Cosmic topology

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https://cosmo.torun.pl/~boud/Cosmic_topology.pdf

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  - ◆ standard model: density perturbations (anisotropy)
  - ◆ verbal averaging: can we do better?

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  - standard model: density perturbations (anisotropy)
  - scalar (GR) averaging: statistically homogeneous spatial slices
- within this model, what is the shape of the Universe?

verbal averaging

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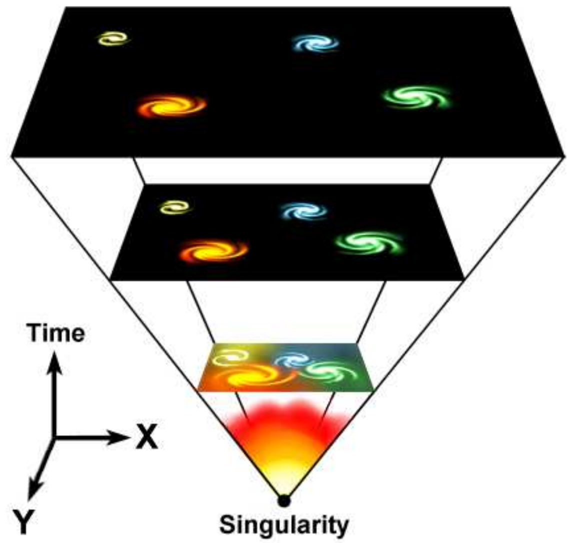
  3. assume that $(M, \mathbf{g})$ remains unchanged if we add density perturbations to an early time slice

verbal averaging

- w:Comoving coordinates

verbal averaging



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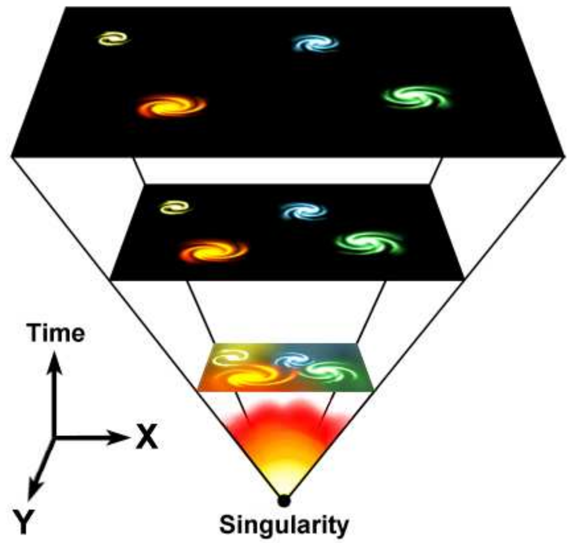

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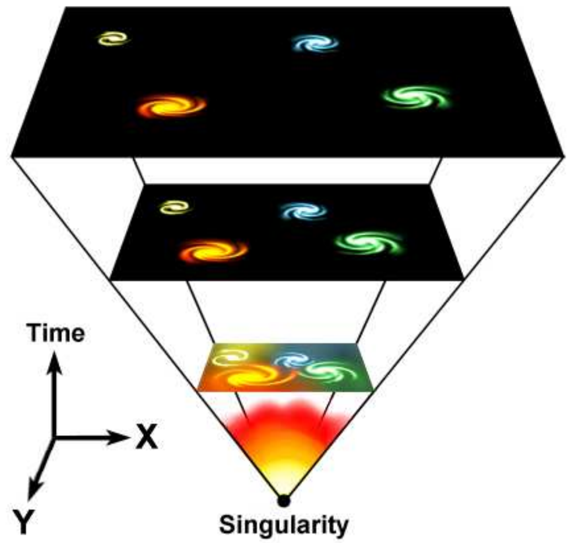
-

$$\Delta x(t) = a(t)\Delta r$$

verbal averaging



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- spherical coordinates for spatial slice

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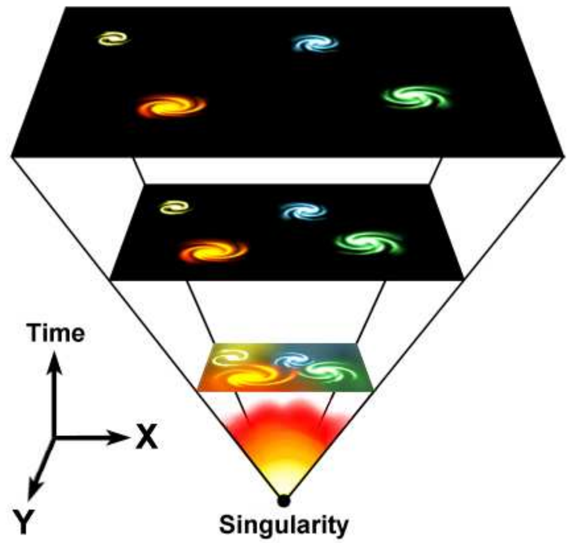

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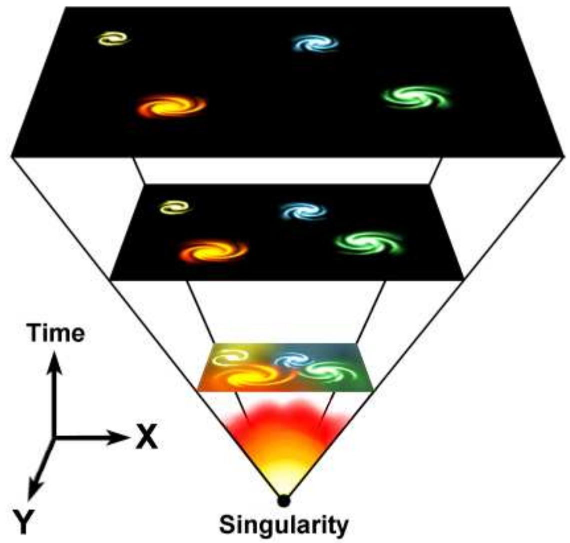

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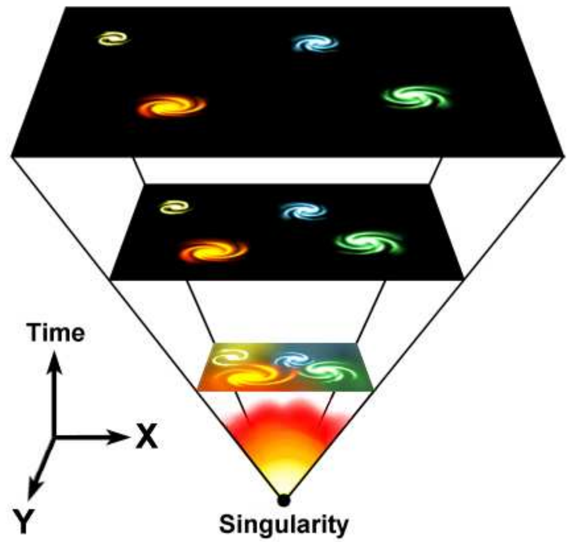

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- universe is static in comoving coordinates (r, θ, ϕ)

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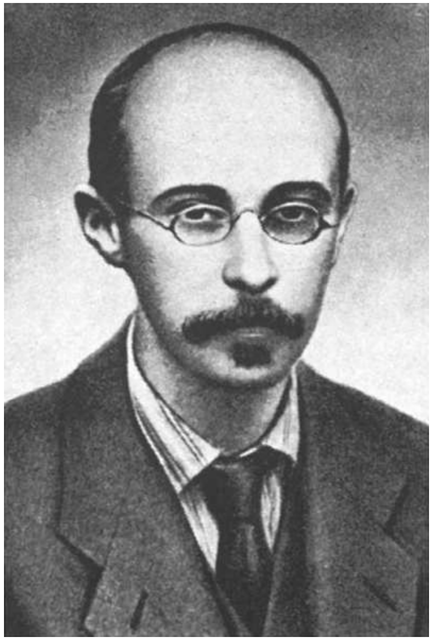
- w: w:
  w:Howard Percy Robertson
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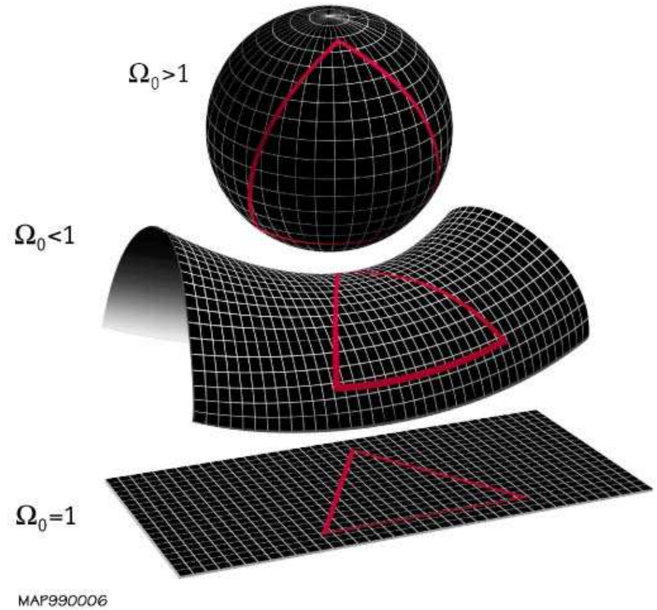

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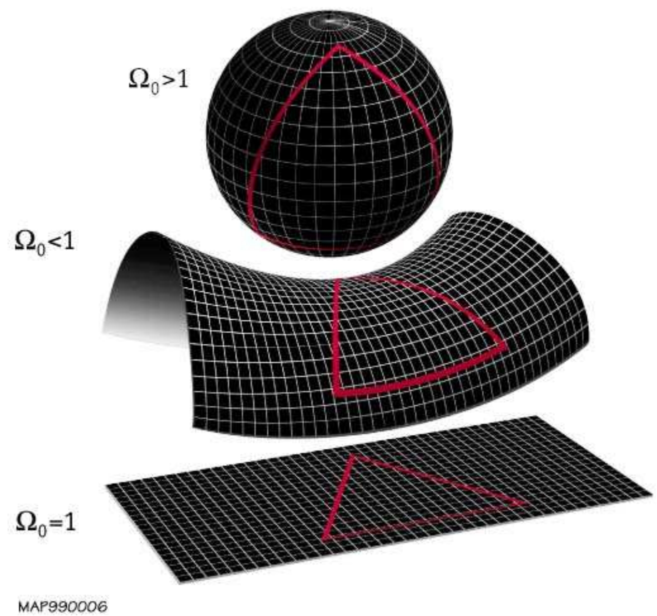

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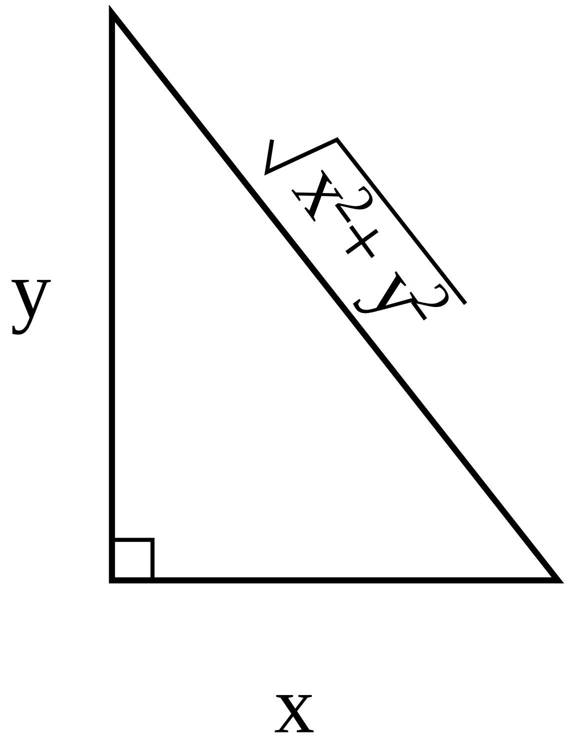
for a comoving radius of curvature R_C and curvature of sign k

curvature

- on a spatial slice (fixed value of t):



$k = 0$

curvature

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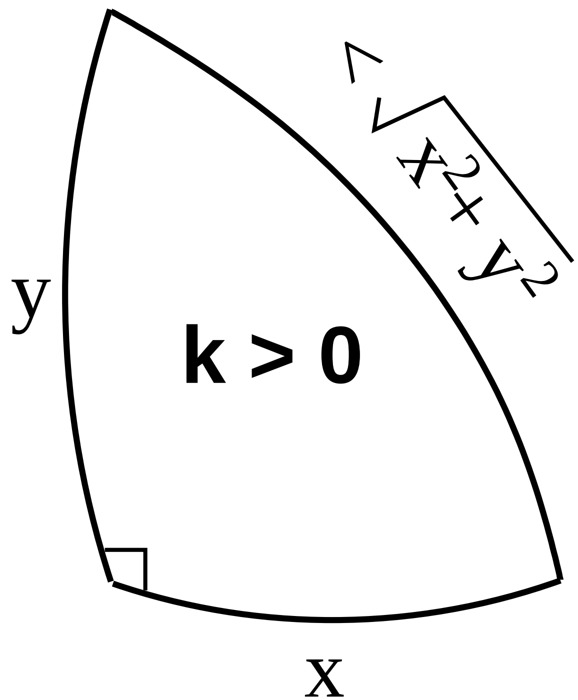

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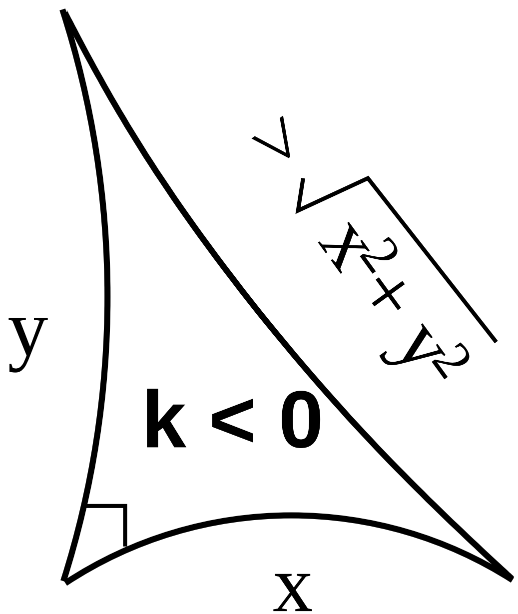

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2D curvature intuition: $k > 0$

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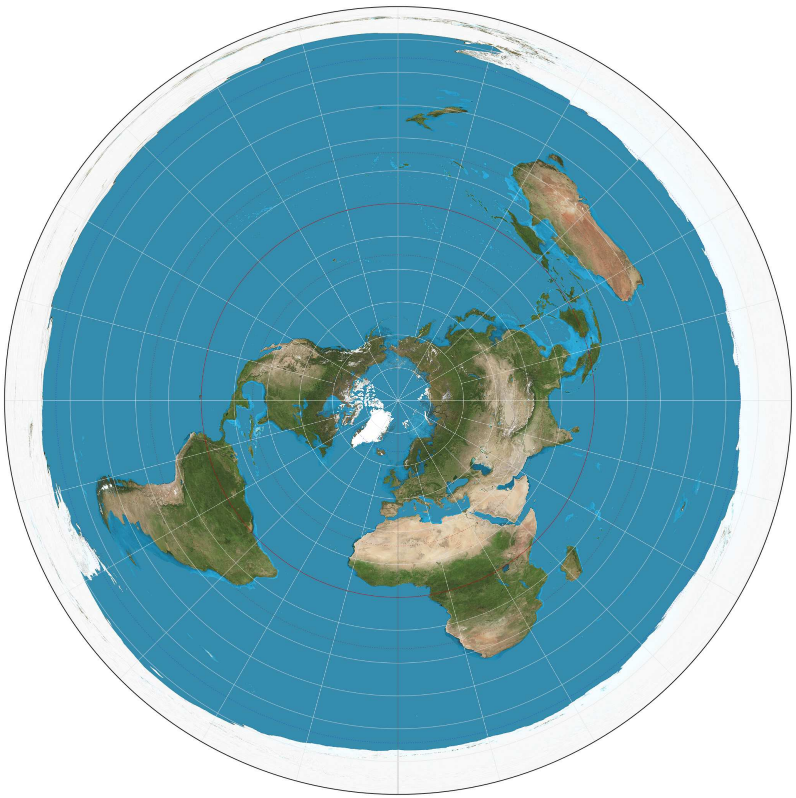

w: (al-Biruni, c. 1000 CE)

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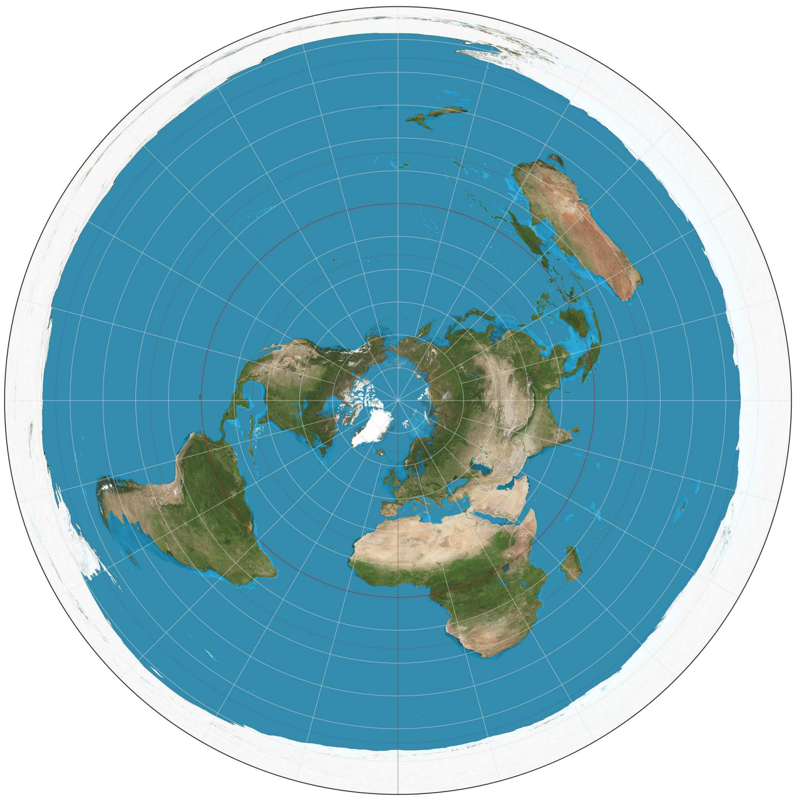

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- intuition switch: S^2 easier vs S^3 more physical

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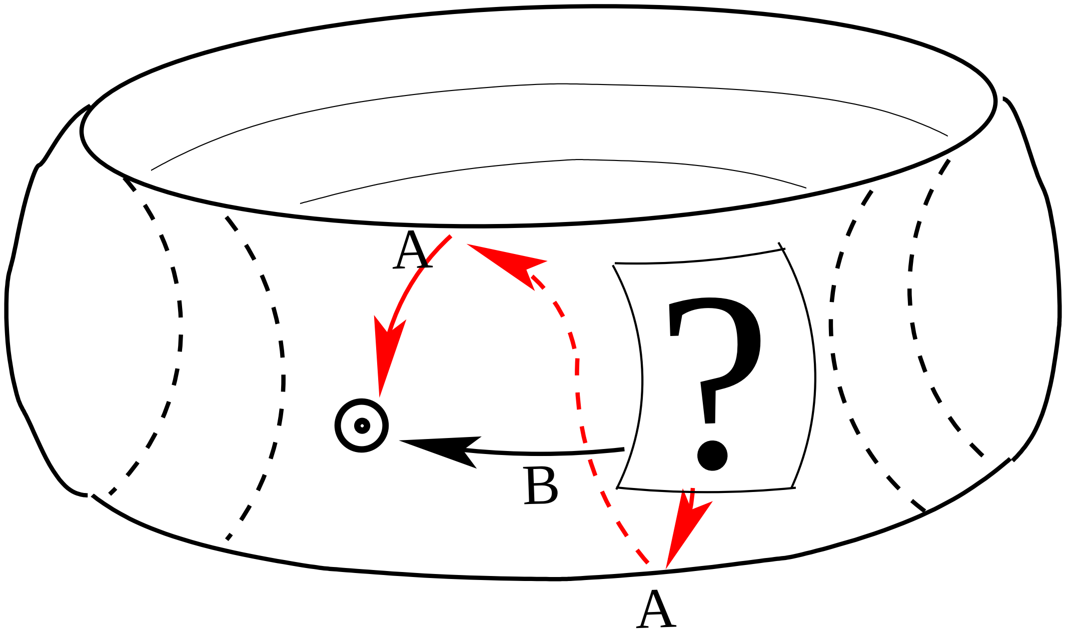

