm{d}}{\mathrm{d}\tau}(x, y, z)^{\mathrm{T}}$$

$$= \gamma \frac{\mathrm{d}}{\mathrm{d}t}(x, y, z)^{\mathrm{T}}$$

SR: four-velocity, four-momentum

Is the 3-component (spatial component) of $\vec{u}$ the same as the non-relativistic velocity?

$$^{(3)}\vec{u} = \frac{\mathrm{d}}{\mathrm{d}\tau}(x, y, z)^{\mathrm{T}}$$

$$= \gamma \frac{\mathrm{d}}{\mathrm{d}t}(x, y, z)^{\mathrm{T}}$$

$\neq \frac{\mathrm{d}}{\mathrm{d}t}(x, y, z)^{\mathrm{T}}$ except if $\beta = 0 \Leftrightarrow \gamma = 1$

SR: four-velocity, four-momentum

momentum: $\vec{p} := m\vec{u} = m\gamma(1, \beta^x, \beta^y, \beta^z)^{\mathrm{T}}$, where m = constant w:invariant mass

^{x} $\ldots$ = tensor-style component notation, not powers

SR: four-velocity, four-momentum

momentum: $\vec{p} := m\vec{u} = m\gamma(1, \beta^x, \beta^y, \beta^z)^{\mathrm{T}}$, where m = constant w:invariant mass

SR: four-velocity, four-momentum

momentum: $\vec{p} := m\vec{u} = m\gamma(1, \beta^x, \beta^y, \beta^z)^{\mathrm{T}}$, where m = constant w:invariant mass

What does the time component of momentum = $p^0 = m\gamma$ mean physically?

- first look at spatial component in a given ref. frame

SR: four-velocity, four-momentum

momentum: $\vec{p} := m\vec{u} = m\gamma(1, \beta^x, \beta^y, \beta^z)^{\mathrm{T}}$, where m = constant w:invariant mass

$$^{(3)}\vec{p} = m\frac{\mathrm{d}}{\mathrm{d}\tau}(x, y, z)^{\mathrm{T}}$$

SR: four-velocity, four-momentum

momentum: $\vec{p} := m\vec{u} = m\gamma(1, \beta^x, \beta^y, \beta^z)^{\mathrm{T}}$, where m = constant w:invariant mass

$$^{(3)}\vec{p} = m\frac{\mathrm{d}}{\mathrm{d}\tau}(x, y, z)^{\mathrm{T}}$$

$$= m\gamma\frac{\mathrm{d}}{\mathrm{d}t}(x, y, z)^{\mathrm{T}}$$

SR: four-velocity, four-momentum

momentum: $\vec{p} := m\vec{u} = m\gamma(1, \beta^x, \beta^y, \beta^z)^{\mathrm{T}}$, where m = constant w:invariant mass

$$^{(3)}\vec{p} = m\frac{\mathrm{d}}{\mathrm{d}\tau}(x, y, z)^{\mathrm{T}}$$

$$= m\gamma\frac{\mathrm{d}}{\mathrm{d}t}(x, y, z)^{\mathrm{T}}$$

$\neq m\frac{\mathrm{d}}{\mathrm{d}t}(x, y, z)^{\mathrm{T}}$ except if $\beta = 0 \Leftrightarrow \gamma = 1$

SR: four-velocity, four-momentum

let us define 4-acceleration, 4-force

SR: four-velocity, four-momentum

$$(u^t, u^x) := \left(\frac{\mathrm{d}}{\mathrm{d}\tau}t(\tau), \frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau)\right)$$

SR: four-velocity, four-momentum

$$\vec{a} := \frac{\mathrm{d}}{\mathrm{d}\tau}\vec{u}$$

SR: four-velocity, four-momentum

$$\vec{a} := \frac{\mathrm{d}}{\mathrm{d}\tau}\vec{u}$$

$$^{(3)}\vec{a} = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}{}^{(3)}\vec{x}$$

SR: four-velocity, four-momentum

$$\vec{a} := \frac{\mathrm{d}}{\mathrm{d}\tau}\vec{u}$$

$$^{(3)}\vec{a} = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(x, y, z)^{\mathrm{T}}$$

SR: four-velocity, four-momentum

$\vec{a} := \frac{\mathrm{d}}{\mathrm{d}\tau}\vec{u}$

${}^{(4)}\vec{f} := m\ {}^{(4)}\vec{a}$ defn w:four-force

SR: four-velocity, four-momentum

$$\vec{a} := \frac{\mathrm{d}}{\mathrm{d}\tau}\vec{u}$$

$${}^{(4)}\vec{f} := m\ {}^{(4)}\vec{a}$$ defn w:four-force

$$= m\frac{\mathrm{d}}{\mathrm{d}\tau}\vec{u}$$

SR: four-velocity, four-momentum

$\vec{a} := \frac{\mathrm{d}}{\mathrm{d}\tau}\vec{u}$

${}^{(4)}\vec{f} := m\ {}^{(4)}\vec{a}$ defn w:four-force

$= \frac{\mathrm{d}}{\mathrm{d}\tau}\vec{p}$

SR: invariance of ${}^{(4)}u, {}^{(4)}a, {}^{(4)}f$

Euclidean norm: $\|\vec{x}\|^2 = \sum_\mu (x^\mu)^2$

SR: invariance of ${}^{(4)}u, {}^{(4)}a, {}^{(4)}f$

Minkowski pseudo-norm: $\|\vec{x}\|^2 = \sum_{\mu,\nu} \eta_{\mu\nu} x^\mu x^\nu$

SR: invariance of ${}^{(4)}u, {}^{(4)}a, {}^{(4)}f$

Minkowski pseudo-norm: $\|\vec{x}\|^2 = \eta_{\mu\nu} x^\mu x^\nu$

w:Einstein summation sum is implicit

SR: invariance of ${}^{(4)}u, {}^{(4)}a, {}^{(4)}f$

Minkowski pseudo-norm: $\|\vec{x}\|^2 = -x^0x^0 + \delta_{ij}x^ix^j$

$\delta_{ij} = 1$ if $i = j$, otherwise $= 0$; $i, j \in 1, 2, 3$

invariance: $\|\vec{x}\|^2$ = same in all reference frames

sign convention: $(-, +, +, +)$ or $(+, -, -, -)$ are common

SR: invariance of ${}^{(4)}u, {}^{(4)}a, {}^{(4)}f$

Minkowski pseudo-norm: $\|\vec{x}\|^2 = -x^0x^0 + \delta_{ij}x^ix^j$

$\delta_{ij} = 1$ if $i = j$, otherwise $= 0$; $i, j \in 1, 2, 3$

invariance: $\|\vec{x}\|^2$ = same in all reference frames

sign convention: $(-, +, +, +)$ or $(+, -, -, -)$ are common

non-rest frame: $\|\vec{u}\|^2 = -\gamma^2 + \gamma^2\beta^2$

SR: invariance of ${}^{(4)}u, {}^{(4)}a, {}^{(4)}f$

Minkowski pseudo-norm: $\|\vec{x}\|^2 = -x^0 x^0 + \delta_{ij} x^i x^j$

$\delta_{ij} = 1$ if $i = j$, otherwise $= 0$; $i, j \in 1, 2, 3$

invariance: $\|\vec{x}\|^2$ = same in all reference frames

sign convention: $(-, +, +, +)$ or $(+, -, -, -)$ are common

non-rest frame: $\|\vec{u}\|^2 = -\gamma^2 + \gamma^2 \beta^2 = -1$

SR: invariance of ${}^{(4)}u, {}^{(4)}a, {}^{(4)}f$

Minkowski pseudo-norm: $\|\vec{x}\|^2 = -x^0 x^0 + \delta_{ij} x^i x^j$

$\delta_{ij} = 1$ if $i = j$, otherwise $= 0$; $i, j \in 1, 2, 3$

invariance: $\|\vec{x}\|^2$ = same in all reference frames

sign convention: $(-, +, +, +)$ or $(+, -, -, -)$ are common

non-rest frame: $\|\vec{u}\|^2 = -\gamma^2 + \gamma^2 \beta^2 = -1$

rest frame: $\|\vec{u}\|^2 = -1^2 + 0^2 = -1$ invariant

SR: invariance of ${}^{(4)}u, {}^{(4)}a, {}^{(4)}f$

Minkowski pseudo-norm: $\|\vec{x}\|^2 = -x^0x^0 + \delta_{ij}x^ix^j$

$\delta_{ij} = 1$ if $i = j$, otherwise $= 0$; $i, j \in 1, 2, 3$

invariance: $\|\vec{x}\|^2$ = same in all reference frames

sign convention: $(-, +, +, +)$ or $(+, -, -, -)$ are common

non-rest frame: $\|\vec{u}\|^2 = -\gamma^2 + \gamma^2\beta^2 = -1$

rest frame: $\|\vec{u}\|^2 = -1^2 + 0^2 = -1$ invariant

similarly: $\|\vec{a}\|^2$, $\|\vec{f}\|^2$ invariant

SR: energy: varies with ref frame

Newtonian $K = (1/2)m\beta^2$ = 0 in rest frame

SR: energy: varies with ref frame

Newtonian $K = (1/2)m\beta^2$ = 0 in rest frame

4-force $\vec{f}$ is invariant, but

3-force usually *defined* to be frame-dependent:

3-force $:= \frac{\mathrm{d}}{\mathrm{d}t}{}^{(3)}\vec{p} \quad \neq \frac{\mathrm{d}}{\mathrm{d}\tau}{}^{(3)}\vec{p}$

SR: energy: varies with ref frame

Newtonian $K = (1/2)m\beta^2$ = 0 in rest frame

4-force $\vec{f}$ is invariant, but

3-force usually *defined* to be frame-dependent:

3-force $:= \frac{\mathrm{d}}{\mathrm{d}t} {}^{(3)}\vec{p} \quad \neq \frac{\mathrm{d}}{\mathrm{d}\tau} {}^{(3)}\vec{p}$

$$\frac{\mathrm{d}}{\mathrm{d}t} {}^{(3)}\vec{p} = \frac{{}^{(3)}\vec{f}}{\gamma}$$

SR: energy: varies with ref frame

in (x, t) frame,

$K =$ work done

SR: energy: varies with ref frame

in (x, t) frame, K = work done $= \int_0^{\beta_2} \frac{^{(3)}\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x}$

SR: energy: varies with ref frame

in (x, t) frame,
$K =$ work done

$$= \int_0^{\beta_2} \frac{{}^{(3)}\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x}$$

$$= \int_0^{\beta_2} \frac{\mathrm{d}}{\mathrm{d}t} {}^{(3)}\vec{p} \cdot \mathrm{d}\vec{x}$$

SR: energy: varies with ref frame

in (x, t) frame,
$K =$ work done

$$= \int_0^{\beta_2} \frac{{}^{(3)}\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x}$$

$$= \int_0^{\beta_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\beta\gamma)\mathrm{d}x$$

(assume ${}^{(3)}\vec{f}/\gamma \parallel \vec{x}$)

SR: energy: varies with ref frame

in (x,t) frame,

$K =$ work done $\displaystyle = \int_0^{\beta_2} \frac{^{(3)}\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x}$

$$= \int_0^{\beta_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\beta\gamma)\mathrm{d}x = m \int_0^{\beta_2} \mathrm{d}(\beta\gamma)\frac{\mathrm{d}x}{\mathrm{d}t}$$

SR: energy: varies with ref frame

in (x,t) frame,

$K =$ work done $$= \int_0^{\beta_2} \frac{{}^{(3)}\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x}$$

$$= \int_0^{\beta_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\beta\gamma)\mathrm{d}x = m \int_0^{\beta_2} \mathrm{d}(\beta\gamma)\beta$$

SR: energy: varies with ref frame

in (x,t) frame,

$K =$ work done

$$= \int_0^{\beta_2} \frac{(3)\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x}$$

$$= \int_0^{\beta_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\beta\gamma)\mathrm{d}x = m \int_0^{\beta_2} \mathrm{d}(\beta\gamma)\beta$$

$$= m \int_0^{\beta_2} \beta(\beta\mathrm{d}\gamma + \gamma\mathrm{d}\beta)$$

SR: energy: varies with ref frame

in (x,t) frame,

$K =$ work done

$$= \int_0^{\beta_2} \frac{{}^{(3)}\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x}$$

$$= \int_0^{\beta_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\beta\gamma)\mathrm{d}x = m \int_0^{\beta_2} \mathrm{d}(\beta\gamma)\beta$$

$$= m \int_0^{\beta_2} \beta(\beta \mathrm{d}\gamma + \gamma \mathrm{d}\beta)$$

$$= m \int_{\gamma=1}^{\gamma=\gamma_2} (\beta^2 + \gamma^{-2})\mathrm{d}\gamma \;\Leftarrow \mathrm{d}\gamma = \beta\gamma^3 \mathrm{d}\beta$$