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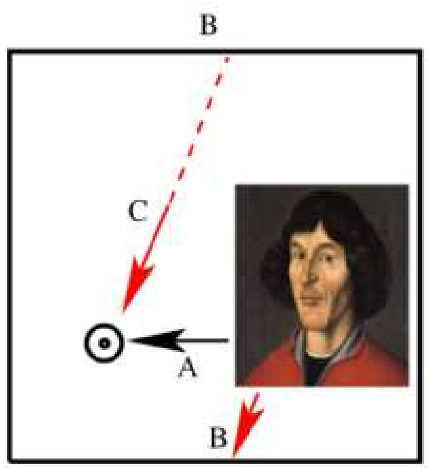

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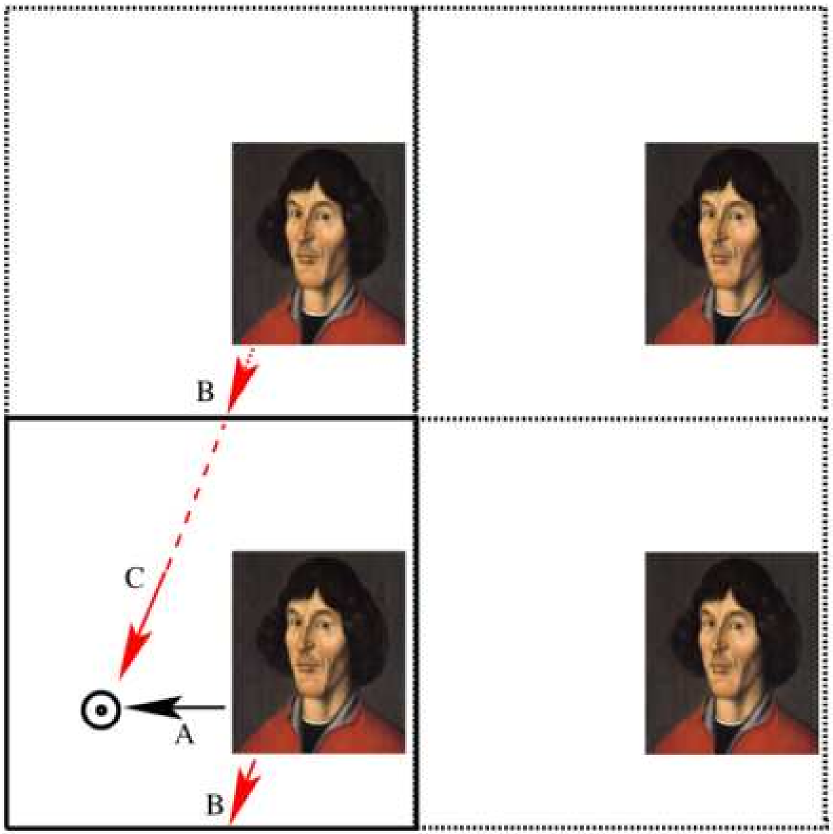

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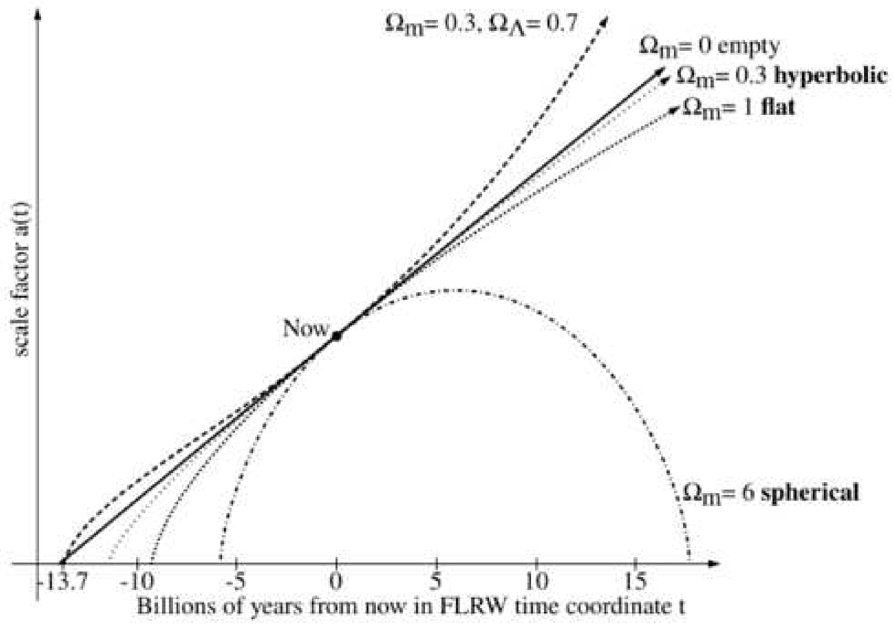

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$\Leftrightarrow$

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  (Defn: $a_0 := 1$)

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  (Defn of redshift z)

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- radiation density: $E = h\nu \Rightarrow \rho_{\rm r} \propto a^{-4} = (1+z)^4$

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CMB discovery: McKellar 1941

- $T \approx 2.3$ K — Andrew McKellar (1941; ADS:1941PDAO....7..251M) from observations by Walter S. Adams (1941; ADS:1941ApJ....93...11A)
- Penzias & Wilson rediscovery (1965 + Nobel prize)

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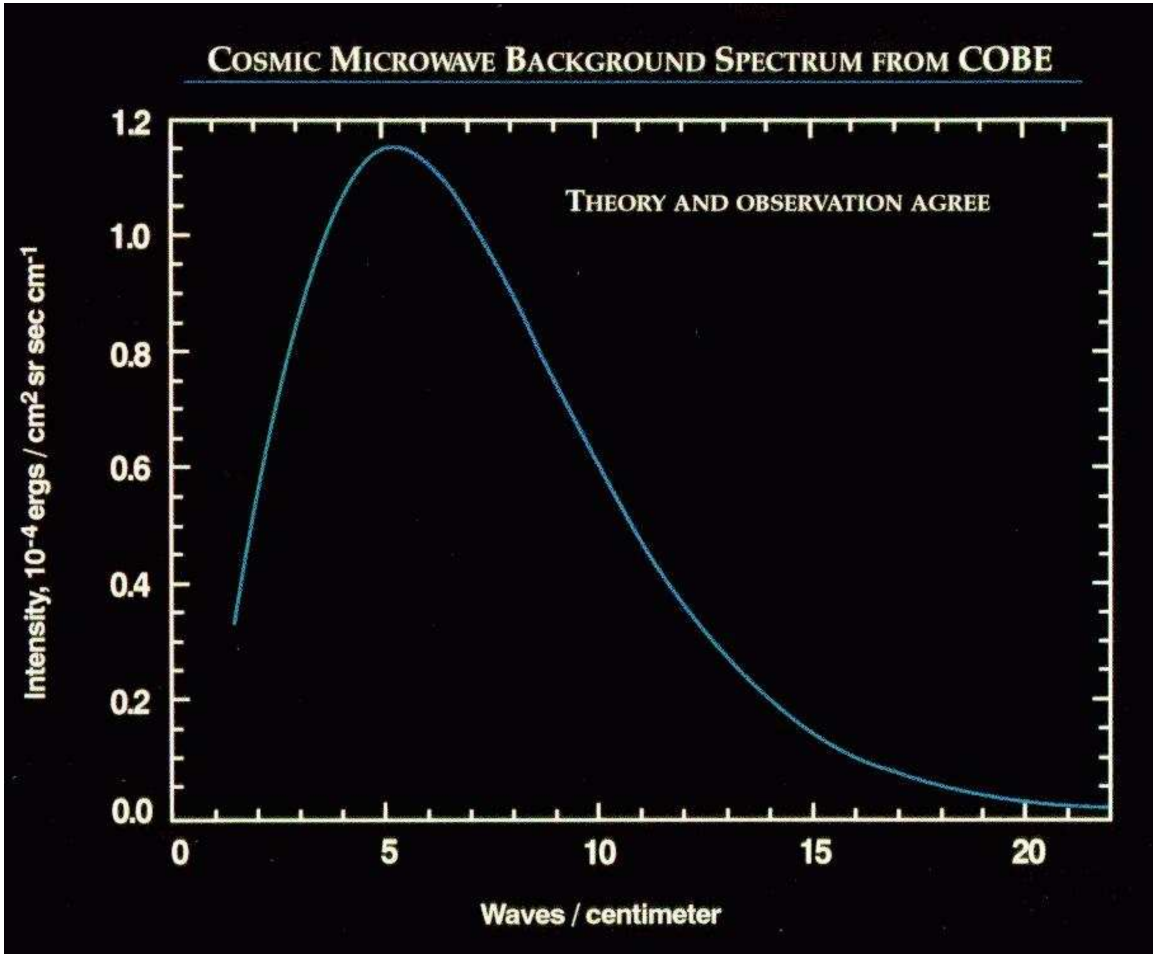

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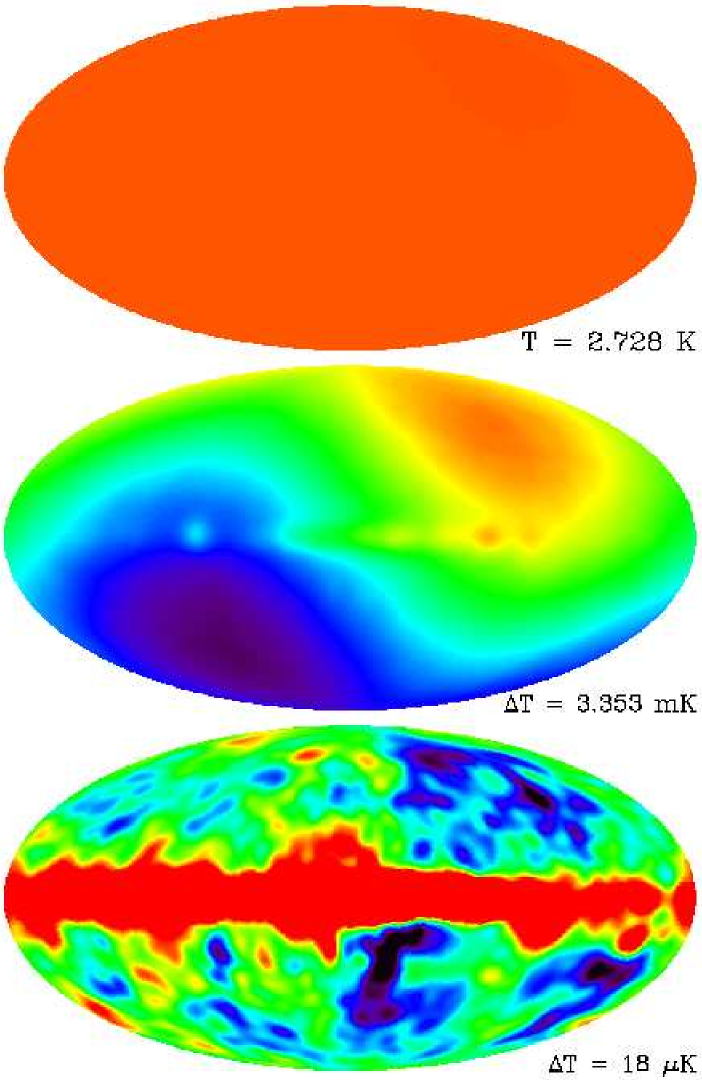
- COBE /DMR (Differential Microwave Radiometer)



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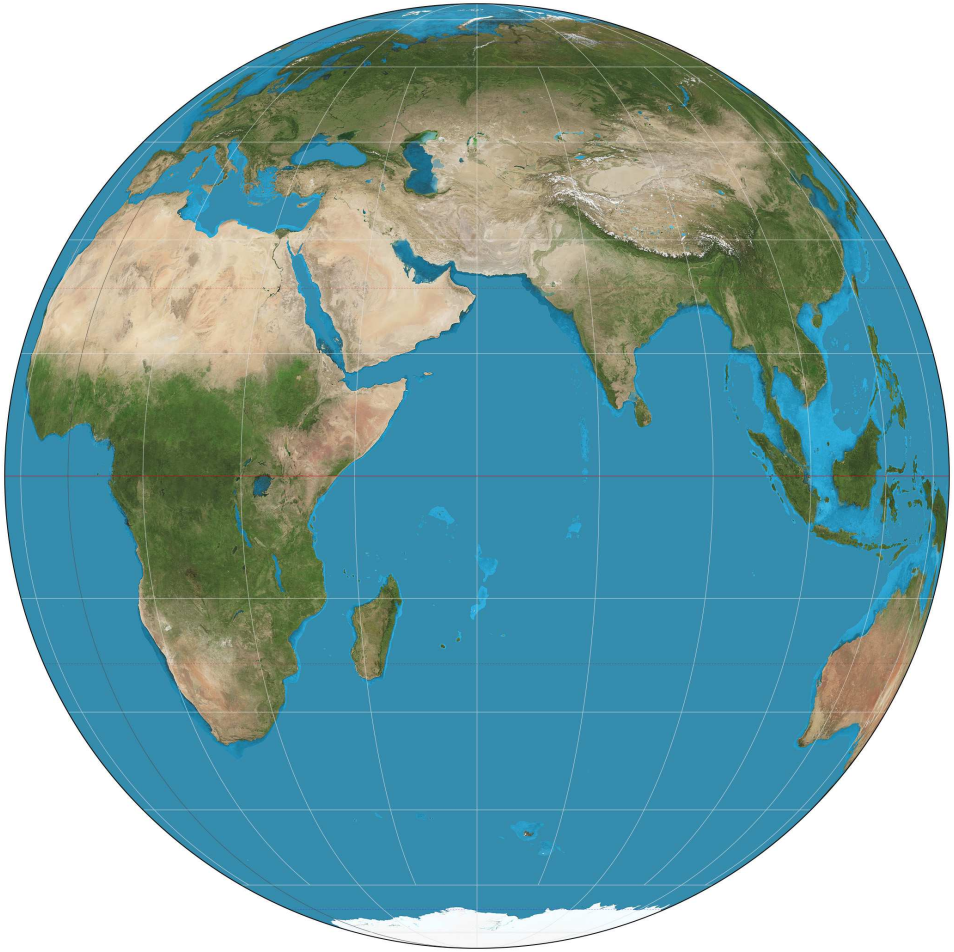

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- convenient conversion: 1 km/s ≈ 1.04 kpc/Gyr ≈ 1 kpc/Gyr

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- 1980's: $H_0 \approx 0.05$ or 0.1 Gyr^{-1} $\Rightarrow t_0(\mathrm{EdS}) \approx 13.0$ or 6.5 Gyr, resp.

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  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$

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Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter

FLRW: ρ_{crit}

- Friedmann Eqn: $H^2 = \Omega_{\mathrm{m}} H^2 - \frac{c^2\, k}{a^2}$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2\, k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

FLRW: ρ_{crit}

- Friedmann Eqn: $H^2 = \Omega_{\mathrm{m}} H^2 + \Omega_k H^2$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

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FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\text{m}} + \Omega_k}$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\text{m}0} = 1 \Leftrightarrow k = 0$

FLRW: ρ_{crit}

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Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\mathrm{m}0} = 1 \Leftrightarrow k = 0$ flat

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm m} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$

FLRW: ρ_{crit}

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Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\mathrm{m}0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\mathrm{m}0} > 1 \Leftrightarrow k > 0$ spherical

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  - $\Omega_{\mathrm{m}0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\mathrm{m}0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\mathrm{m}0} < 1 \Leftrightarrow k < 0$

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\text{m}} + \Omega_k}$

Defn: $\boxed{\rho_{\text{crit}} := \frac{3H^2}{8\pi G}}$ critical density

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Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\text{m}0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\text{m}0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\text{m}0} < 1 \Leftrightarrow k < 0$ hyperbolic

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm tot} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\rm tot} :=$

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\text{tot}} + \Omega_k}$

Defn: $\boxed{\rho_{\text{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\text{m}} := \frac{\rho}{\rho_{\text{crit}}}}$ matter density parameter

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\text{m}0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\text{m}0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\text{m}0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\text{tot}} := \Omega_{\text{m}} +$

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm tot} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\rm tot} := \Omega_{\rm m} + \Omega_{\rm r} +$

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm tot} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\rm tot} := \Omega_{\rm m} + \Omega_{\rm r} + \Omega_\Lambda$

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm tot} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\rm tot} := \Omega_{\rm b} + \Omega_{\rm nbDM} + \Omega_{\rm r} + \Omega_\Lambda$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: $R_{\rm C}$
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\rm C}^2 = 1 \Rightarrow k = 1/R_{\rm C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; \Omega_{k0} = -\frac{c^2 k}{H_0^2}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_C
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_\mathrm{C}^2 = 1 \Rightarrow k = 1/R_\mathrm{C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; k = -\frac{\Omega_{k\,0} H_0^2}{c^2}$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; R_{\mathrm{C}}^2 = -\frac{c^2}{\Omega_{k0} H_0^2}$

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  - azimuthal equidistant coords: R_{C}
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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; R_{\mathrm{C}}^2 = -\frac{c^2}{H_0^2}\frac{1}{\Omega_{k0}}$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $R_{\mathrm{C}}^2 = -\frac{c^2}{H_0^2}\frac{1}{1-\Omega_{\mathrm{tot}\,0}}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: $R_{\rm C}$
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\rm C}^2 = 1 \Rightarrow k = 1/R_{\rm C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\rm C} = \frac{c}{H_0}\frac{1}{\sqrt{\Omega_{\rm tot0} - 1}}}$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- orthographic: $1 - kr^2 = 0$ coord singularity at equator
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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$

FLRW curvature constant

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  - azimuthal equidistant coords: R_C
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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_\mathrm{C} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_\mathrm{tot0} - 1}}}$
- $\Omega_\mathrm{tot0} > 1$ *spherical*

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real

FLRW curvature constant

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  - azimuthal equidistant coords: R_C
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_\mathrm{C}^2 = 1 \Rightarrow k = 1/R_\mathrm{C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $$R_\mathrm{C} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}$$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\mathrm{tot}0} = 1$

FLRW curvature constant

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  - azimuthal equidistant coords: R_C
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_\mathrm{C}^2 = 1 \Rightarrow k = 1/R_\mathrm{C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_\mathrm{C} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\mathrm{tot}0} = 1$ *flat*

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_C
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_C undefined
- $\Omega_{\mathrm{tot}0} < 1$

FLRW curvature constant

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined
- $\Omega_{\mathrm{tot}0} < 1$ *hyperbolic*

FLRW curvature constant

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  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_\mathrm{C}^2 = 1 \Rightarrow k = 1/R_\mathrm{C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_\mathrm{C} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_C undefined
- $\Omega_{\mathrm{tot}0} < 1$ *hyperbolic* R_C imaginary (or use $|R_\mathrm{C}|$)

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Einstein's free parameter: Λ

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} + \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
  https://cosmo.torun.pl/Cosmo/FLRWEquationsGR

Einstein's free parameter: Λ

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} + \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
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- *hint:* mixed index form of $\mathbf{g}$ is easy

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  Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

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Friedmann Eqn ($\Lambda \neq 0$):

$$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

acceleration Eqn ($\Lambda \neq 0$):

$$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G\;(\rho + 3\,p/c^2)}{3} + \frac{c^2\,\Lambda}{3}$$

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Friedmann Eqn ($\Lambda \neq 0$):
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Defn: "dust solution": $p(t) = 0$

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Friedmann Eqn ($\Lambda \neq 0$):

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acceleration Eqn ($\Lambda \neq 0$):

$$\frac{\ddot{a}}{a} = -\frac{4\,\pi\, G\, (\rho + 3\, p/c^2)}{3} + \frac{c^2\, \Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\, \Lambda}{3\, H^2}$

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{H^2}{2}\frac{\rho}{\rho_{\mathrm{crit}}} + \Omega_\Lambda\,H^2$$

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\, k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\, G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{H^2\,\Omega_{\rm m}}{2} + \Omega_\Lambda\, H^2$$

Einstein's free parameter: Λ

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2} = -\frac{\Omega_{\mathrm{m}}}{2} + \Omega_\Lambda$$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2} = -\frac{\Omega_{\mathrm{m}}}{2} + \Omega_\Lambda$$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$

Einstein's free parameter: Λ

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

acceleration Eqn ($\Lambda \neq 0$): $$q = \frac{\Omega_{\mathrm{m}}}{2} - \Omega_\Lambda$$

Einstein's free parameter: Λ

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  ADS:1917SPAW.......142E

  Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

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distances in FLRW cosmology

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Low z limit Hubble law

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$\Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}z} \approx \frac{c}{H_0}$

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&\approx -\int_z^0 \frac{c}{H_0} \frac{1\,\mathrm{d}z}{\sqrt{1}} \\
&= -\int_z^0 \frac{c}{H_0}\,\mathrm{d}z
\end{aligned}
$$

$\Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}z} \approx \frac{c}{H_0}$

$\Rightarrow \quad$ when $0 < z \ll 1$, $\beta := v/c$, $\; r \approx \dfrac{z\,c}{H_0}$

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$\Rightarrow$ $\frac{\mathrm{d}r}{\mathrm{d}z} \approx \frac{c}{H_0}$