SR: energy: varies with ref frame

in (x, t) frame,

$K =$ work done $= \int_0^{\beta_2} \frac{^{(3)}\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x}$

$$= \int_0^{\beta_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\beta\gamma)\mathrm{d}x = m \int_0^{\beta_2} \mathrm{d}(\beta\gamma)\beta$$

$$= m \int_0^{\beta_2} \beta(\beta \mathrm{d}\gamma + \gamma \mathrm{d}\beta)$$

$$= m \int_{\gamma=1}^{\gamma=\gamma_2} (\beta^2 + \gamma^{-2})\mathrm{d}\gamma \ \Leftarrow \mathrm{d}\gamma = \beta\gamma^3 \mathrm{d}\beta$$

$$= m \int_{\gamma=1}^{\gamma=\gamma_2} [\beta^2 + (1 - \beta^2)]\mathrm{d}\gamma$$

SR: energy: varies with ref frame

in (x,t) frame,

$K =$ work done

$$
\begin{aligned}
&= \int_0^{\beta_2} \frac{^{(3)}\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x} \\
&= \int_0^{\beta_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\beta\gamma)\mathrm{d}x = m\int_0^{\beta_2} \mathrm{d}(\beta\gamma)\beta \\
&= m\int_0^{\beta_2} \beta(\beta\mathrm{d}\gamma + \gamma\mathrm{d}\beta) \\
&= m\int_{\gamma=1}^{\gamma=\gamma_2} (\beta^2 + \gamma^{-2})\mathrm{d}\gamma \ \Leftarrow \mathrm{d}\gamma = \beta\gamma^3\mathrm{d}\beta \\
&= m\int_{\gamma=1}^{\gamma=\gamma_2} \mathrm{d}\gamma
\end{aligned}
$$

SR: energy: varies with ref frame

in (x, t) frame,

$K =$ work done $= \int_0^{\beta_2} \frac{(3)\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x}$

$$= \int_0^{\beta_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\beta\gamma)\mathrm{d}x = m \int_0^{\beta_2} \mathrm{d}(\beta\gamma)\beta$$

$$= m \int_0^{\beta_2} \beta(\beta\mathrm{d}\gamma + \gamma\mathrm{d}\beta)$$

$$= m \int_{\gamma=1}^{\gamma=\gamma_2} (\beta^2 + \gamma^{-2})\mathrm{d}\gamma \ \Leftarrow \mathrm{d}\gamma = \beta\gamma^3\mathrm{d}\beta$$

$$= m \int_{\gamma=1}^{\gamma=\gamma_2} \mathrm{d}\gamma = m\gamma_2 - m$$

SR: energy: varies with ref frame

in (x,t) frame, $K =$ work done

$$= \int_0^{\beta_2} \frac{^{(3)}\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x}$$

$$= \int_0^{\beta_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\beta\gamma)\mathrm{d}x = m\int_0^{\beta_2} \mathrm{d}(\beta\gamma)\beta$$

$$= m\int_0^{\beta_2} \beta(\beta\mathrm{d}\gamma + \gamma\mathrm{d}\beta)$$

$$= m\int_{\gamma=1}^{\gamma=\gamma_2} (\beta^2 + \gamma^{-2})\mathrm{d}\gamma \ \Leftarrow \mathrm{d}\gamma = \beta\gamma^3\mathrm{d}\beta$$

$$= m\int_{\gamma=1}^{\gamma=\gamma_2} \mathrm{d}\gamma = m\gamma_2 - m$$

$\Rightarrow K + m = m\gamma$ *drop* "$_2$"

SR: energy: varies with ref frame

in (x,t) frame,
$K =$ work done

$$= \int_0^{\beta_2} \frac{^{(3)}\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x}$$

$$= \int_0^{\beta_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\beta\gamma)\mathrm{d}x = m \int_0^{\beta_2} \mathrm{d}(\beta\gamma)\beta$$

$$= m \int_0^{\beta_2} \beta(\beta\mathrm{d}\gamma + \gamma\mathrm{d}\beta)$$

$$= m \int_{\gamma=1}^{\gamma=\gamma_2} (\beta^2 + \gamma^{-2})\mathrm{d}\gamma \ \Leftarrow \mathrm{d}\gamma = \beta\gamma^3\mathrm{d}\beta$$

$$= m \int_{\gamma=1}^{\gamma=\gamma_2} \mathrm{d}\gamma = m\gamma_2 - m$$

$$\Rightarrow K + m = m\gamma = p^0$$

SR: energy: varies with ref frame

in (x,t) frame,
$K =$ work done $\displaystyle = \int_0^{\beta_2} \frac{^{(3)}\vec{f}}{\gamma} \cdot \mathrm{d}\vec{x}$

$$= \int_0^{\beta_2} \frac{\mathrm{d}}{\mathrm{d}t}(m\beta\gamma)\mathrm{d}x = m \int_0^{\beta_2} \mathrm{d}(\beta\gamma)\beta$$

$$= m \int_0^{\beta_2} \beta(\beta \mathrm{d}\gamma + \gamma \mathrm{d}\beta)$$

$$= m \int_{\gamma=1}^{\gamma=\gamma_2} (\beta^2 + \gamma^{-2})\mathrm{d}\gamma \;\Leftarrow \mathrm{d}\gamma = \beta\gamma^3 \mathrm{d}\beta$$

$$= m \int_{\gamma=1}^{\gamma=\gamma_2} \mathrm{d}\gamma = m\gamma_2 - m$$

$\Rightarrow K + m = m\gamma = p^0$
so $p^0 =$ kinetic energy + rest mass

SR: energy: varies with ref frame

Does small β limit agree with Newtonian K?

SR: energy: varies with ref frame

Does small β limit agree with Newtonian K?

momentum time component:

$$p^0 = m\gamma = m(1-\beta^2)^{-1/2}$$

$$= m[1-(1/2)(-\beta^2)+\mathcal{O}(\beta^4)] \text{ if } \beta \ll 1$$

SR: energy: varies with ref frame

Does small β limit agree with Newtonian K?

momentum time component:

$p^0 = m\gamma = m(1 - \beta^2)^{-1/2}$

$\approx m[1 + (1/2)\beta^2]$ if $\beta \ll 1$

SR: energy: varies with ref frame

Does small β limit agree with Newtonian K?

momentum time component:

$p^0 = m\gamma = m(1 - \beta^2)^{-1/2}$

$\approx m + (1/2)m\beta^2$ if $\beta \ll 1$

Yes.