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$\Rightarrow$ when $0 < z \ll 1$, $\beta := v/c$, $\boxed{r \approx \frac{z\,c}{H_0} \approx \frac{\beta\,c}{H_0} = \frac{v}{H_0}}$

EdS: radial comoving distance

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$

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$$= \frac{c}{H_0} \left[2a^{1/2} \right]_{1/(1+z)}^{1}$$

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$$= \boxed{\frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)}$$

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- GPL numerical package: `cosmdist`
  https://codeberg.org/boud/cosmdist

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- high-level frontends (e.g. python) should be easy to write

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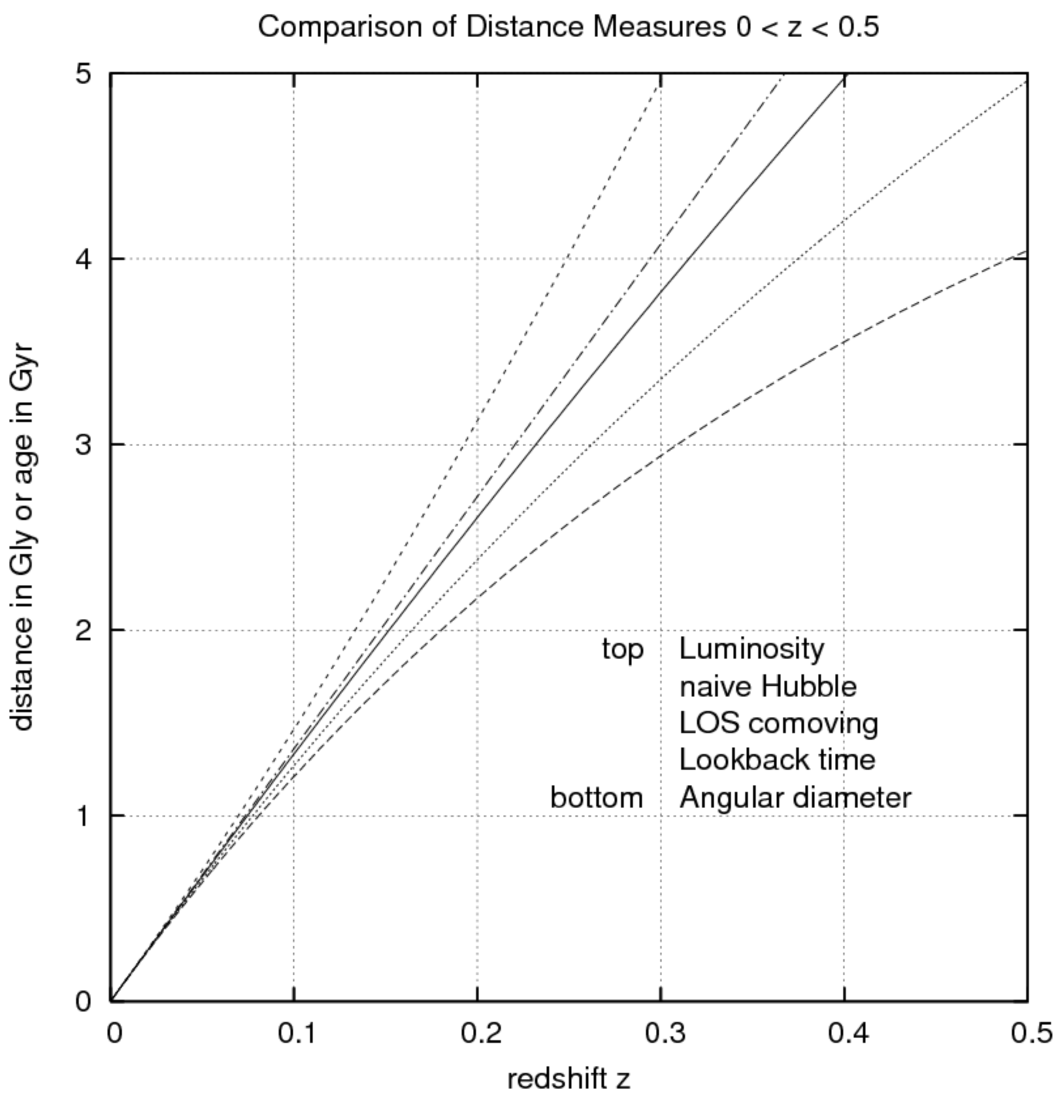
- w:Distance measures (cosmology)

FLRW distances e.g. ΛCDM



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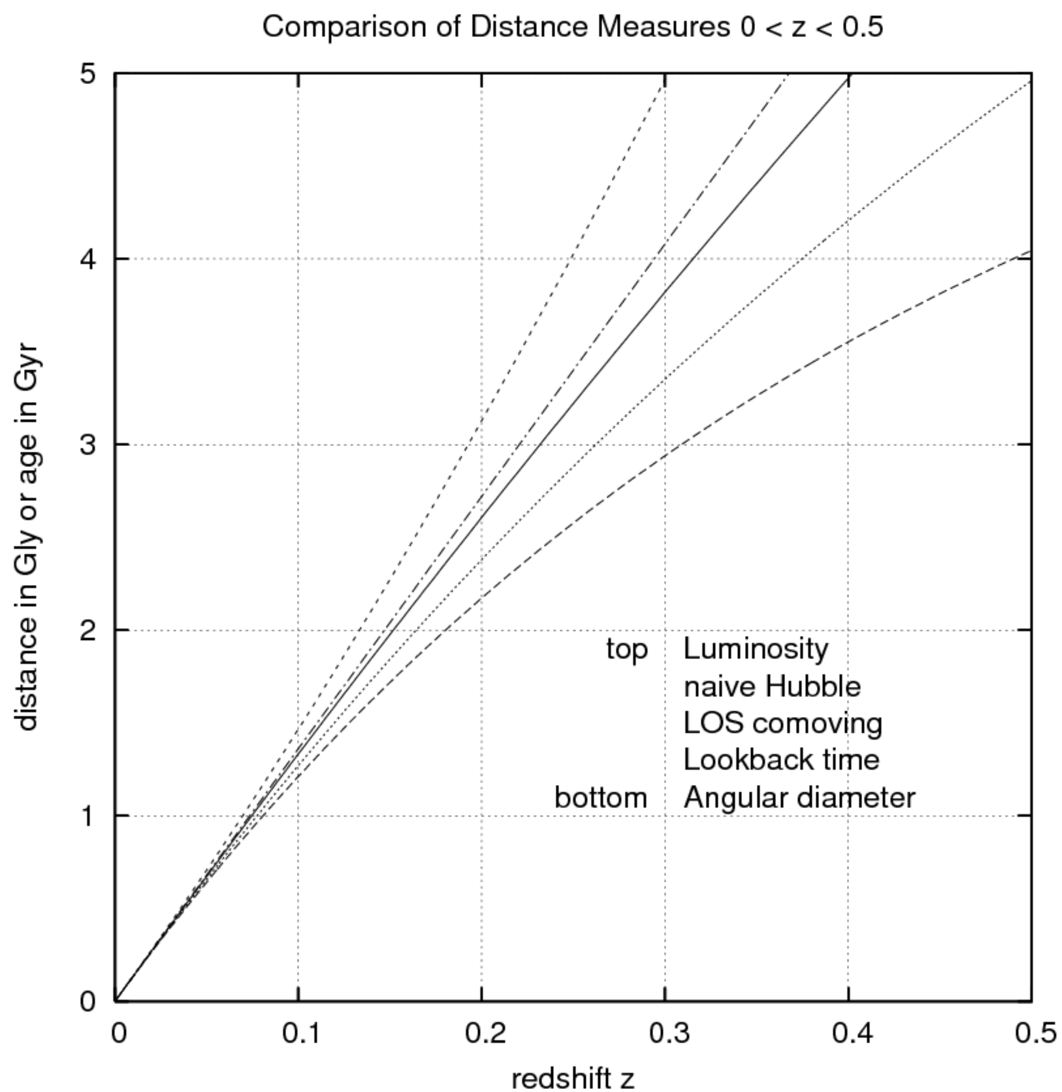

Defn: $\boxed{h := H_0/100 \text{ km/s/Mpc}}$ (without a "0" subscript on h)

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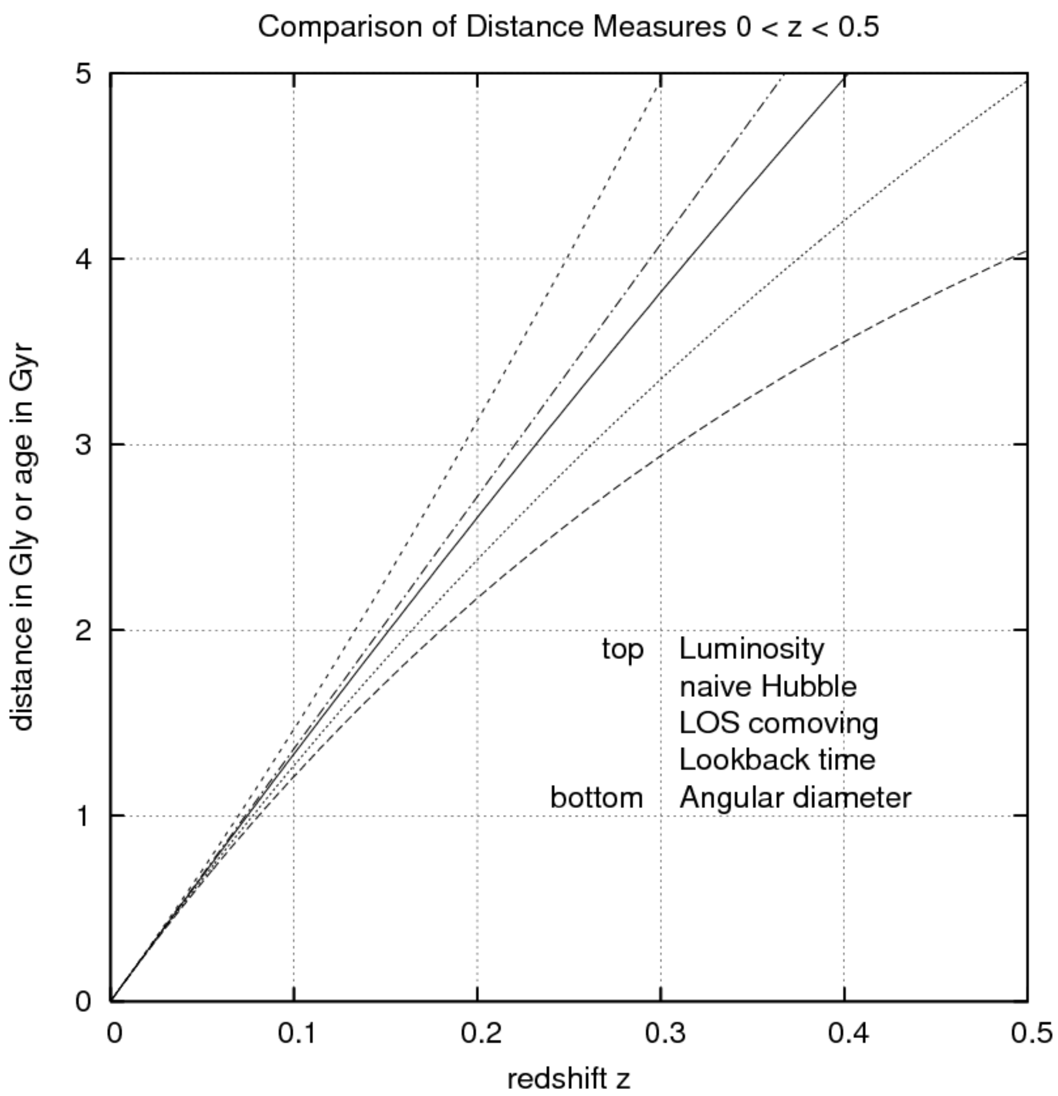

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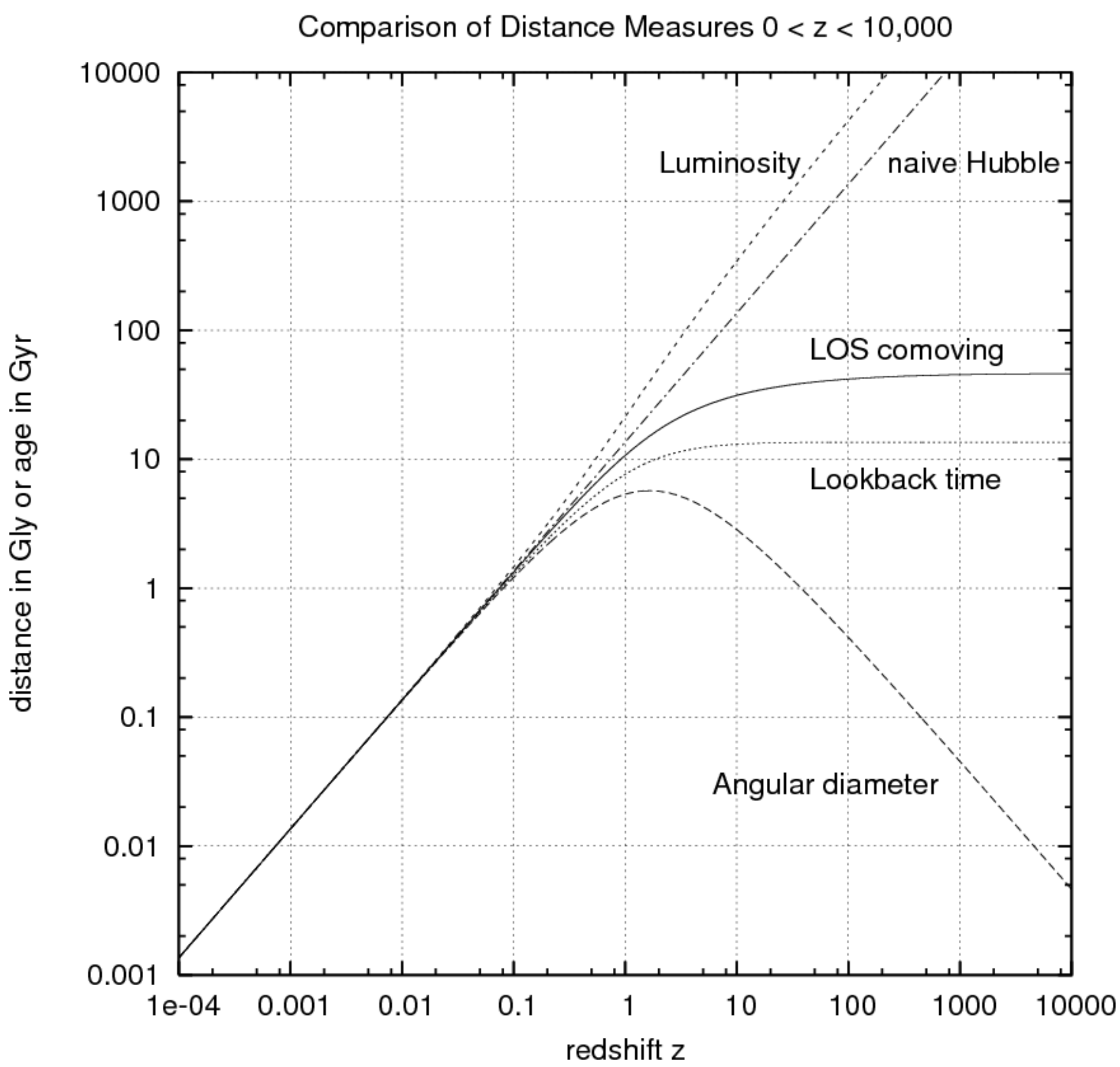

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"Velocity" $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of "velocity":
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption ($\|$ transport + Mink.) or
  3. $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = r H_0$ or
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$\Rightarrow$ no conflict with locally Lorentzian (SR) spacetime

Non-radial spatial geodesics

- What is the comoving distance between two objects at different celestial positions and different redshifts, for an arbitrary curvature $(+, 0, -)$?

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- distances on the 2-sphere, embedded in $\mathbb{R}^3$

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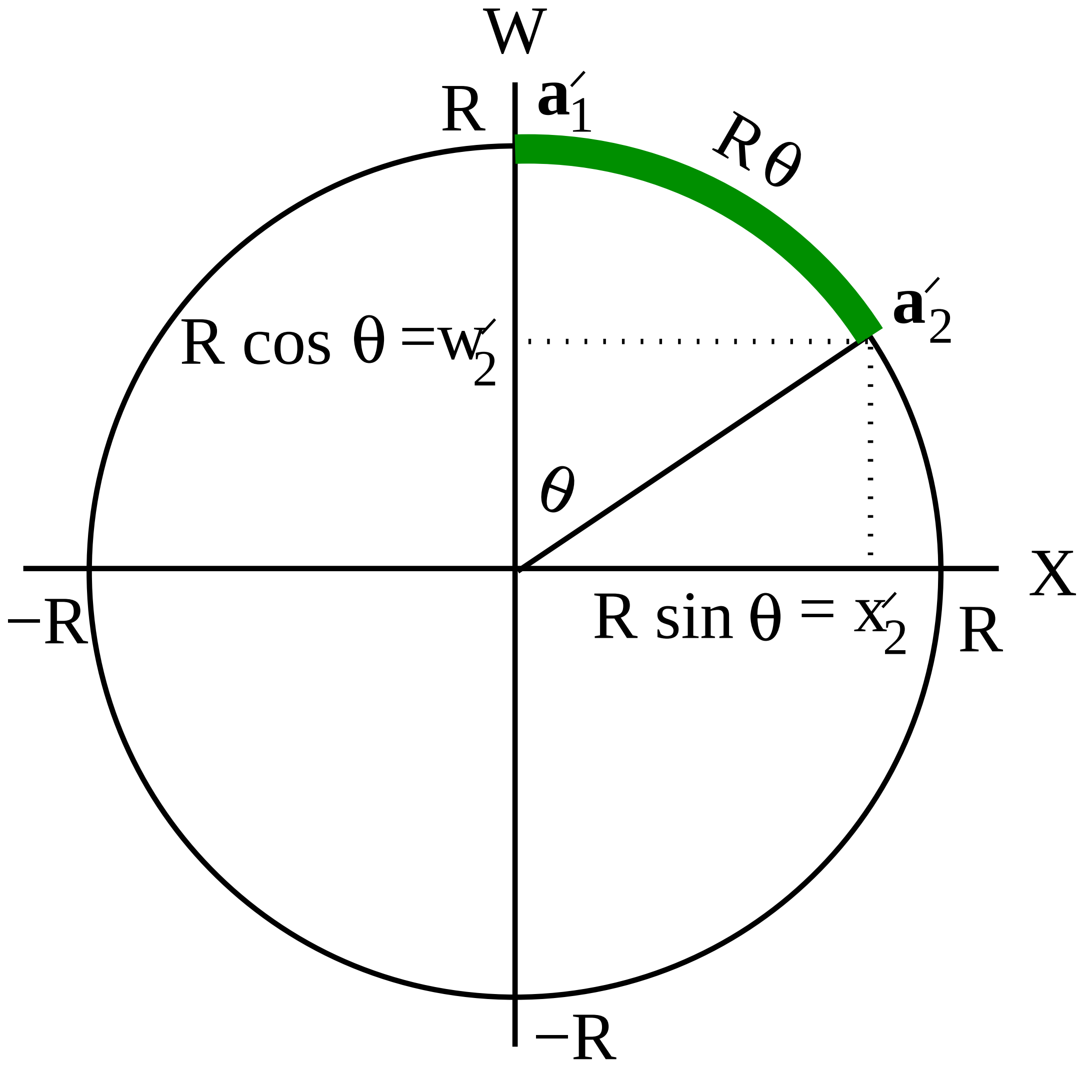

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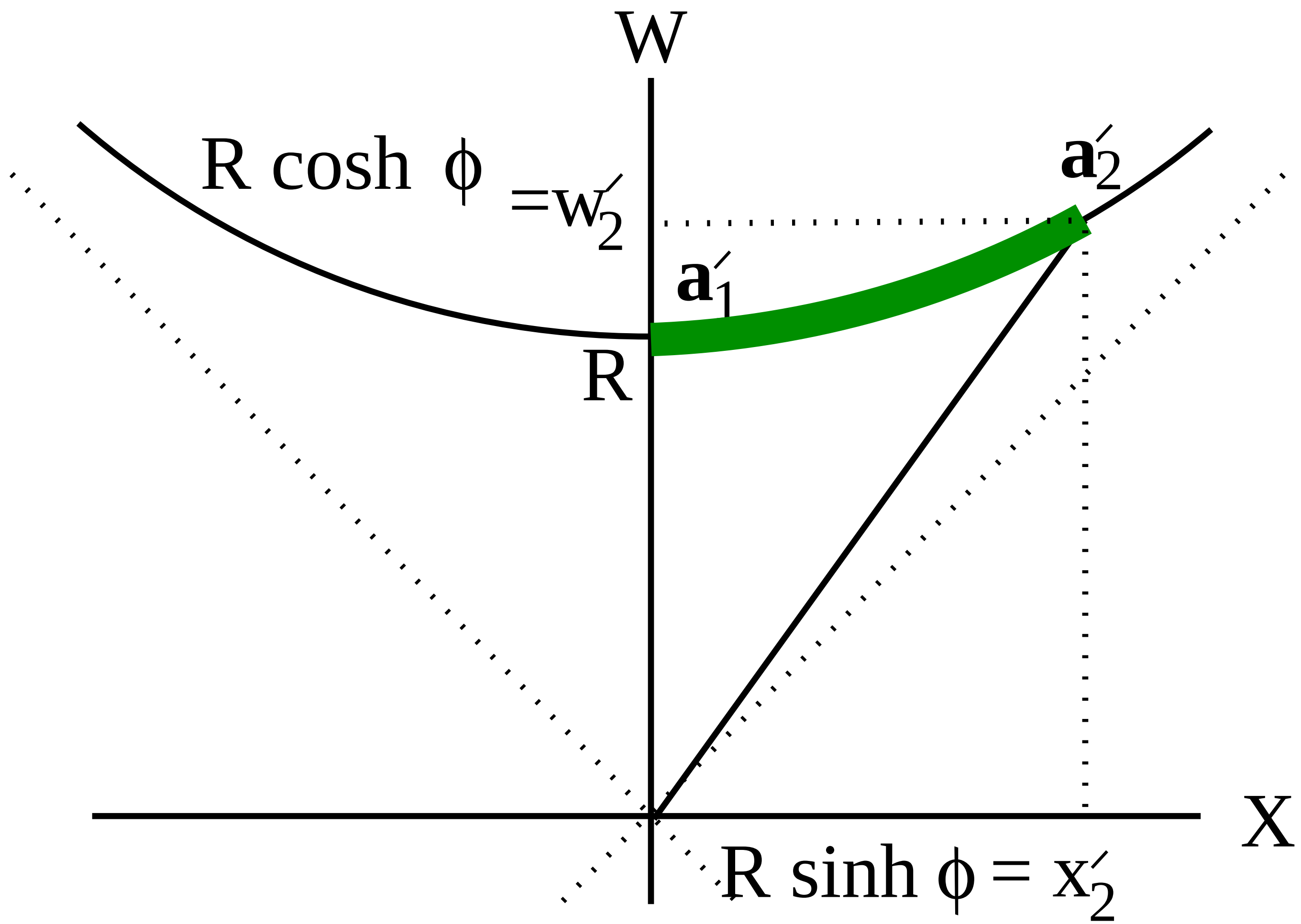

-

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$$\Sigma(\chi_i) := \begin{cases} R_{\mathrm{C}}\ \sinh(\chi_i/R_{\mathrm{C}}) & k < 0 \\ \chi_i & k = 0 \\ R_{\mathrm{C}}\ \sin(\chi_i/R_{\mathrm{C}}) & k > 0 \end{cases}$$

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- metric on S^3 (or $\mathbb{R}^3$ or H^3):

$$\mathrm{d}s^2 = \begin{cases} (k/|k|)\ (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \mathrm{d}w^2 & k \neq 0 \\ \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 & k = 0 \end{cases}$$

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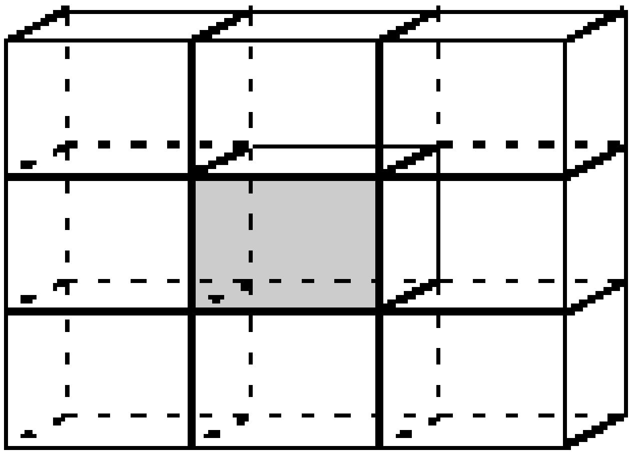
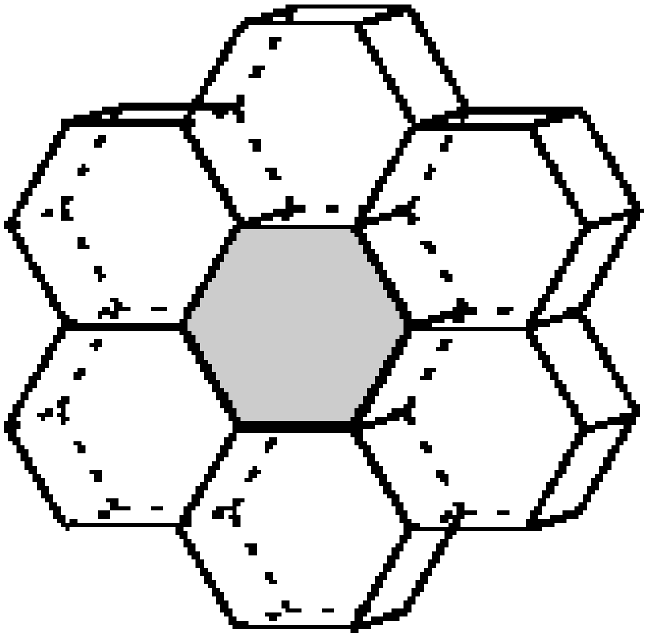
Cosmic topology: definitions





3D flat examples arXiv:astro-ph/9901364

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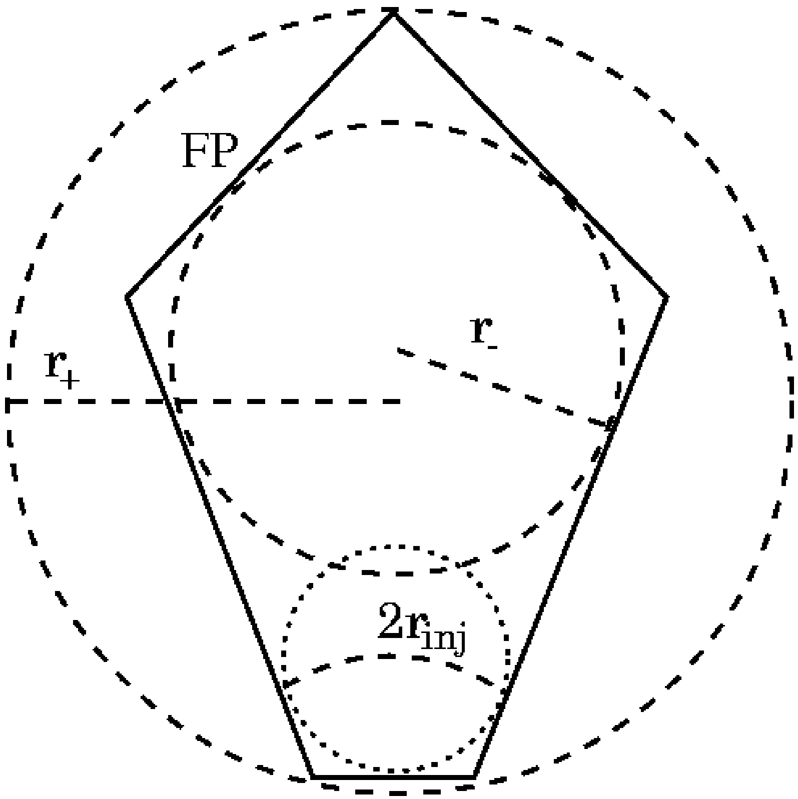

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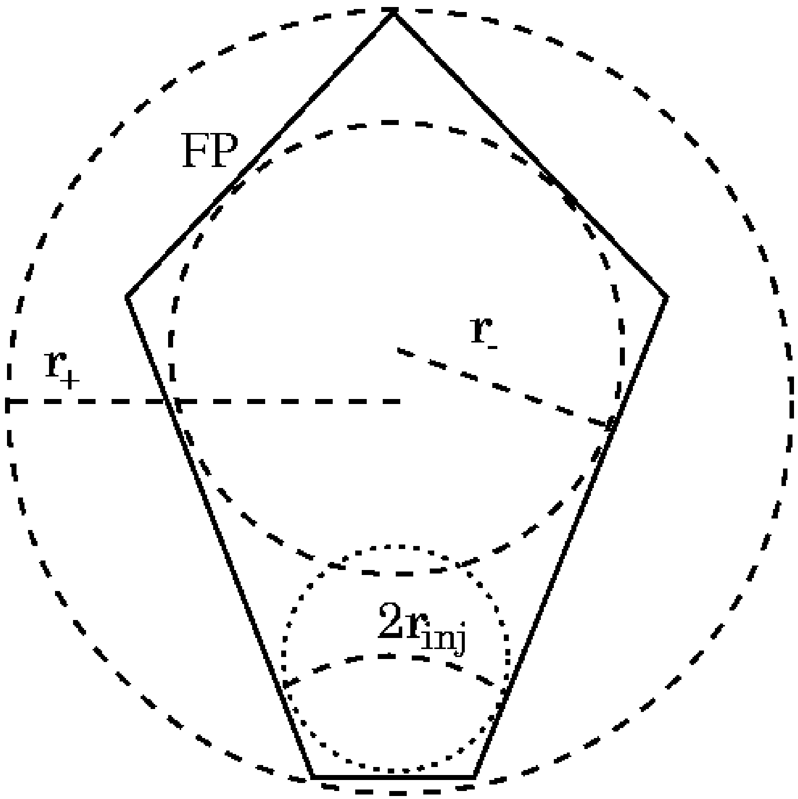

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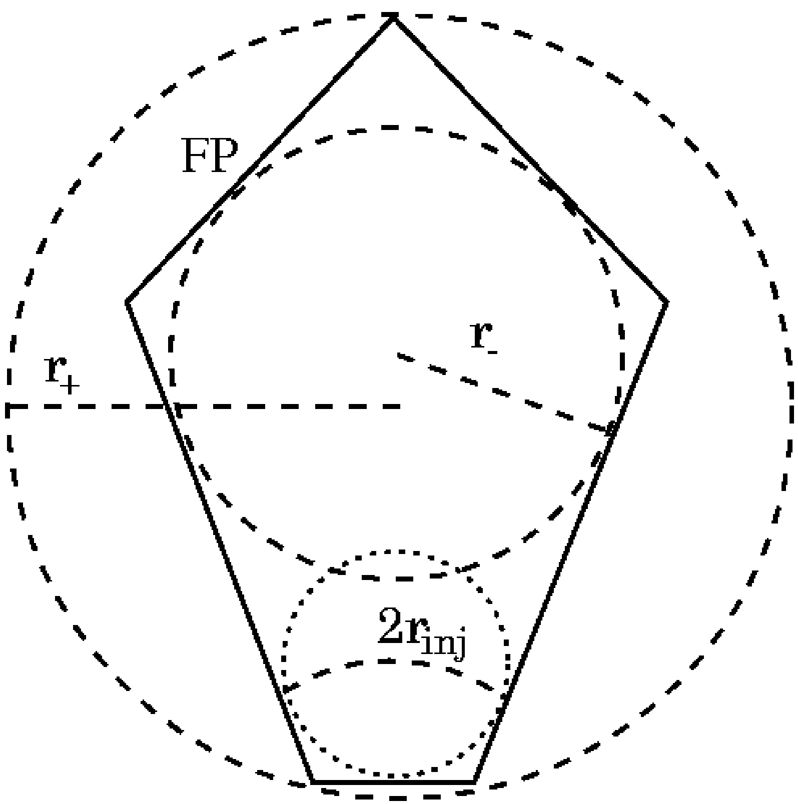

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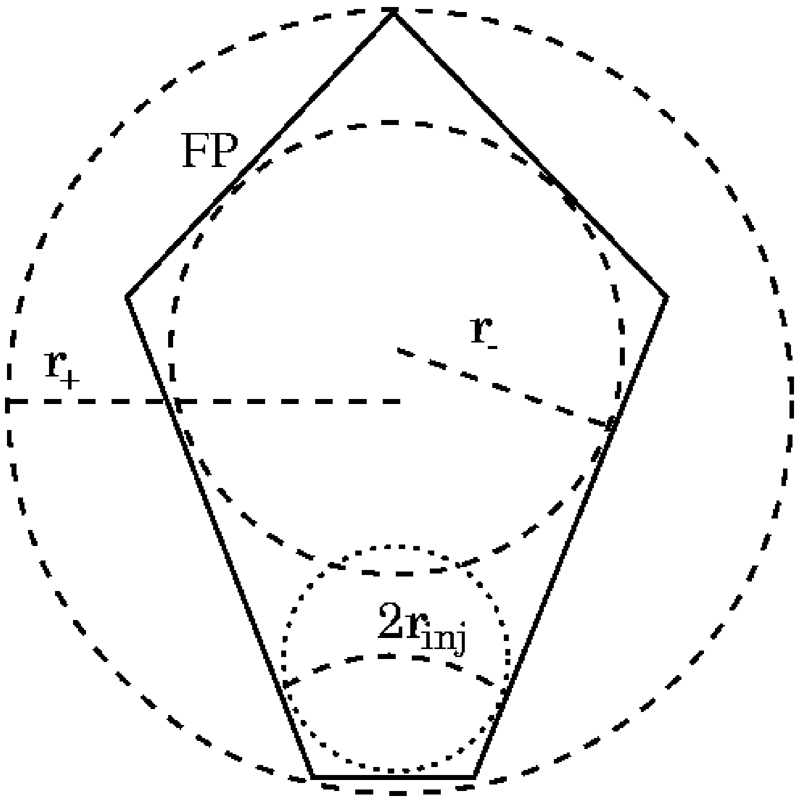

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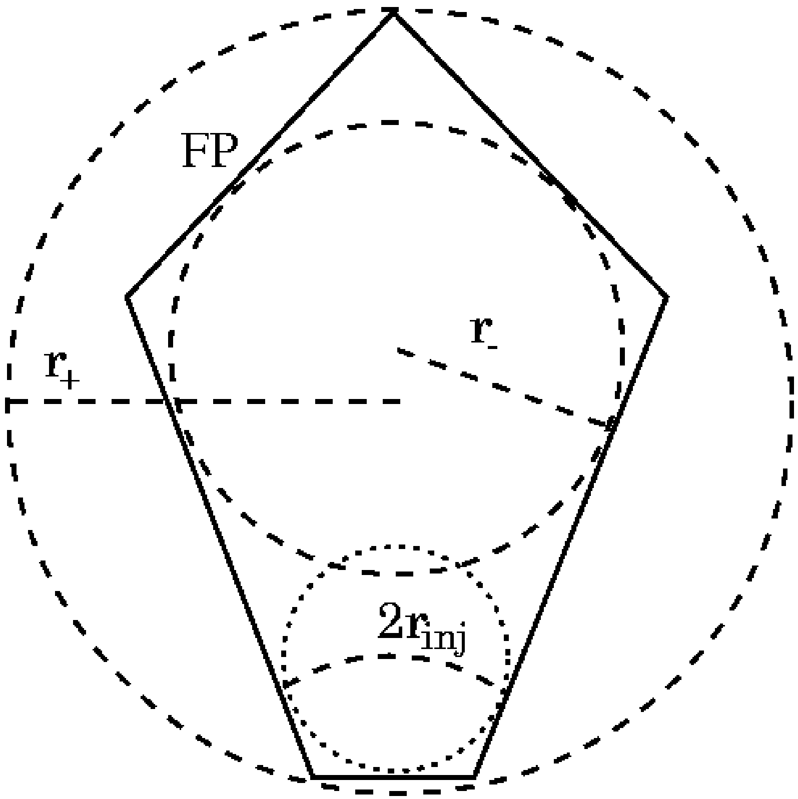

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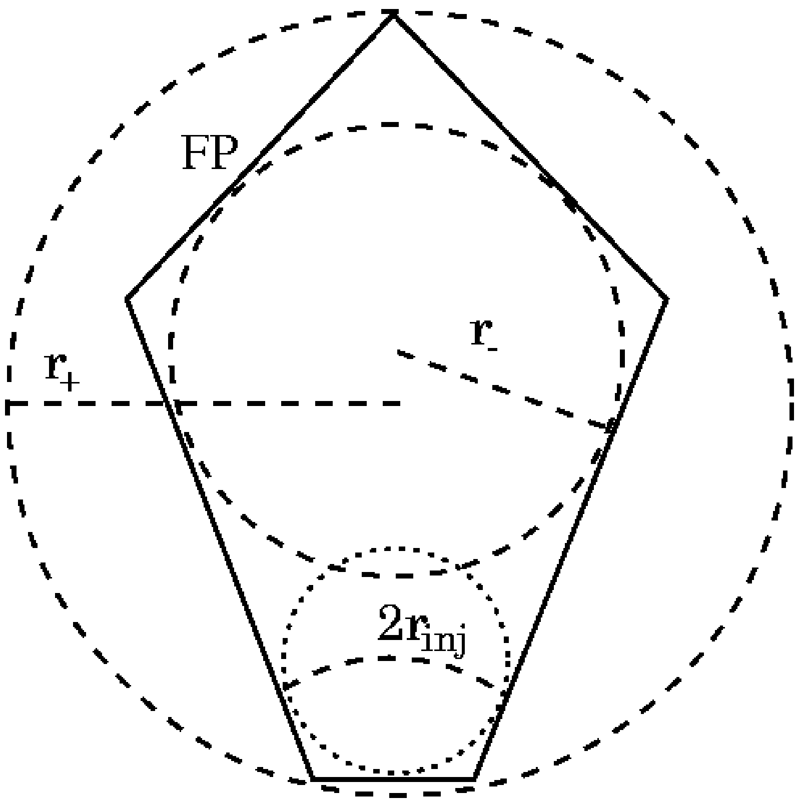

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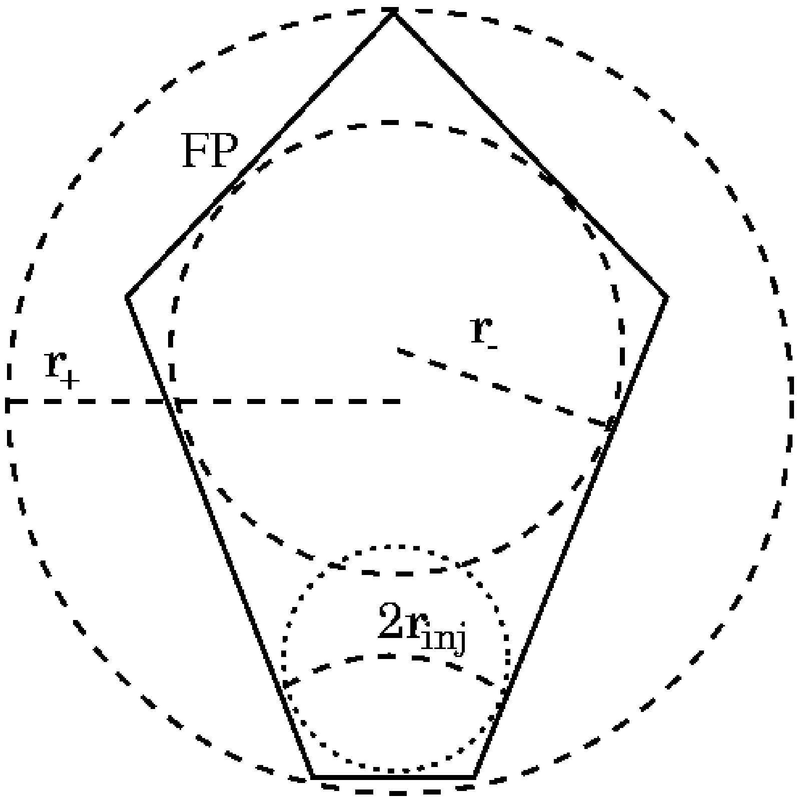

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  - w:Poincaré Conjecture *"Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere."* w:Grigori Perelman, arXiv:math/0211159 + arXiv:math/0303109 + arXiv:math/0307245

Families of const k 3-spaces

- flat 3-spaces: 18
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  - active research area, e.g. arXiv:0705.4325 min. vol.
- 5 other Thurston classes – w:Geometrization conjecture

Cosmic topol: top. accel.