SR: $\vec{p}\ldots$: invariant or not?

momentum: $\vec{p} = m\gamma(1, \beta^x, \beta^y, \beta^z)^{\mathrm{T}}$

$p^0 = m + K = m\gamma$

SR: $\vec{p}\ldots$: invariant or not?

momentum: $\vec{p} = m\gamma(1, \beta^x, \beta^y, \beta^z)^{\mathrm{T}}$

$p^0 = m + K = m\gamma$

non-rest frame: $\|\vec{u}\|^2 = -\gamma^2 + \gamma^2\beta^2 = -1$

rest frame: $\|\vec{u}\|^2 = -1^2 + 0^2 = -1$ invariant

SR: $\vec{p}\ldots$: invariant or not?

momentum: $\vec{p} = m\gamma(1, \beta^x, \beta^y, \beta^z)^{\mathrm{T}}$

$p^0 = m + K = m\gamma$

non-rest frame: $\|\vec{p}\|^2 = -m^2\gamma^2 + m^2\gamma^2\beta^2 = -m^2$

rest frame: $\|\vec{p}\|^2 = -m^2 + 0^2 = -m^2$ invariant

SR: $\vec{p}\ldots$: invariant or not?

momentum: $\vec{p} = m\gamma(1, \beta^x, \beta^y, \beta^z)^{\mathrm{T}}$

$p^0 = m + K = m\gamma$

non-rest frame: $\|\vec{p}\|^2 = -m^2\gamma^2 + m^2\gamma^2\beta^2 = -m^2$

rest frame: $\|\vec{p}\|^2 = -m^2 + 0^2 = -m^2$ invariant

m = w:invariant mass $\equiv$ rest mass: invariant

AND conserved (in interactions): $\|\vec{p} + \vec{q}\|^2 = \|\vec{r}\|^2$

SR: $\vec{p}\ldots$: invariant or not?

momentum: $\vec{p} = m\gamma(1, \beta^x, \beta^y, \beta^z)^{\mathrm{T}}$

$p^0 = m + K = m\gamma$

non-rest frame: $\|\vec{p}\|^2 = -m^2\gamma^2 + m^2\gamma^2\beta^2 = -m^2$

rest frame: $\|\vec{p}\|^2 = -m^2 + 0^2 = -m^2$ invariant

m = w:invariant mass $\equiv$ rest mass: invariant

AND conserved (in interactions): $\|\vec{p} + \vec{q}\|^2 = \|\vec{r}\|^2$

where interaction is (4-momenta):

$\vec{p} + \vec{q} \longrightarrow \vec{r}$

SR: $\vec{p}\ldots$: invariant or not?

momentum: $\vec{p} = m\gamma(1, \beta^x, \beta^y, \beta^z)^{\mathrm{T}}$

$p^0 = m + K = m\gamma$

non-rest frame: $\|\vec{p}\|^2 = -m^2\gamma^2 + m^2\gamma^2\beta^2 = -m^2$

rest frame: $\|\vec{p}\|^2 = -m^2 + 0^2 = -m^2$ invariant

m = w:invariant mass $\equiv$ rest mass: invariant

AND conserved (in interactions): $\|\vec{p} + \vec{q}\|^2 = \|\vec{r}\|^2$

where interaction is (4-momenta):

$\vec{p} + \vec{q} \longrightarrow \vec{r}$

WARNING: assume that 4-momentum vectors at different space-time positions can be parallel-transported; not the case in curved spacetime

SR: $\vec{p}\ldots$: invariant or not?

vector space $\Rightarrow p^i + q^i = r^i$ ($i = 1, 2, 3$)

SR: $\vec{p}\ldots$: invariant or not?

vector space $\Rightarrow p^i + q^i = r^i$ ($i = 1, 2, 3$)

= conservation of (relativistic) 3-momentum (Newtonian: conserved)

but NOT invariant (Newtonian: not invariant)

SR: $\vec{p} \ldots$: invariant or not?

vector space $\Rightarrow p^i + q^i = r^i$ ($i = 1, 2, 3$)

= conservation of (relativistic) 3-momentum (Newtonian: conserved)

but NOT invariant (Newtonian: not invariant)

vector space $\Rightarrow p^0 + q^0 = r^0$

SR: $\vec{p}\ldots$: invariant or not?

vector space $\Rightarrow p^i + q^i = r^i$ ($i = 1, 2, 3$)

= conservation of (relativistic) 3-momentum (Newtonian: conserved)

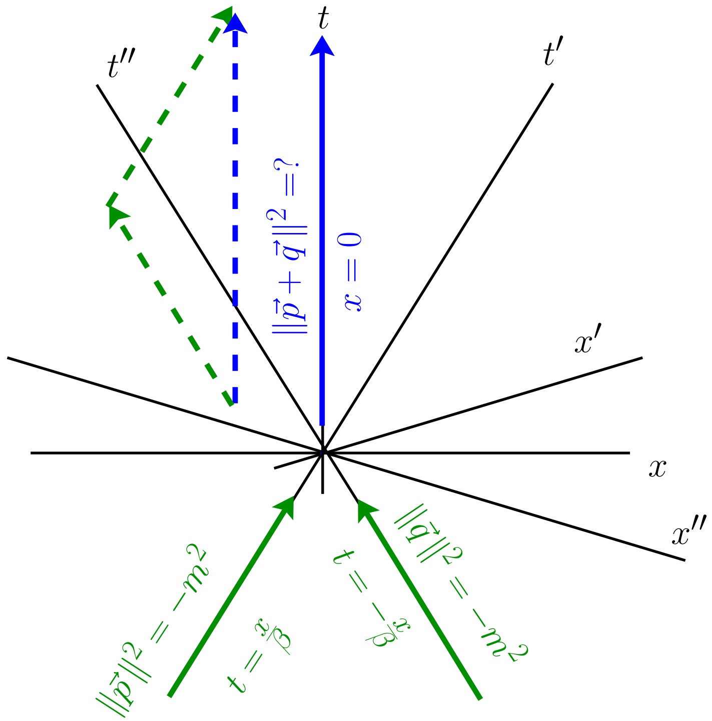
but NOT invariant (Newtonian: not invariant)