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- scalar averaging and dynamical topology change (e.g. black holes): Brunswic & Buchert (CQG, 2020) arXiv:2002.08336

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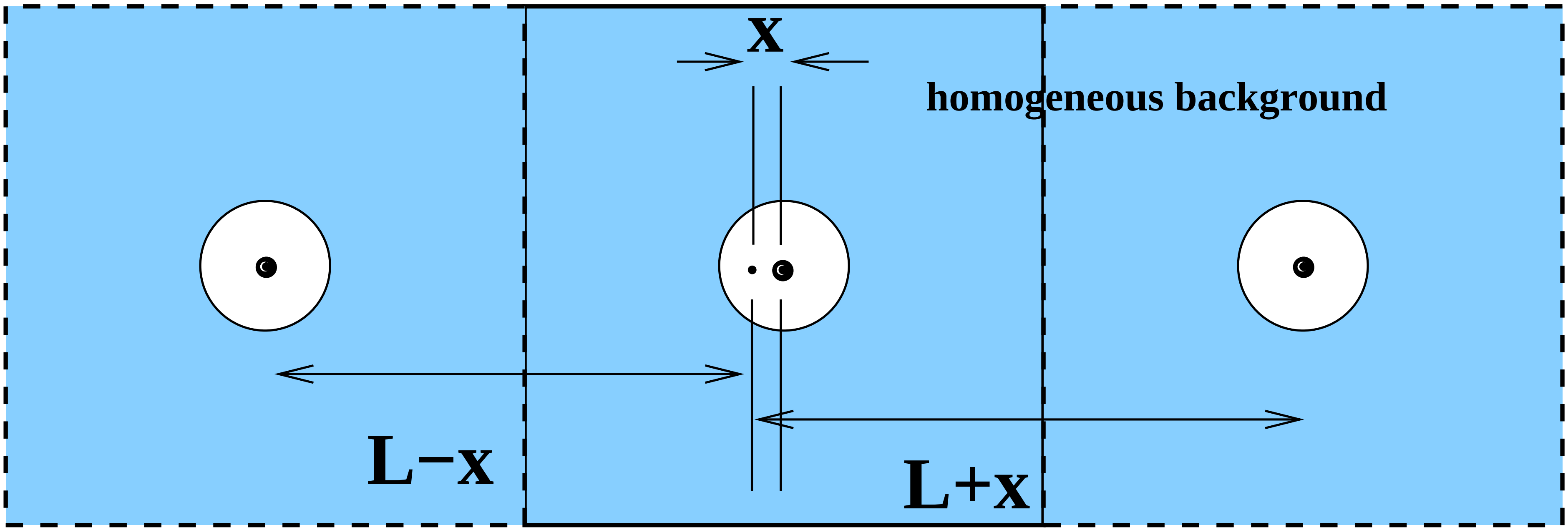

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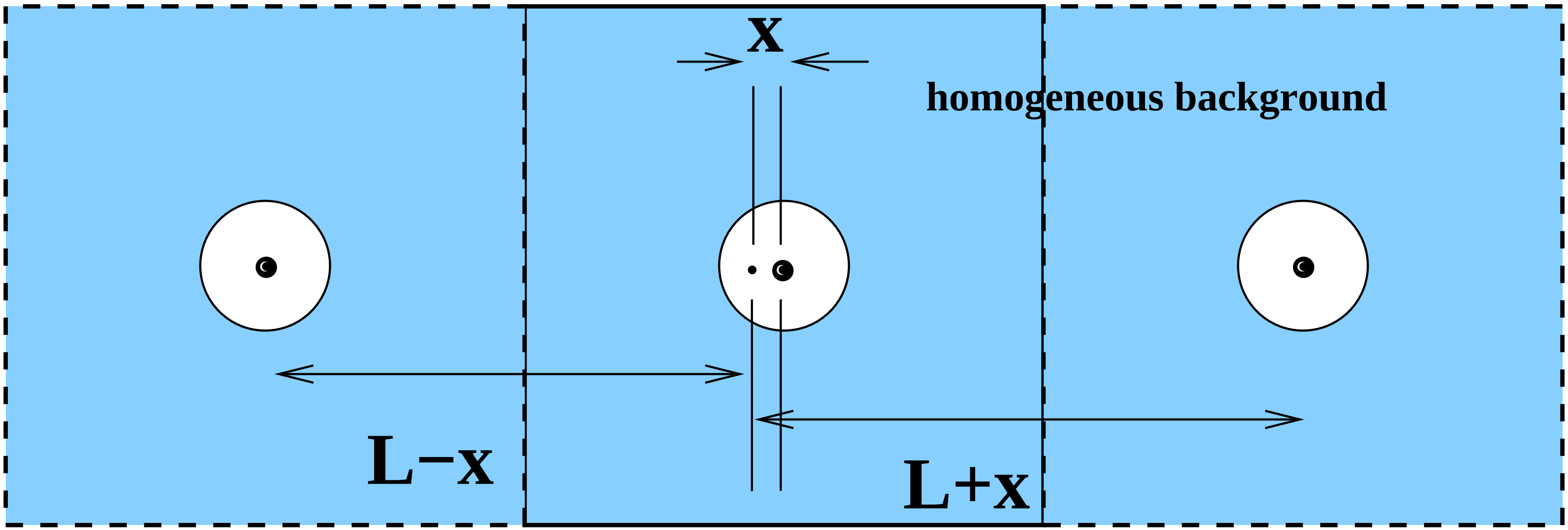

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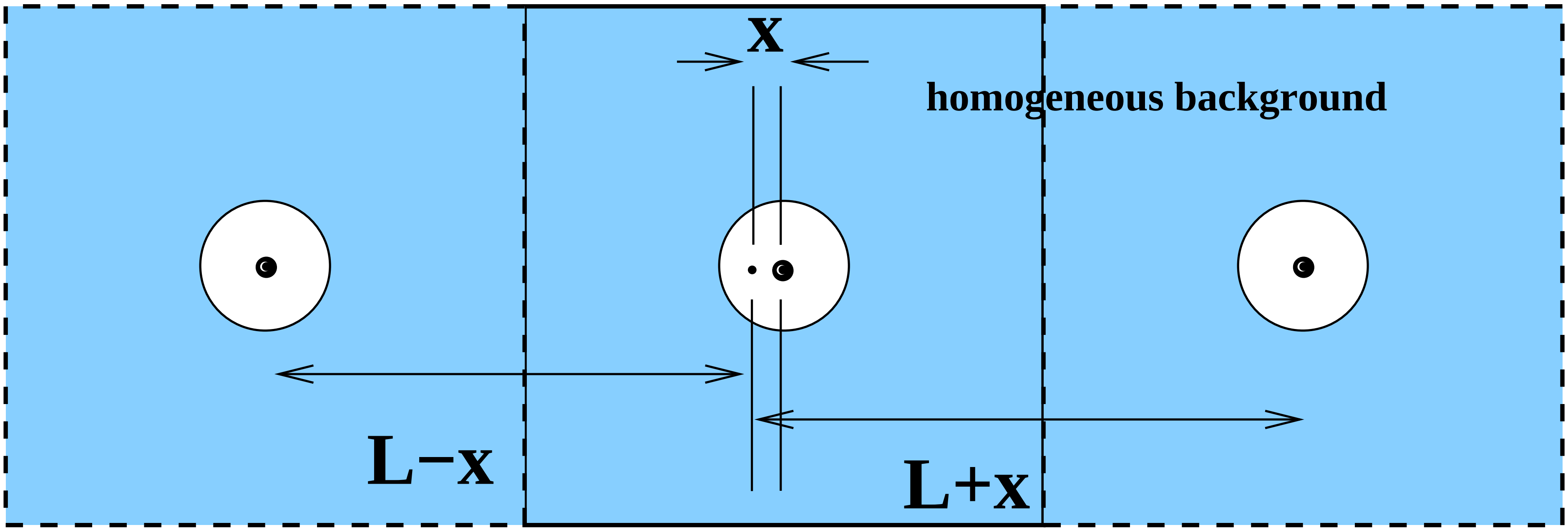

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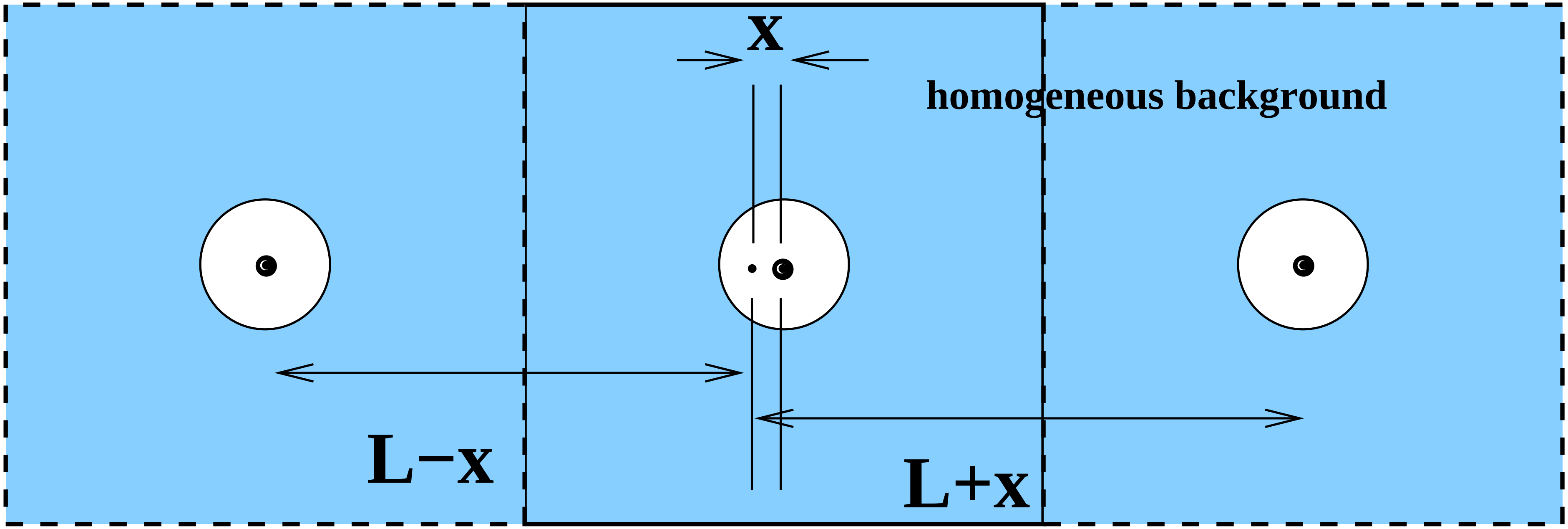

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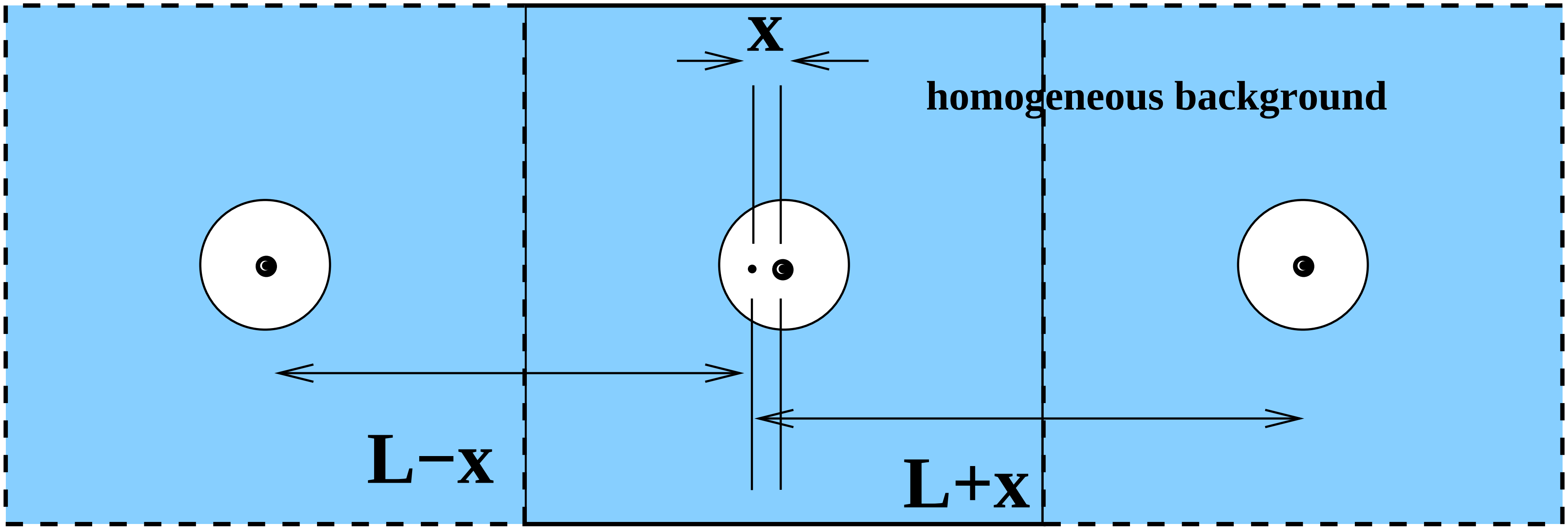

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original heuristic — Roukema+2007 A&A arXiv:astro-ph/0602159

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- covering space $\mathbb{E}^3$ or $\mathbb{S}^3$
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- $\mathbb{T}^3 = \mathbb{E}^3/\mathbb{Z}^3 \;\Rightarrow\; \ddot{x}_{\mathrm{resid}} \propto (x/L)^3 + \ldots$
- $\mathbb{S}^3/T^* \equiv M_6$ (octahedral space) $\;\Rightarrow\; \ddot{x}_{\mathrm{resid}} \propto (x/R_C)^3 + \ldots$
- $\mathbb{S}^3/O^*$ (truncated cube space) $\;\Rightarrow\; \ddot{x}_{\mathrm{resid}} \propto (x/R_C)^3 + \ldots$
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  $\Rightarrow\; \ddot{x}_{\mathrm{resid}} \propto (x/R_C)^5 + \ldots$
- *topological acceleration is manifold-dependent*
  Roukema & Różański arXiv:0902.3402, A&A, 502, 27

Newt. non-Euclid. top.accel.

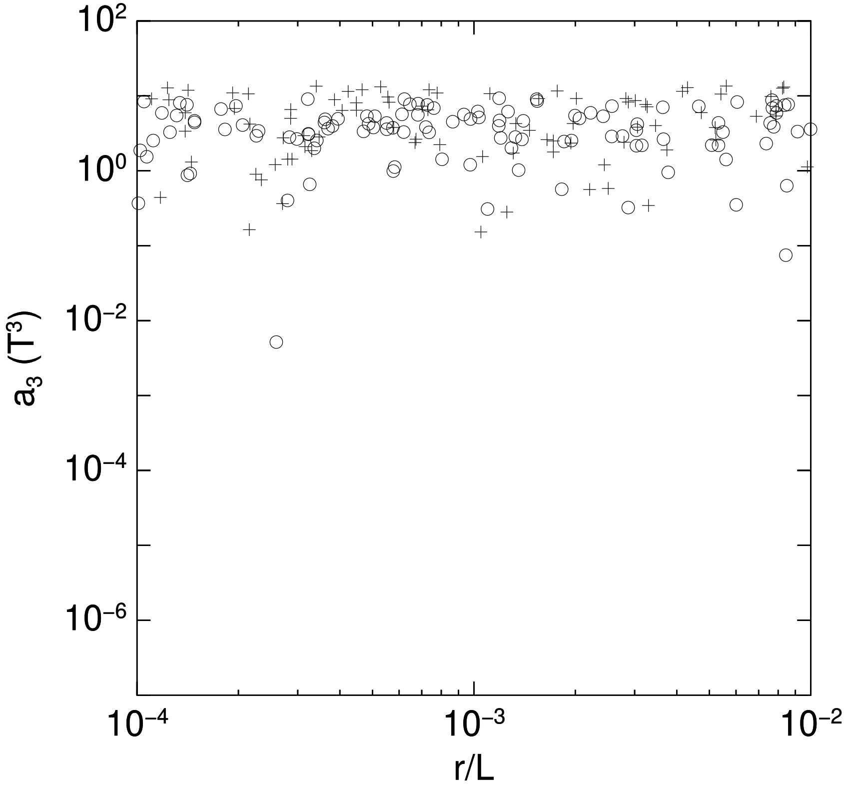
NEN: $\Phi_{\mathbb{S}}(\xi) \propto -\cot\xi\,(1-\xi/\pi) + A$

Topology	N_Σ	Φ_{-1}	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5
Euclidean (infinite or Thurston-type)							
$\mathbb{E}^3$		-1	0	0	0	0	0
$\mathbb{T}^3$		-1	0	$-\frac{2\pi}{3}\frac{1}{V_\Sigma}$	0	–	0
Spherical							
$\mathbb{S}^3$	1	-1	$\frac{1}{3}\frac{\mathcal{R}}{6}$	$-\frac{2\pi}{3}\frac{1}{V_\Sigma}$	$\frac{1}{45}\left(\frac{\mathcal{R}}{6}\right)^2$	$-\frac{2\pi}{45}\frac{\mathcal{R}/6}{V_\Sigma}$	$\frac{2}{945}\left(\frac{\mathcal{R}}{6}\right)^3$
M_3	8	-1	$\frac{1}{3}\frac{\mathcal{R}}{6}$	$-\frac{2\pi}{3}\frac{1}{V_\Sigma}$	$\frac{1}{45}\left(\frac{\mathcal{R}}{6}\right)^2$	–	$\frac{2}{945}\left(\frac{\mathcal{R}}{6}\right)^3$
M_6	24	-1	$\frac{1}{3}\frac{\mathcal{R}}{6}$	$-\frac{2\pi}{3}\frac{1}{V_\Sigma}$	$\frac{1}{45}\left(\frac{\mathcal{R}}{6}\right)^2$	–	$\frac{2}{945}\left(\frac{\mathcal{R}}{6}\right)^3$
M_7	120	-1	$\frac{1}{3}\frac{\mathcal{R}}{6}$	$-\frac{2\pi}{3}\frac{1}{V_\Sigma}$	$\frac{1}{45}\left(\frac{\mathcal{R}}{6}\right)^2$	$-\frac{2\pi}{45}\frac{\mathcal{R}/6}{V_\Sigma}$	$\frac{2}{945}\left(\frac{\mathcal{R}}{6}\right)^3$
Hyperbolic (infinite)							
$\mathbb{H}^3$		-1	$\frac{1}{3}\frac{\mathcal{R}}{6}$	0	$\frac{1}{45}\left(\frac{\mathcal{R}}{6}\right)^2$	0	$\frac{2}{945}\left(\frac{\mathcal{R}}{6}\right)^3$

even terms $\Rightarrow$ closed; odd terms $\Rightarrow$ curved

Vigneron & Roukema (2022) arXiv:2201.09102

$T^3, S^3/\Gamma$



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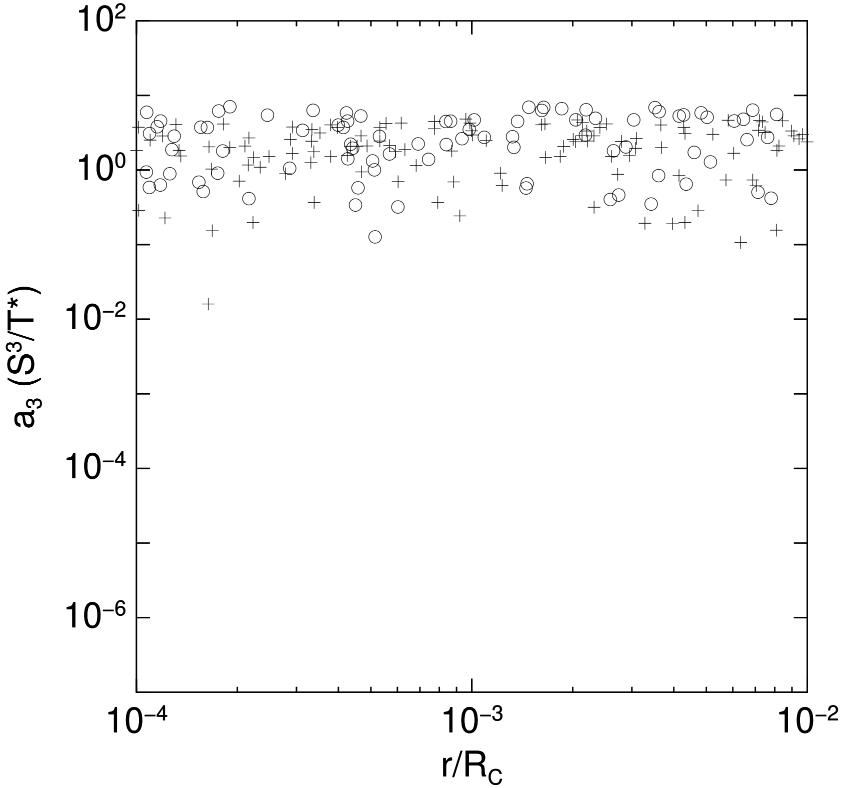