vector space $\Rightarrow p^0 + q^0 = r^0$

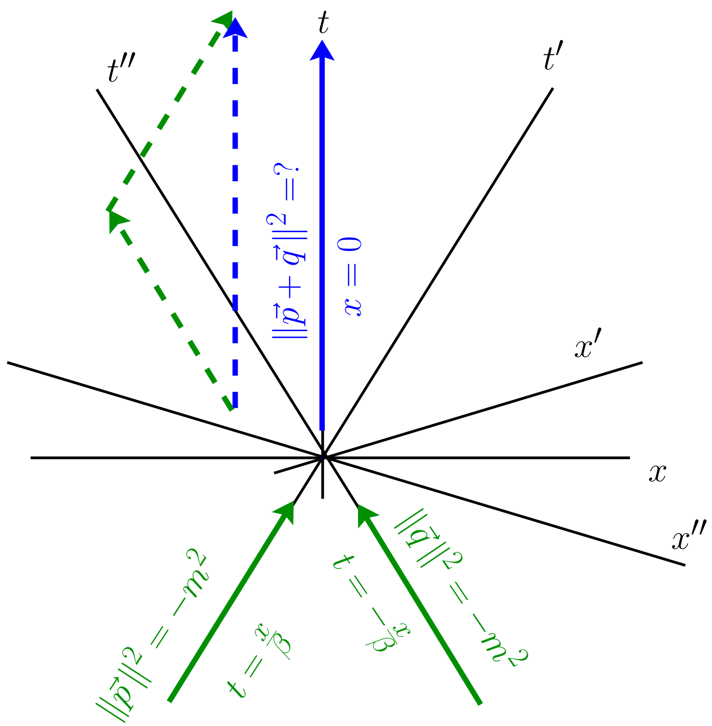
= conservation of (relativistic) "total energy" = $m + K$ (Newtonian: m conserved, K not conserved, $K+$ potential energy conserved)

but NOT invariant (Newtonian: m invariant, K not invariant)

SR: $\vec{p}\ldots$: invariant or not?

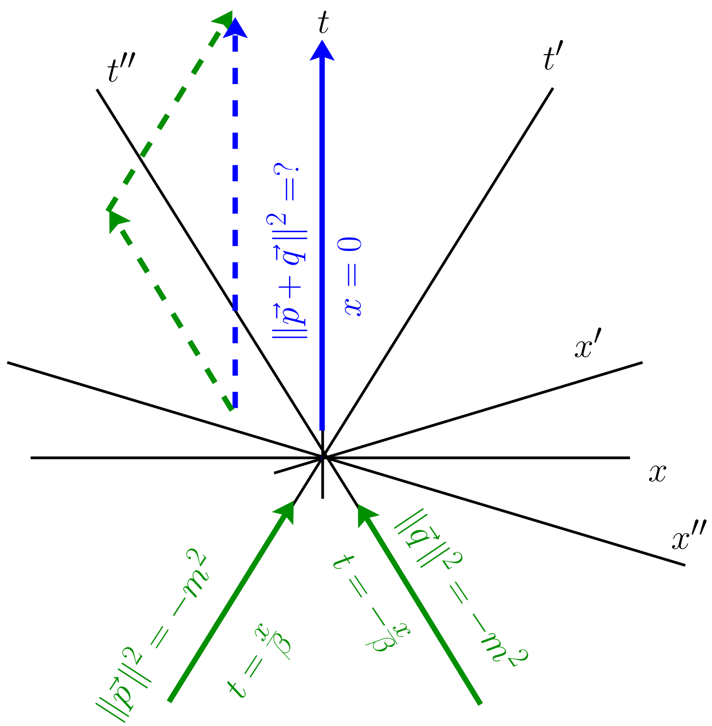


SR: $\vec{p}\ldots$: invariant or not?



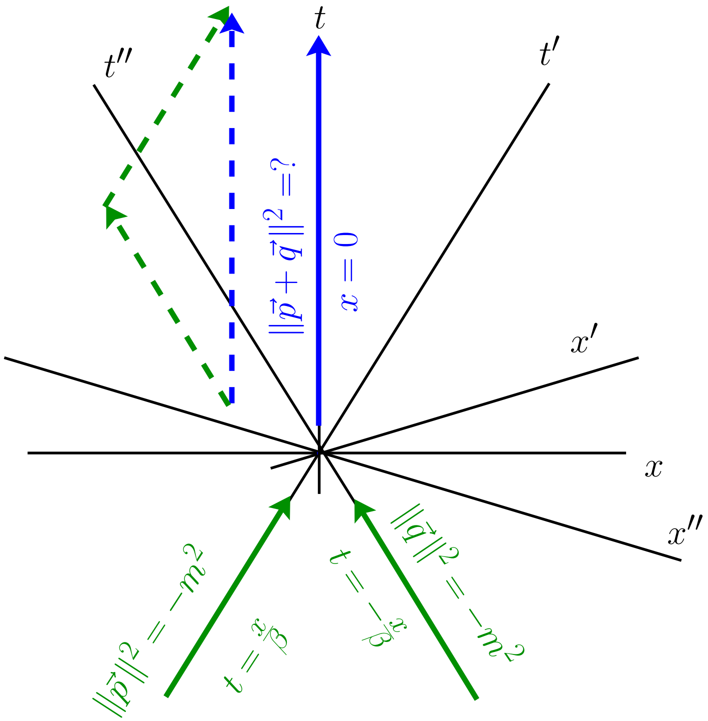
$$\vec{p} = m(\gamma, +\beta\gamma, 0, 0)^{\mathrm{T}}$$

SR: $\vec{p}\ldots$: invariant or not?