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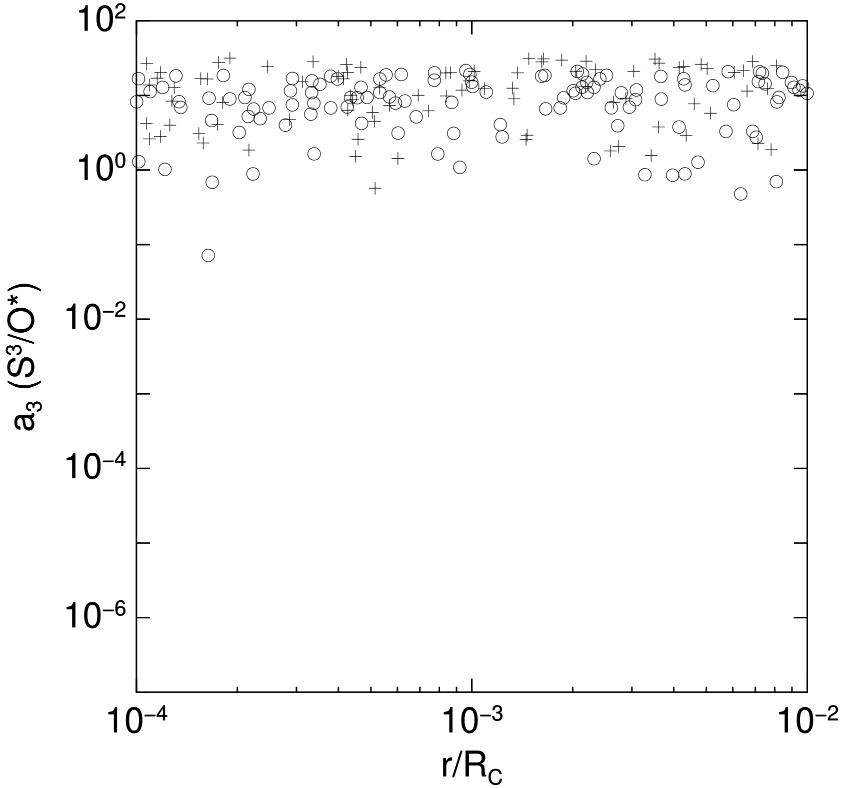

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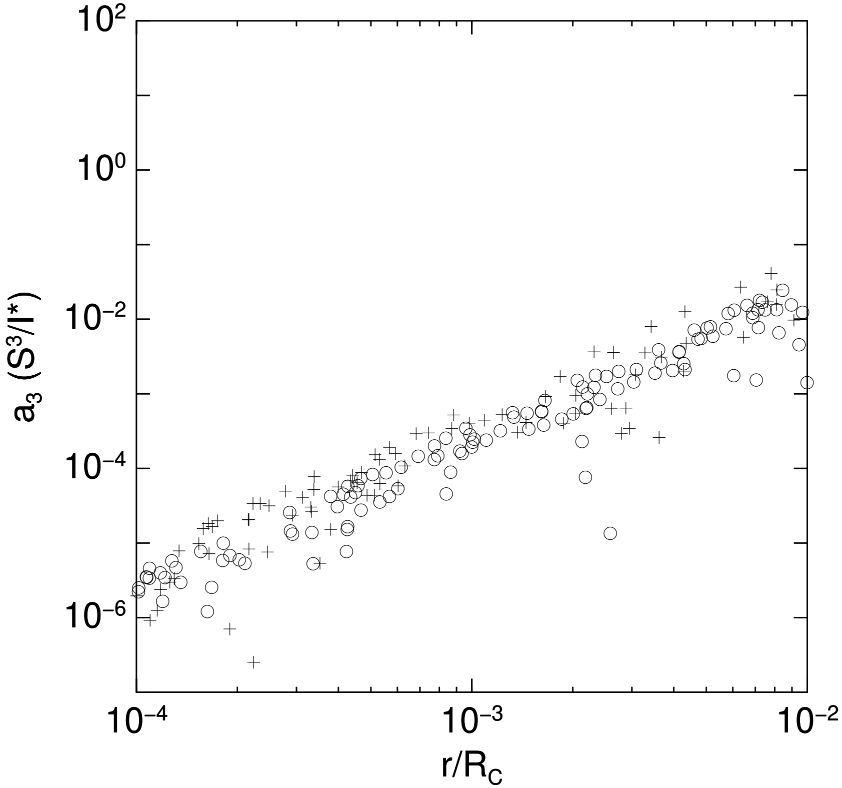

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$T^3, S^3/\Gamma$

- Some spaces are more equal than others.
- Roukema & Różański arXiv:0902.3402, A&A, 502, 27
- Newton–Cartan approach for preparing for full GR approach: Vigneron (2020, PRD) arXiv:2010.10247; Vigneron (2021, PRD) arXiv:2012.10213; Vigneron (2022a, PRD) arXiv:2109.10336; Vigneron (2022b, CQG) arXiv:2201.02112; Vigneron & Roukema (2022) arXiv:2201.09102

Is topolog. acceleration relativistic?

- Korotkin & Nikolai (1994) arXiv:gr-qc/9403029 solution: Schwarzschild-like BH in $S^1 \times E^2$ (slab space $= T^1$)
- outside event horizon, inside topology scale:

$$\ddot{x} = 4\zeta(3)G\,\frac{M}{L^3}x \propto x$$

Ostrowski, Roukema & Buliński (2012) arXiv:1109.1596

$\Rightarrow$ Yes.

Cosmic topol: obs. strategies

empirical strategies: arXiv:astro-ph/0010189

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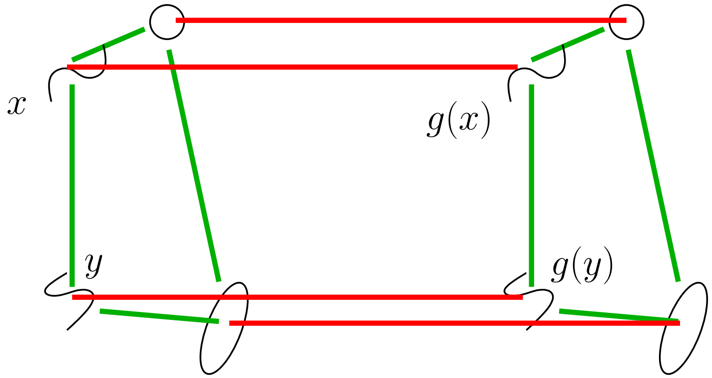
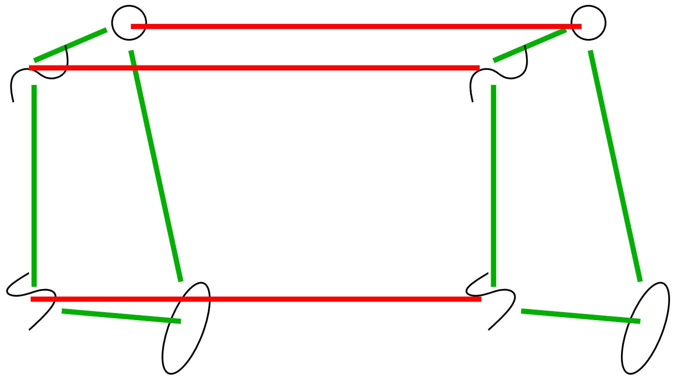
3D strategies—pair types





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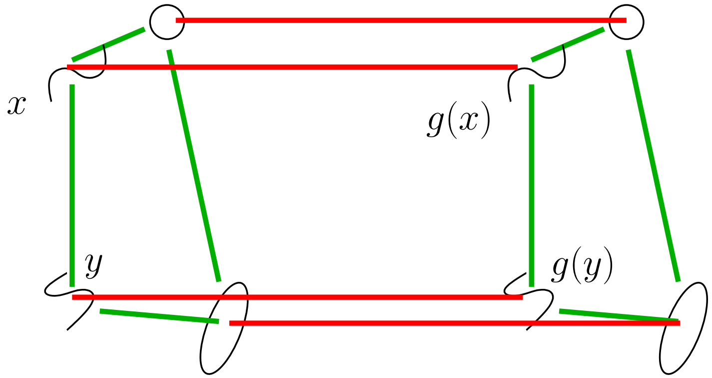

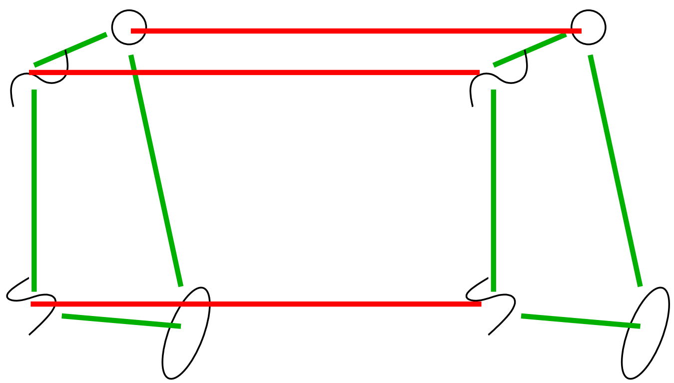

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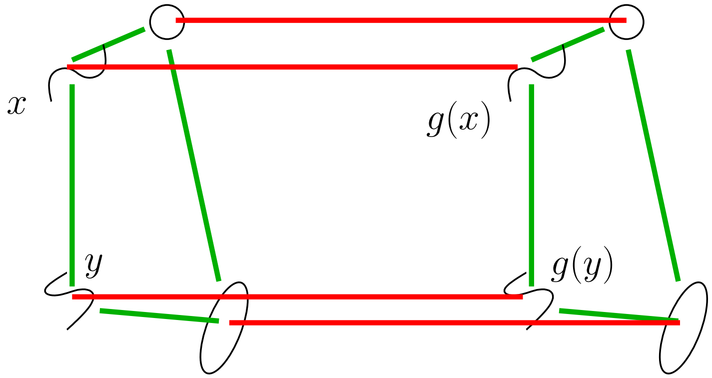

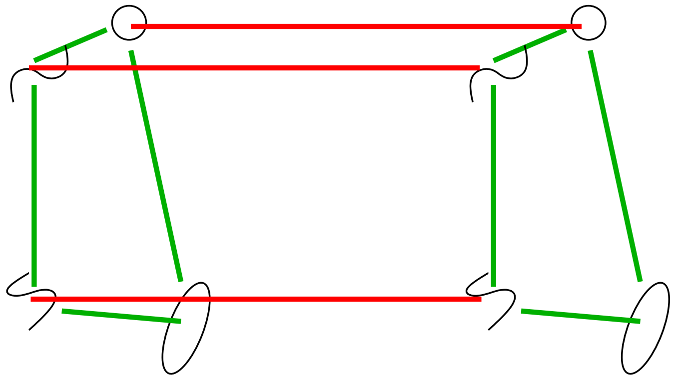

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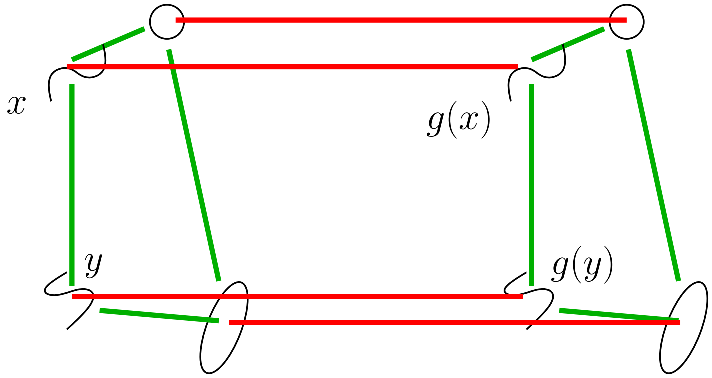

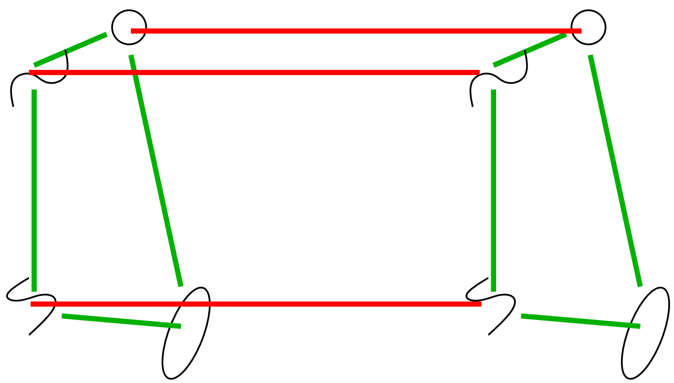

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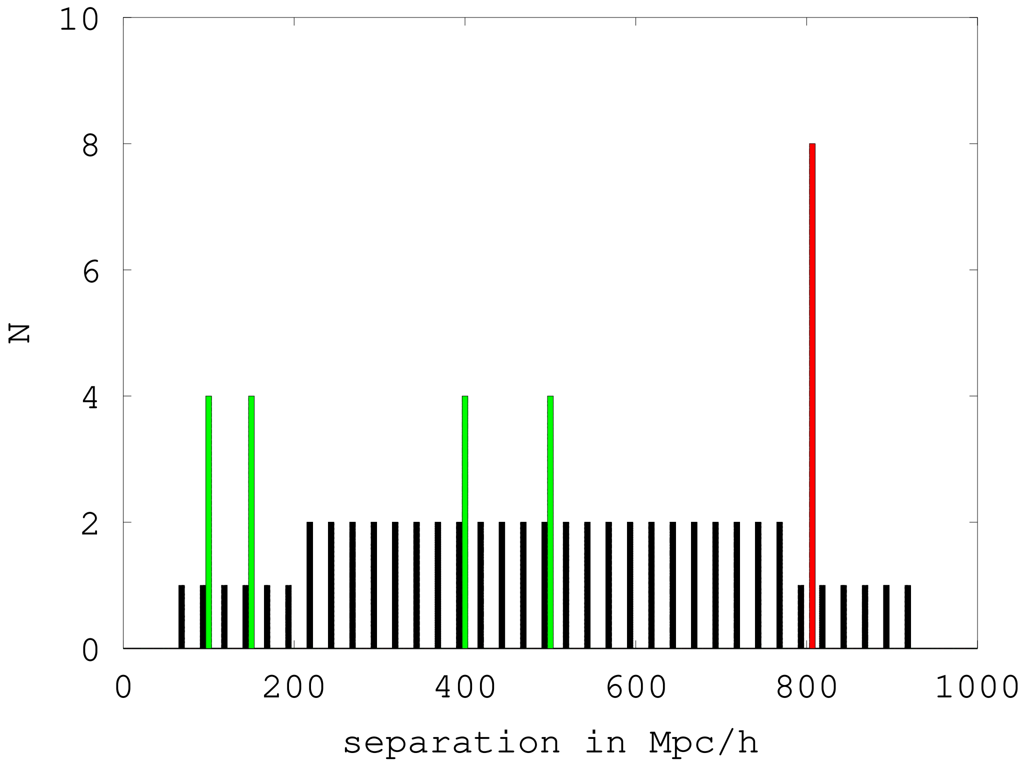

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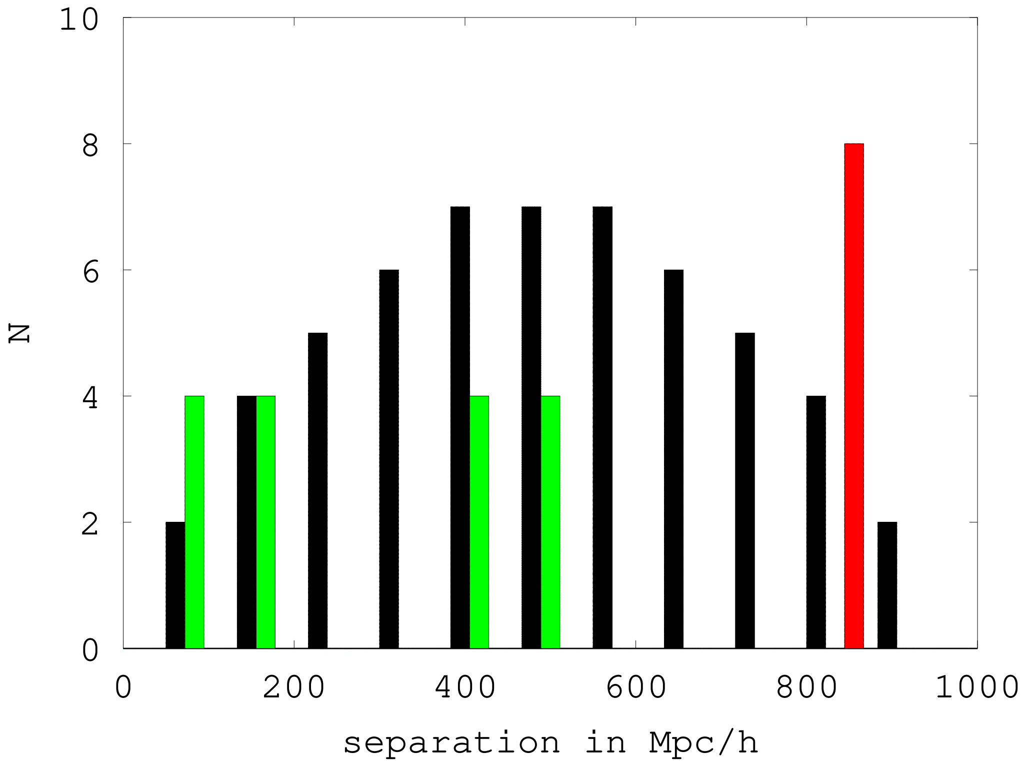

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- quadruples + successive filters + collect membership s of quadruples Fujii & Yoshii (2013)

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Marecki, Roukema, Bajtlik (2005) arXiv:astro-ph/0412181