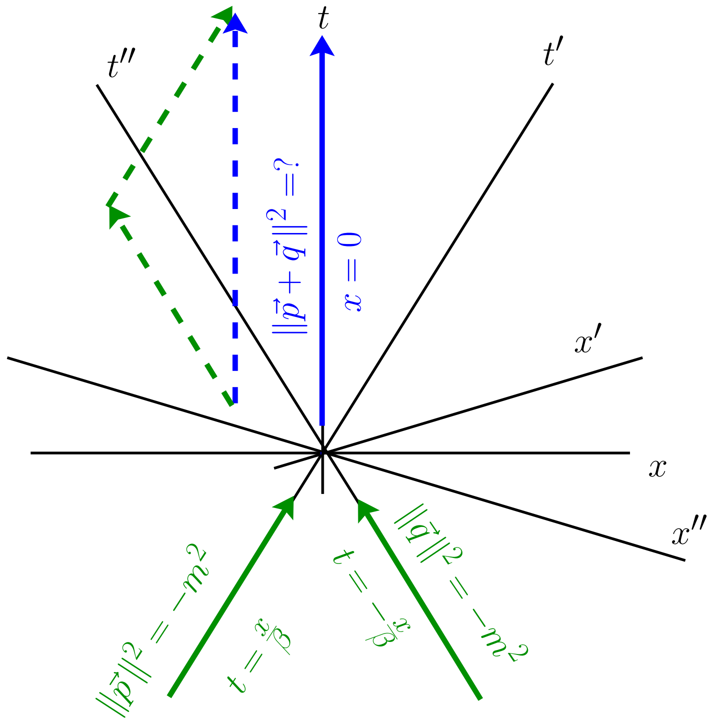
$$\vec{p} = m(\gamma, +\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{q} = m(\gamma, -\beta\gamma, 0, 0)^{\mathrm{T}}$$

SR: $\vec{p}\ldots$: invariant or not?



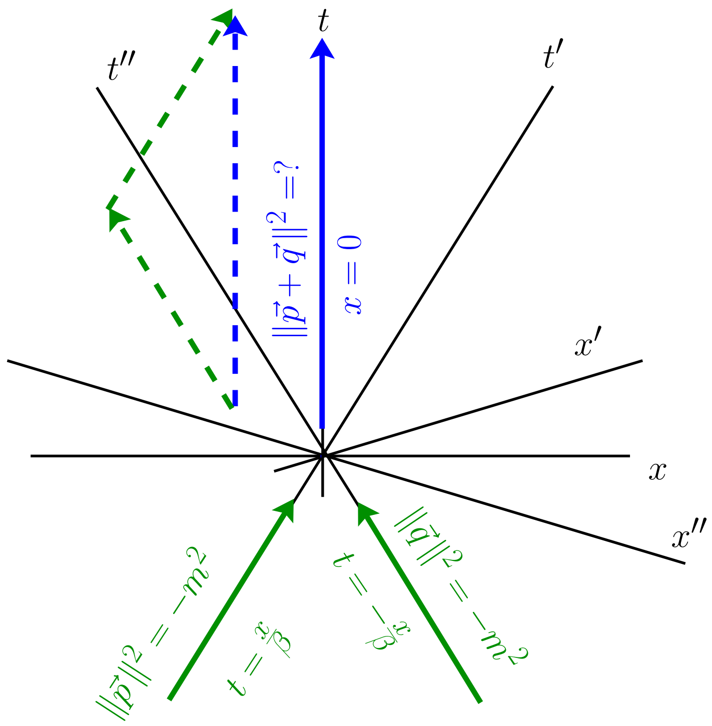
$$\vec{p} = m(\gamma, +\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{q} = m(\gamma, -\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{r} = \vec{p} + \vec{q}$$

SR: $\vec{p}\ldots$: invariant or not?



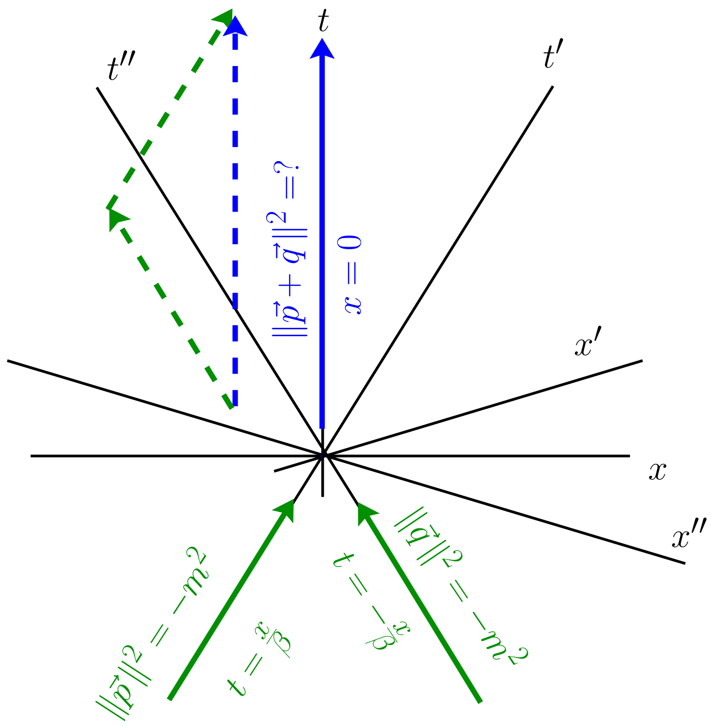
$$\vec{p} = m(\gamma, +\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{q} = m(\gamma, -\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{r} = \vec{p} + \vec{q}$$
$$= m[2\gamma, (-\beta + \beta)\gamma, 0, 0]^{\mathrm{T}}$$

SR: $\vec{p}\ldots$: invariant or not?



$$\vec{p} = m(\gamma, +\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{q} = m(\gamma, -\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{r} = \vec{p} + \vec{q} = 2m\gamma(1, 0, 0, 0)^{\mathrm{T}}$$

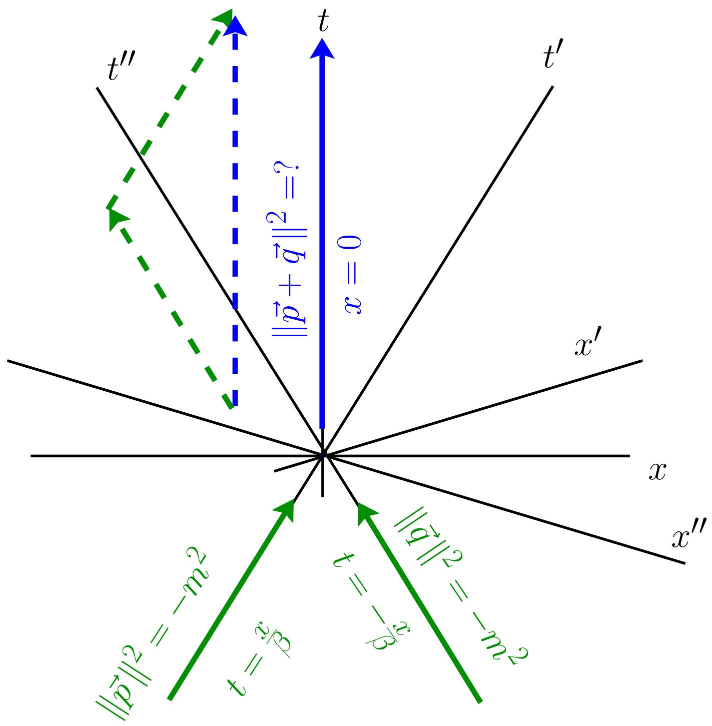
SR: $\vec{p}\ldots$: invariant or not?



system rest mass before and after: $2m\gamma$

$$\vec{p} = m(\gamma, +\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{q} = m(\gamma, -\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{r} = \vec{p} + \vec{q} = 2m\gamma(1, 0, 0, 0)^{\mathrm{T}}$$

SR: $\vec{p}\ldots$: invariant or not?



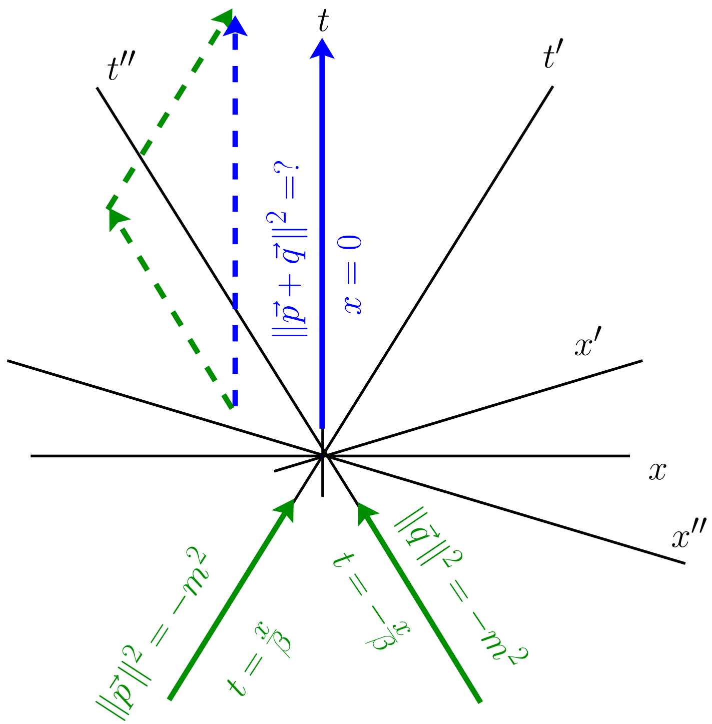
system rest mass before and after: $2m\gamma$

rest masses in many different frames:

$m + m \neq 2m\gamma$ if $\gamma \neq 1$

$$\vec{p} = m(\gamma, +\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{q} = m(\gamma, -\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{r} = \vec{p} + \vec{q} = 2m\gamma(1, 0, 0, 0)^{\mathrm{T}}$$

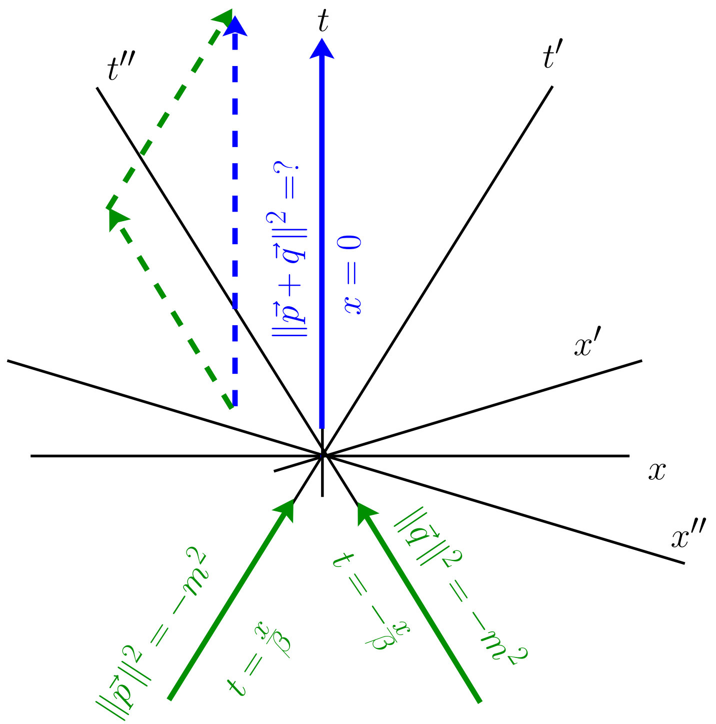
SR: $\vec{p}\ldots$: invariant or not?



$$\vec{p} = m(\gamma, +\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{q} = m(\gamma, -\beta\gamma, 0, 0)^{\mathrm{T}}$$
$$\vec{r} = \vec{p}+\vec{q} = 2m\gamma(1, 0, 0, 0)^{\mathrm{T}}$$

system rest mass before and after: $2m\gamma$
rest masses in many different frames:
$m + m \neq 2m\gamma$ if $\gamma \neq 1$
$\sqrt{-\|\vec{p}\|^2} + \sqrt{-\|\vec{q}\|^2} \neq \sqrt{-\|\vec{r}\|^2}$

SR: $\vec{p} \ldots$: invariant or not?



$\vec{p} = m(\gamma, +\beta\gamma, 0, 0)^{\mathrm{T}}$
$\vec{q} = m(\gamma, -\beta\gamma, 0, 0)^{\mathrm{T}}$
$\vec{r} = \vec{p} + \vec{q} = 2m\gamma(1, 0, 0, 0)^{\mathrm{T}}$

system rest mass before and after: $2m\gamma$
rest masses in many different frames:
$m + m \neq 2m\gamma$ if $\gamma \neq 1$
$\sqrt{-\|\vec{p}\|^2} + \sqrt{-\|\vec{q}\|^2} \neq \sqrt{-\|\vec{r}\|^2}$
system mass is invariant, but can be divided into p^0 and $p^i, i \in \{1, 2, 3\}$ components in many different ways

SR: $\vec{p}\ldots$: invariant or not?

interaction: momenta: $\vec{p} + \vec{q} \to \vec{r}$

moduli: m, m, $2m\gamma$

SR: $\vec{p}\ldots$: invariant or not?