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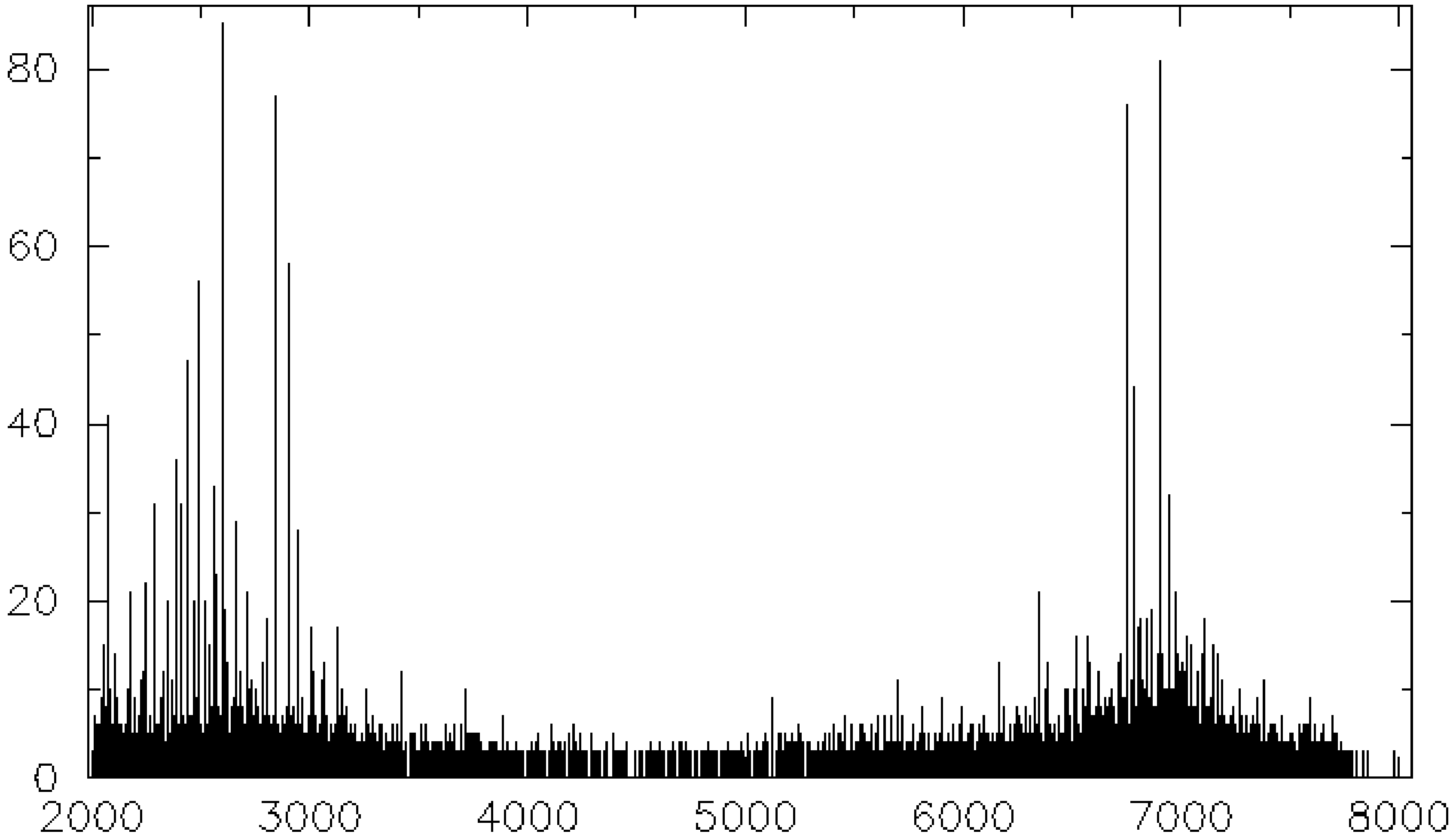
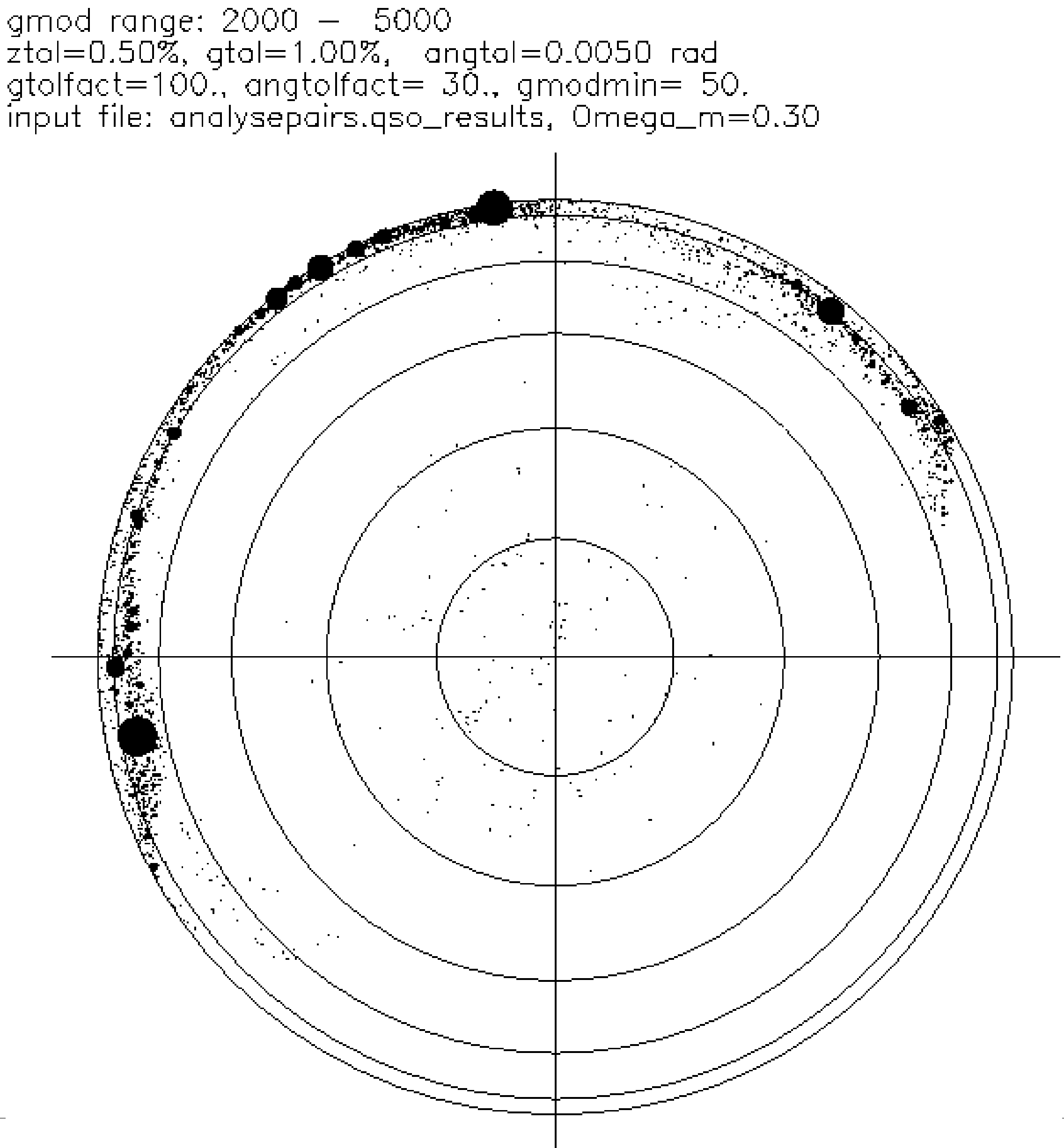
AGNs—successive filters

Marecki, Roukema, Bajtlik (2005) arXiv:astro-ph/0412181



AGN Catalogues



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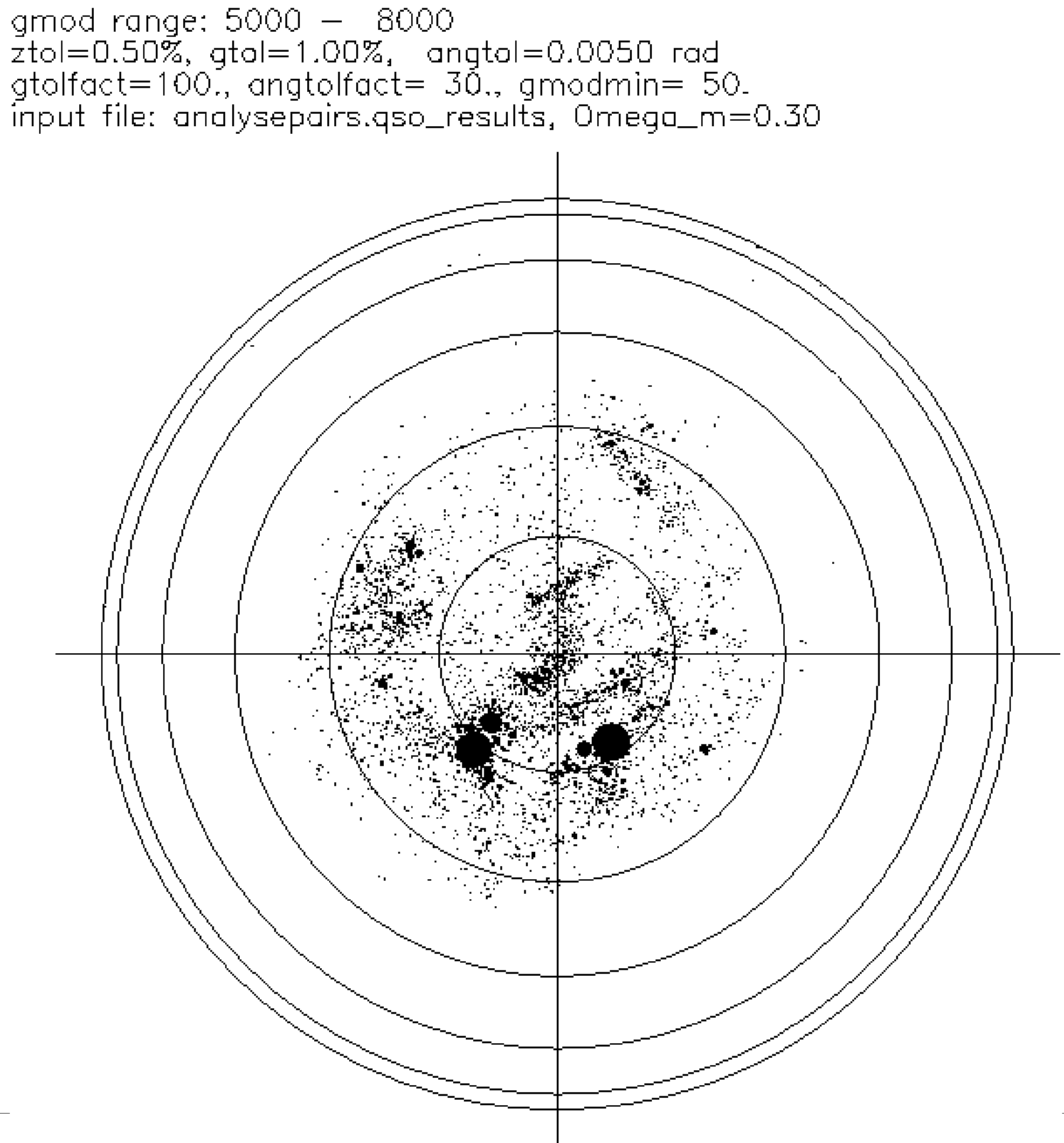

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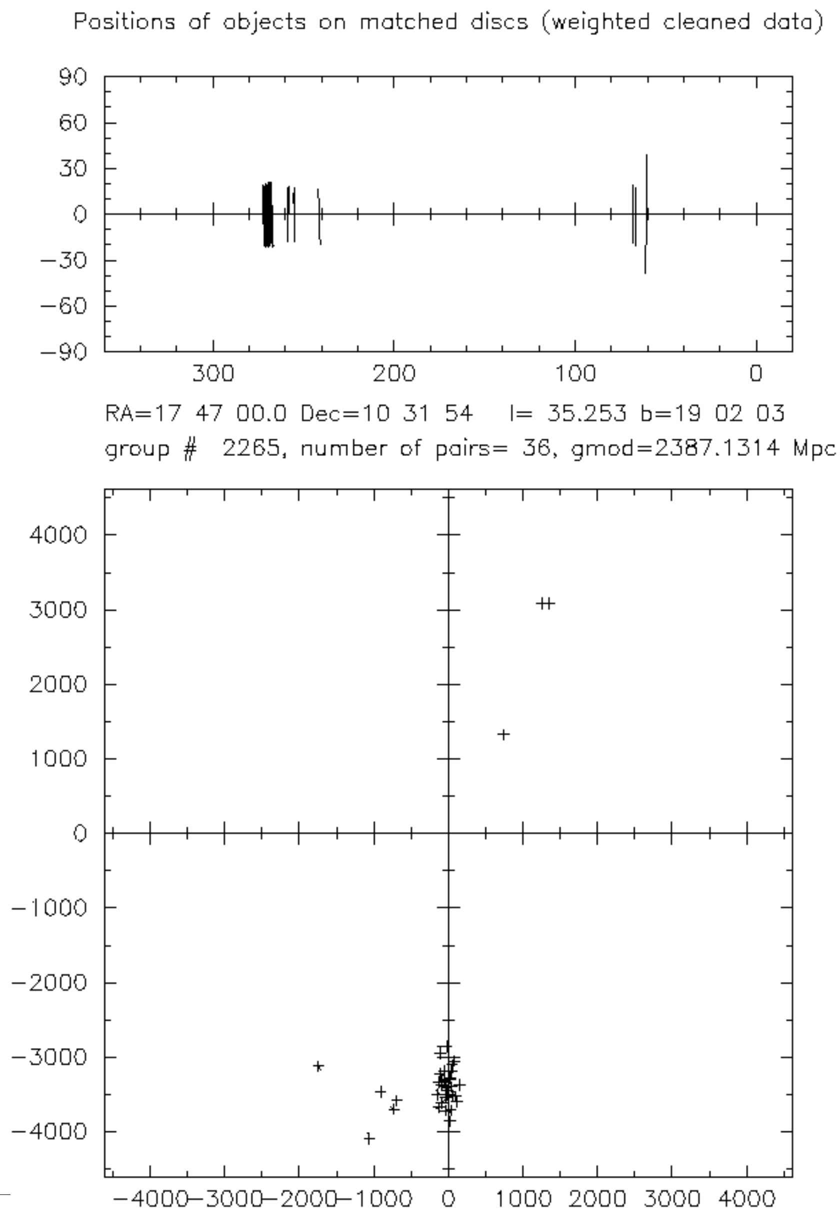

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AGNs—successive filters

Fujii & Yoshii (2013) arXiv:1103.1466

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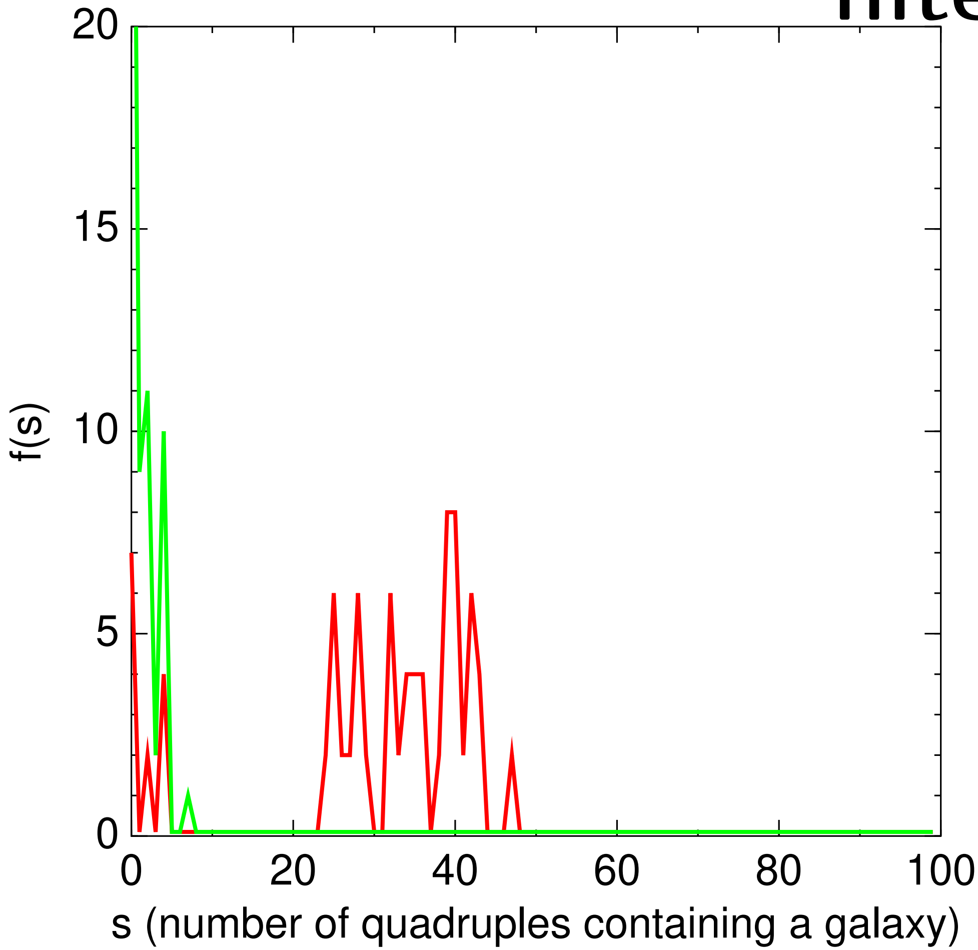
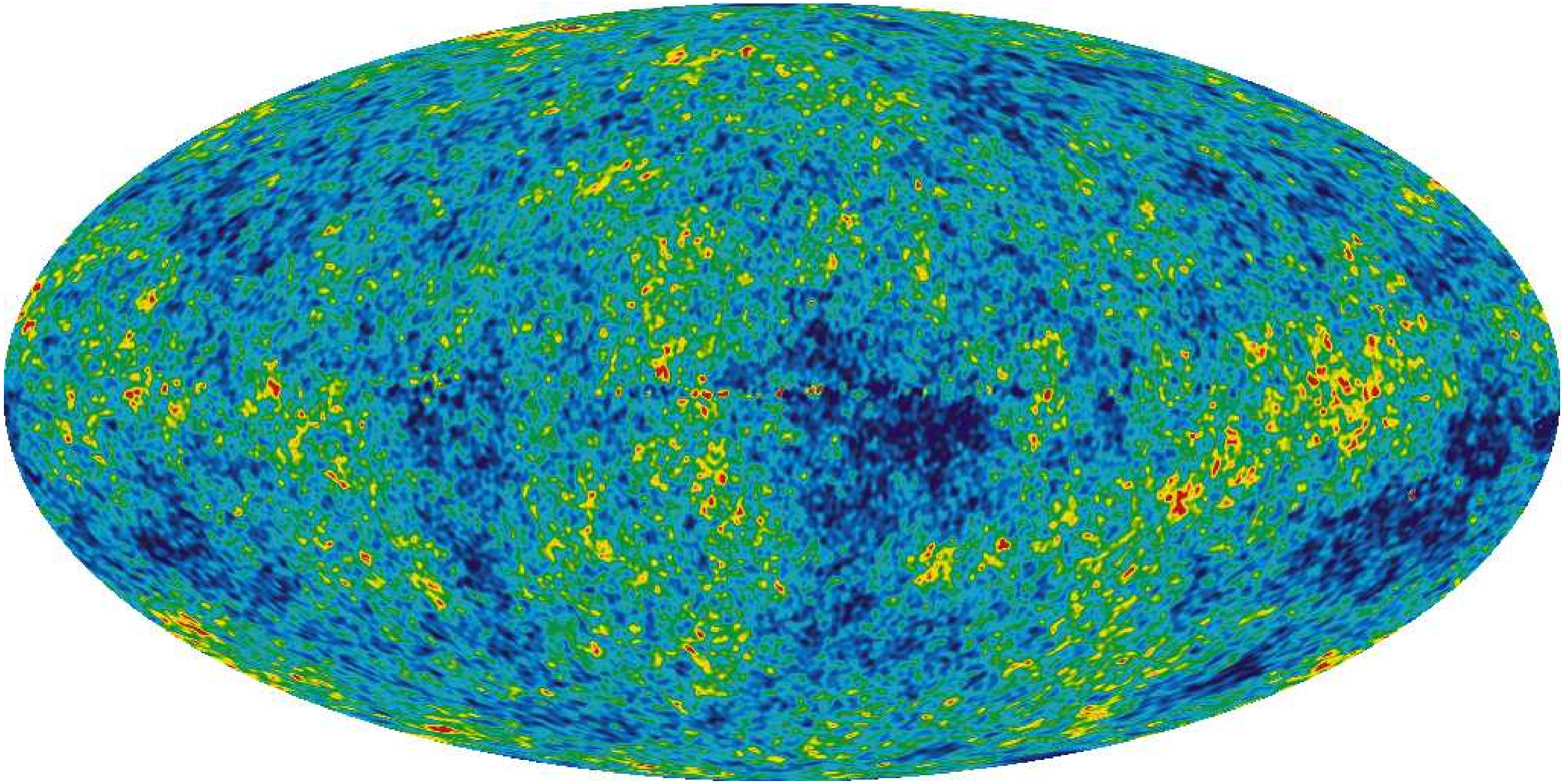


simulation of s histogram for Lyman break galaxies (LBGs) at $z \approx 6$

green: simply connected; red: T^3

ADS:2014MNRAS.437.1096R

2D methods: structure cutoff



WMAP 5yr ILC (internal linear combination)

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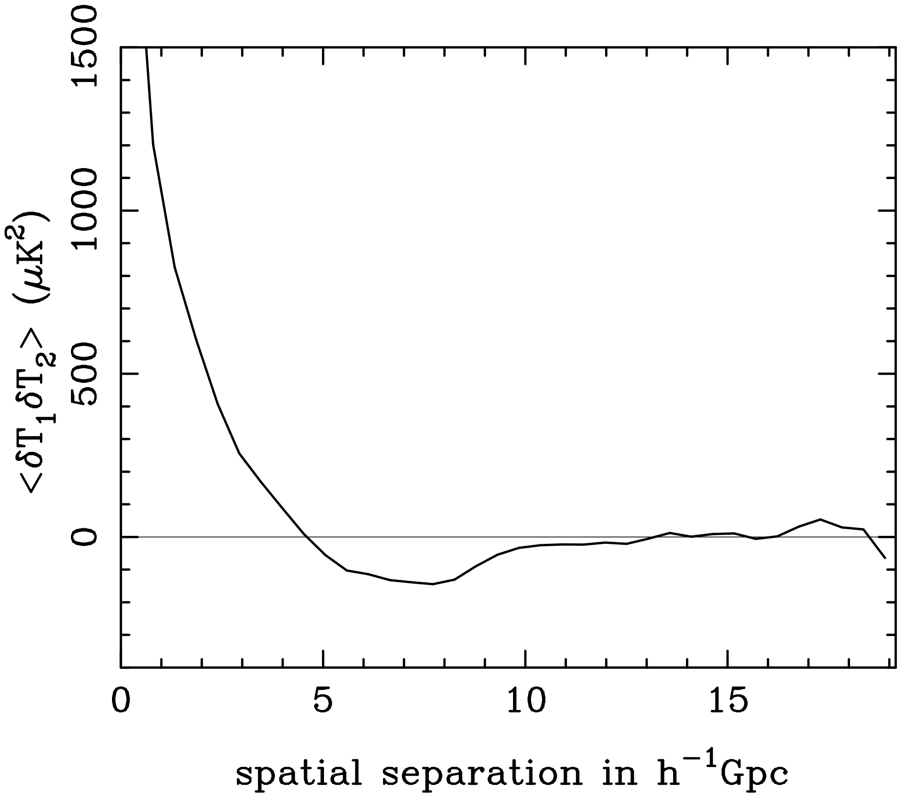