interaction: momenta: $\vec{p} + \vec{q} \rightarrow \vec{r}$

moduli: m, m, $2m\gamma$

total energy $E := p^0$

SR: $\vec{p}\ldots$: invariant or not?

interaction: momenta: $\vec{p} + \vec{q} \to \vec{r}$

moduli: m, m, $2m\gamma$

total energy $E := p^0$

rest frame: $E := p^0$ = t component of $m(1, 0, 0, 0)^{\mathrm{T}}$

$E = m$ (the famous equation)

SR: $\vec{p}\ldots$: invariant or not?

interaction: momenta: $\vec{p} + \vec{q} \to \vec{r}$

moduli: m, m, $2m\gamma$

total energy $E := p^0$

rest frame: $E := p^0$ = t component of $m(1, 0, 0, 0)^{\mathrm{T}}$

$E = m$ (the famous equation)

this means: in the rest frame, $K + m = m$ (trivial)

SR: $\vec{p}\ldots$: invariant or not?

interaction: momenta: $\vec{p} + \vec{q} \to \vec{r}$

moduli: m, m, $2m\gamma$

total energy $E := p^0$

rest frame: $E := p^0$ = t component of $m(1,0,0,0)^{\mathrm{T}}$

$E = m$ (the famous equation)

this means: in the rest frame, $K + m = m$ (trivial)

more interesting: non-rest frame:

SR: $\vec{p}\ldots$: invariant or not?

interaction: momenta: $\vec{p} + \vec{q} \to \vec{r}$

moduli: m, m, $2m\gamma$

total energy $E := p^0$

rest frame: $E := p^0$ = t component of $m(1,0,0,0)^{\mathrm{T}}$

$E = m$ (the famous equation)

this means: in the rest frame, $K + m = m$ (trivial)

more interesting: non-rest frame:

$-m^2 = \|\vec{p}\|^2$

SR: $\vec{p}$ . . .: invariant or not?

interaction: momenta: $\vec{p} + \vec{q} \to \vec{r}$

moduli: m, m, $2m\gamma$

total energy $E := p^0$

rest frame: $E := p^0$ = t component of $m(1,0,0,0)^{\mathrm{T}}$

$E = m$ (the famous equation)

this means: in the rest frame, $K + m = m$ (trivial)

more interesting: non-rest frame:

$-m^2 = \|\vec{p}\|^2$

$-m^2 = \|(E, p^x, p^y, p^z)^{\mathrm{T}}\|^2$

SR: $\vec{p}\ldots$: invariant or not?

interaction: momenta: $\vec{p} + \vec{q} \rightarrow \vec{r}$

moduli: m, m, $2m\gamma$

total energy $E := p^0$

rest frame: $E := p^0$ = t component of $m(1, 0, 0, 0)^{\mathrm{T}}$

$E = m$ (the famous equation)

this means: in the rest frame, $K + m = m$ (trivial)

more interesting: non-rest frame:

$$-m^2 = \|\vec{p}\|^2$$

$$-m^2 = \|(E, p^x, p^y, p^z)^{\mathrm{T}}\|^2$$

$$-m^2 = -E^2 + p^2$$

SR: $\vec{p}\ldots$: invariant or not?

interaction: momenta: $\vec{p} + \vec{q} \to \vec{r}$

moduli: m, m, $2m\gamma$

total energy $E := p^0$

rest frame: $E := p^0$ = t component of $m(1,0,0,0)^{\mathrm{T}}$

$E = m$ (the famous equation)

this means: in the rest frame, $K + m = m$ (trivial)

more interesting: non-rest frame:

$-m^2 = \|\vec{p}\|^2$

$-m^2 = \|(E, p^x, p^y, p^z)^{\mathrm{T}}\|^2$

$m^2 = E^2 - p^2$

SR: $\vec{p}\ldots$: invariant or not?

interaction: momenta: $\vec{p} + \vec{q} \to \vec{r}$

moduli: m, m, $2m\gamma$

total energy $E := p^0$

rest frame: $E := p^0$ = t component of $m(1,0,0,0)^{\mathrm{T}}$

$E = m$ (the famous equation)

this means: in the rest frame, $K + m = m$ (trivial)

more interesting: non-rest frame:

$-m^2 = \|\vec{p}\|^2$

$-m^2 = \|(E, p^x, p^y, p^z)^{\mathrm{T}}\|^2$

$m^2 = E^2 - \|^{(3)}\vec{p}\|^2$

SR: null 4-momentum

photon: $m = 0$

SR: null 4-momentum

photon: $m = 0$

extend defn of 4-momentum to $\vec{p}$ for a photon

SR: null 4-momentum

photon: $m = 0$

extend defn of 4-momentum to $\vec{p}$ for a photon

$\not\Rightarrow \vec{u} = \frac{\vec{p}}{m}$ since $m = 0$; no 4-velocity

SR: null 4-momentum

photon: $m = 0$

extend defn of 4-momentum to $\vec{p}$ for a photon

$\not\Rightarrow \vec{u} = \frac{\vec{p}}{m}$ since $m = 0$; no 4-velocity

so $0 = E^2 - p^2$

SR: null 4-momentum

photon: $m = 0$

extend defn of 4-momentum to $\vec{p}$ for a photon

$\not\Rightarrow \vec{u} = \frac{\vec{p}}{m}$ since $m = 0$; no 4-velocity

so $0 = E^2 - p^2$

i.e. $\vec{p} = (E, E, 0, 0)^{\mathrm{T}} = (p, p, 0, 0)^{\mathrm{T}}$ (if x direction)

SR: null 4-momentum

photon: $m = 0$

extend defn of 4-momentum to $\vec{p}$ for a photon

$\not\Rightarrow \vec{u} = \frac{\vec{p}}{m}$ since $m = 0$; no 4-velocity

so $0 = E^2 - p^2$

i.e. $\vec{p} = (E, E, 0, 0)^{\mathrm{T}} = (p, p, 0, 0)^{\mathrm{T}}$ (if x direction)

so $E = p$ for any massless particle

SR: model summary

Minkowski spacetime: draw a correct diagram

SR: model summary

Minkowski spacetime: draw a correct diagram

Lorentz transformation (boost) $\Lambda(\phi)$ or $\Lambda(\beta)$

SR: model summary

Minkowski spacetime: draw a correct diagram

Lorentz transformation (boost) $\Lambda(\phi)$ or $\Lambda(\beta)$

refuse the assumption of absolute simultaneity (time)