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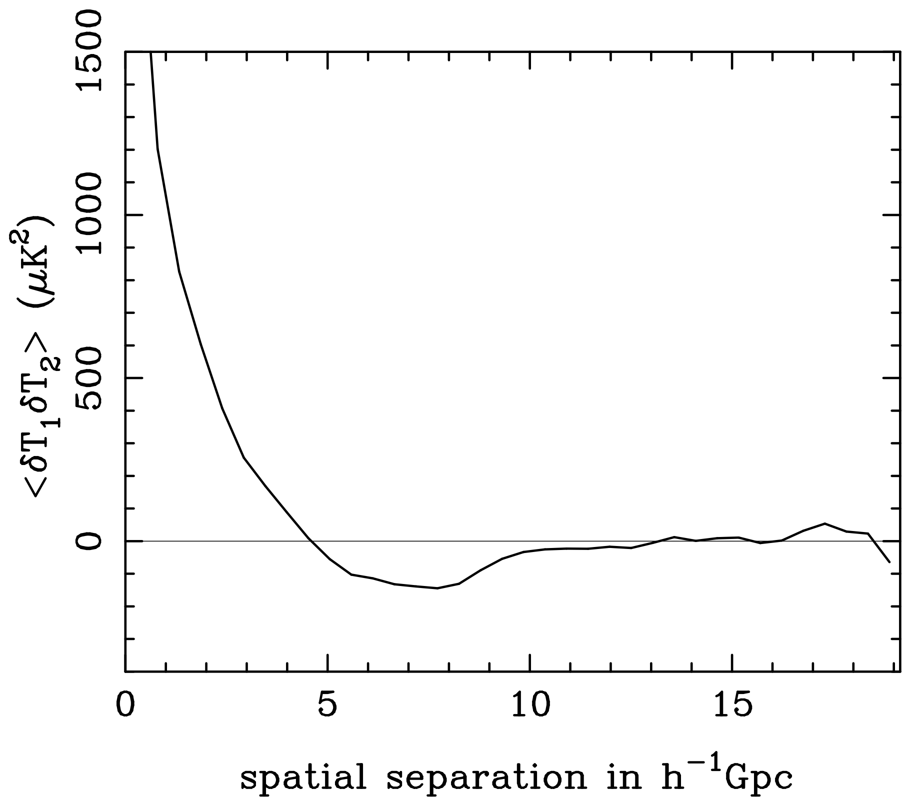

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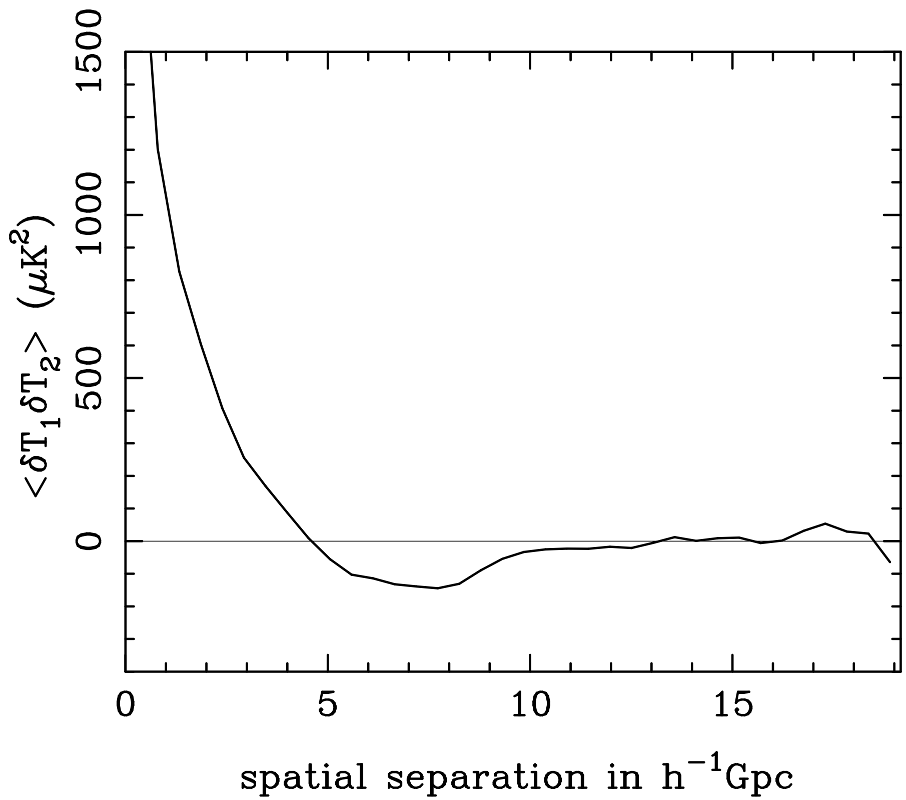

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Starobinsky (1993); Stevens et al. (1993)

The Identified Circles Principle

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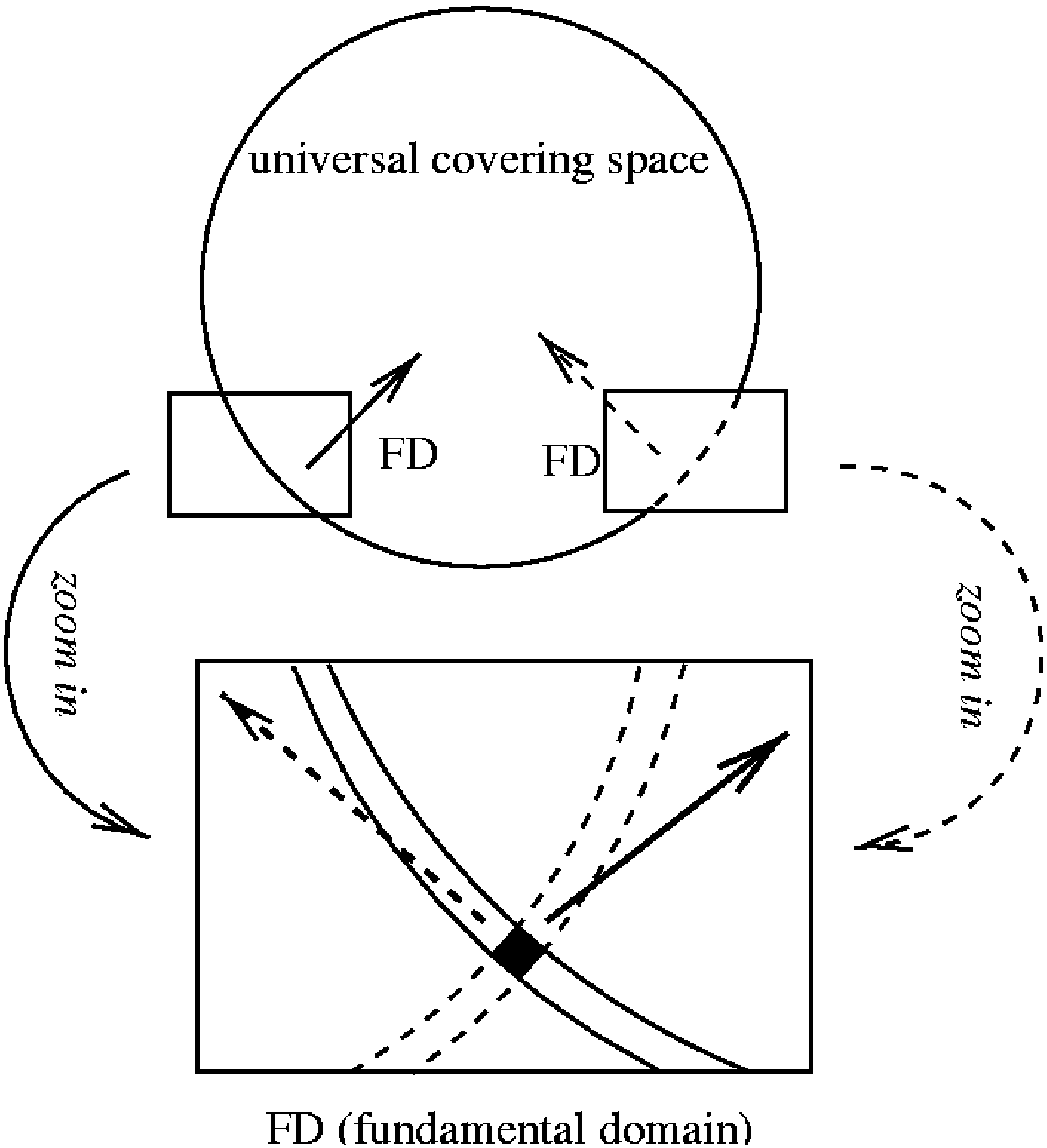

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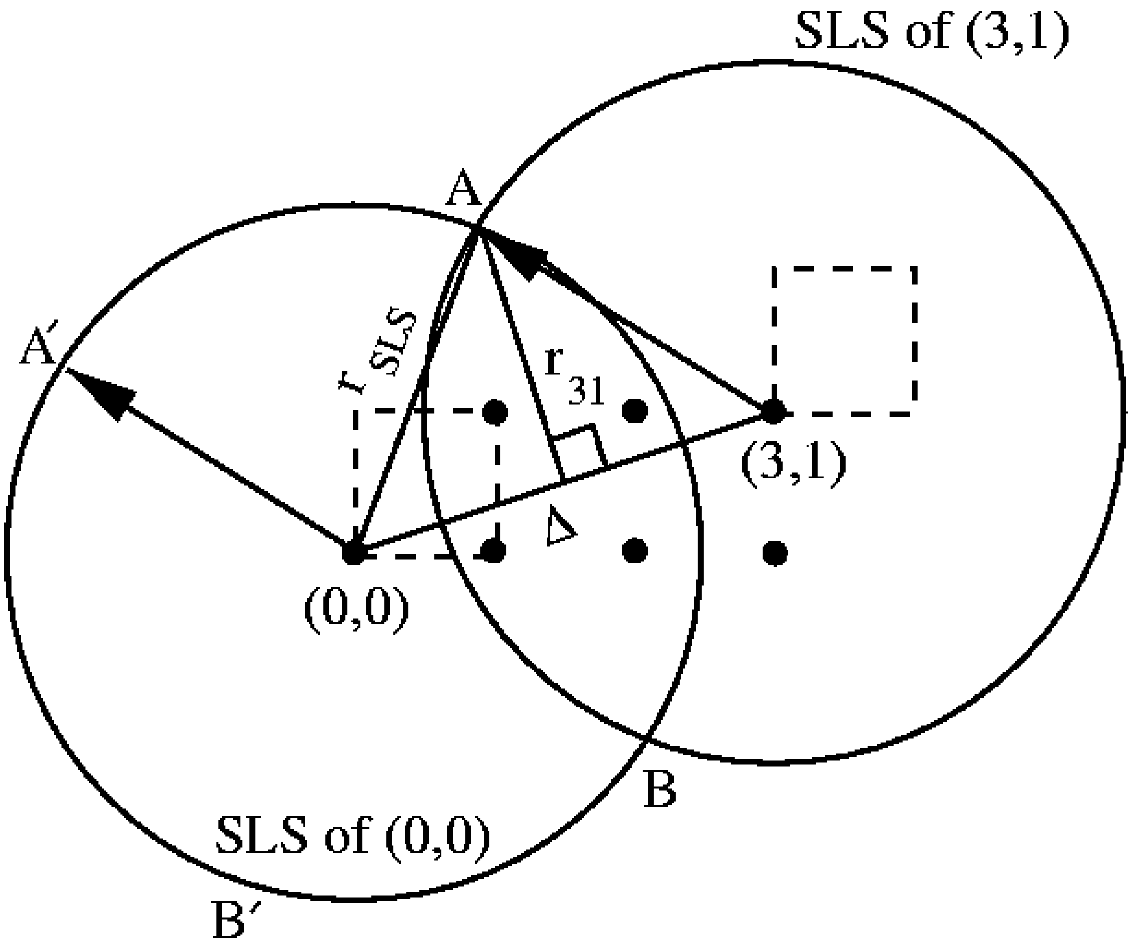
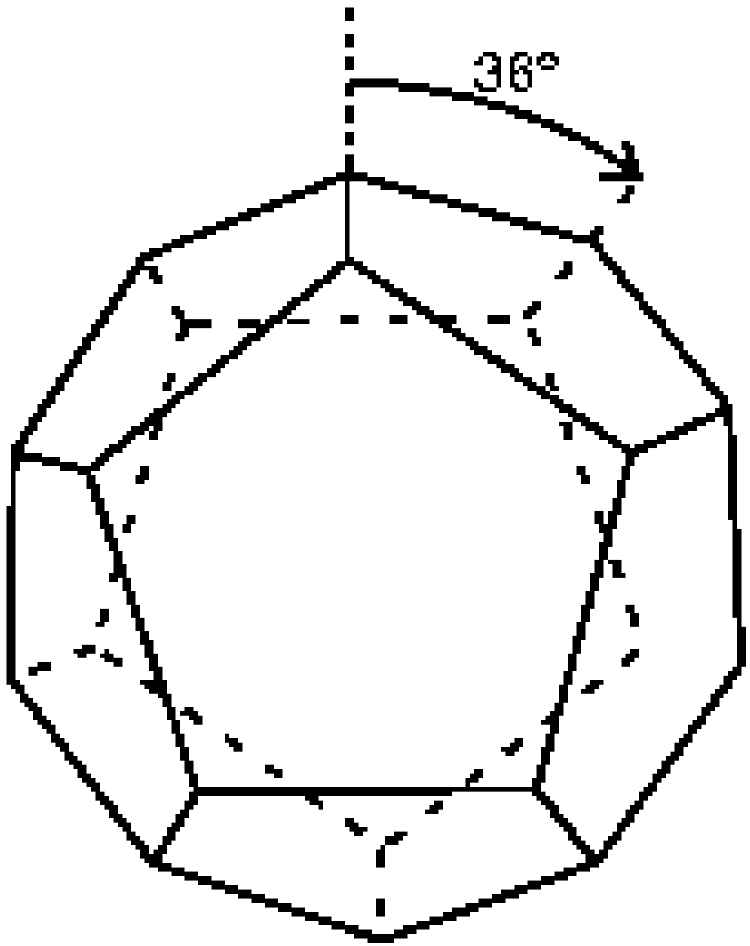


The Poincaré Dodecahedral 3-Manifold



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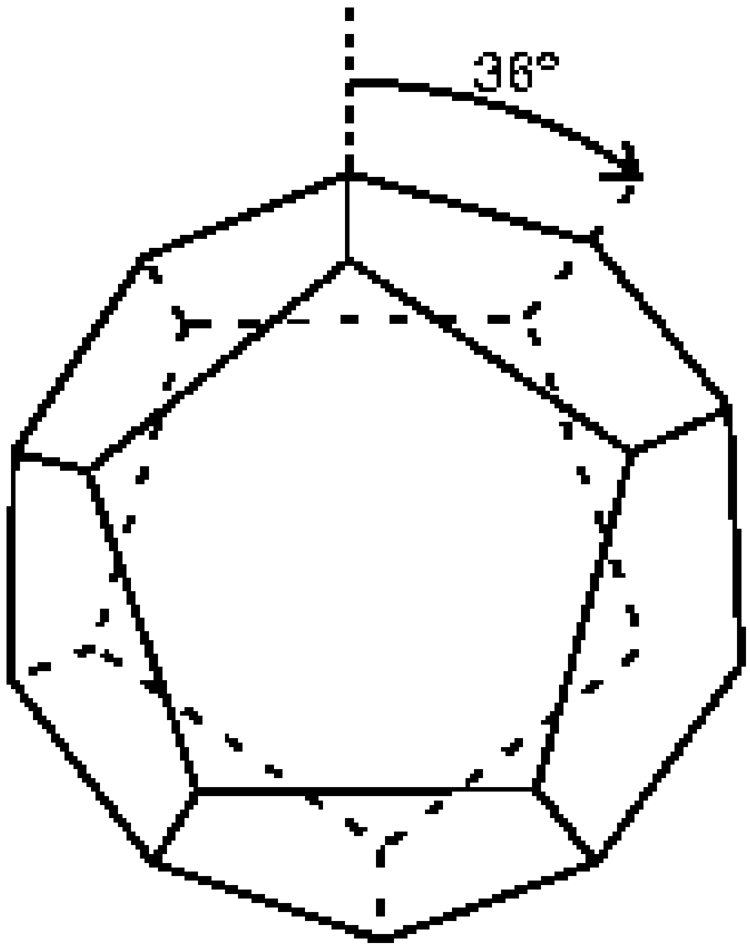

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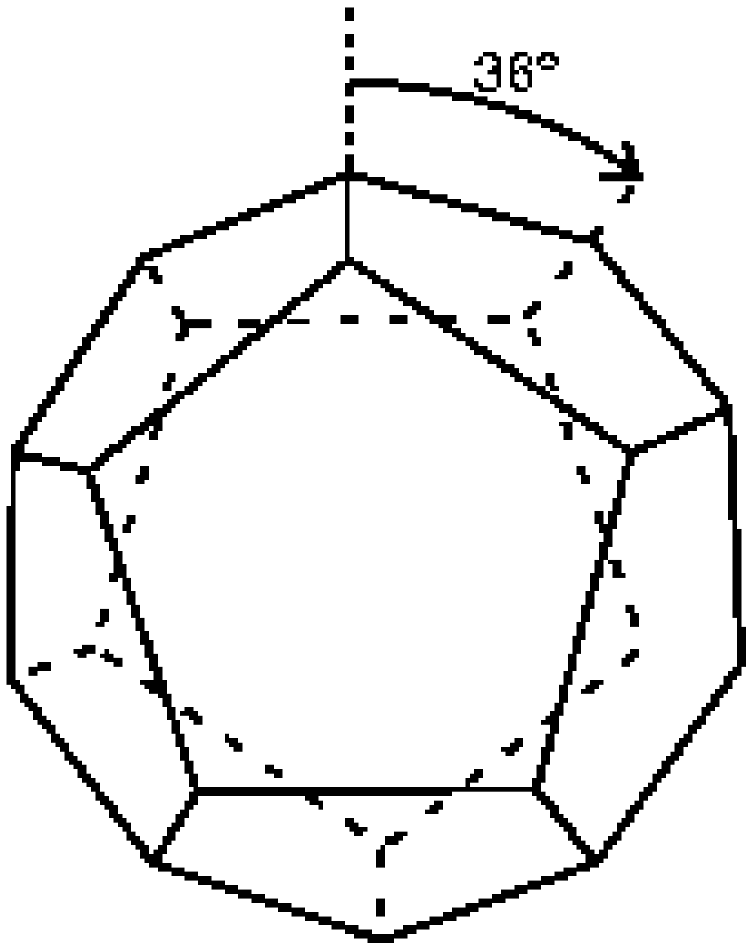

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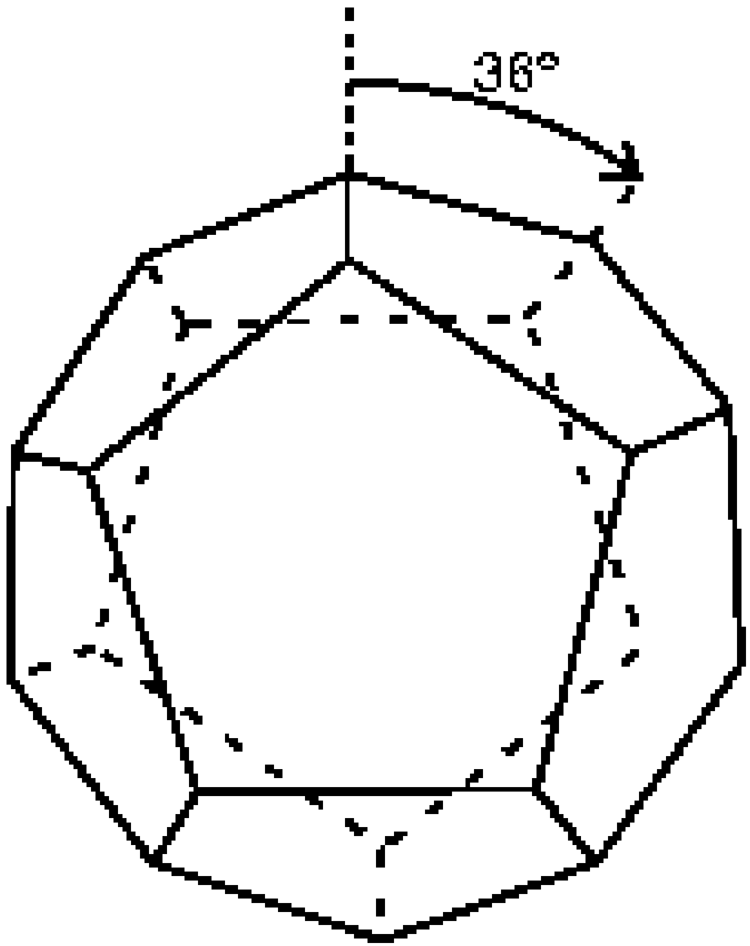

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- Luminet et al. (2003): S^3/I^* favoured by WMAP statistics

Optimal cross-correlation method

- extension to identified circles principle:

Optimal cross-correlation method

- for a given manifold, e.g. S^3/I^*:

Optimal cross-correlation method

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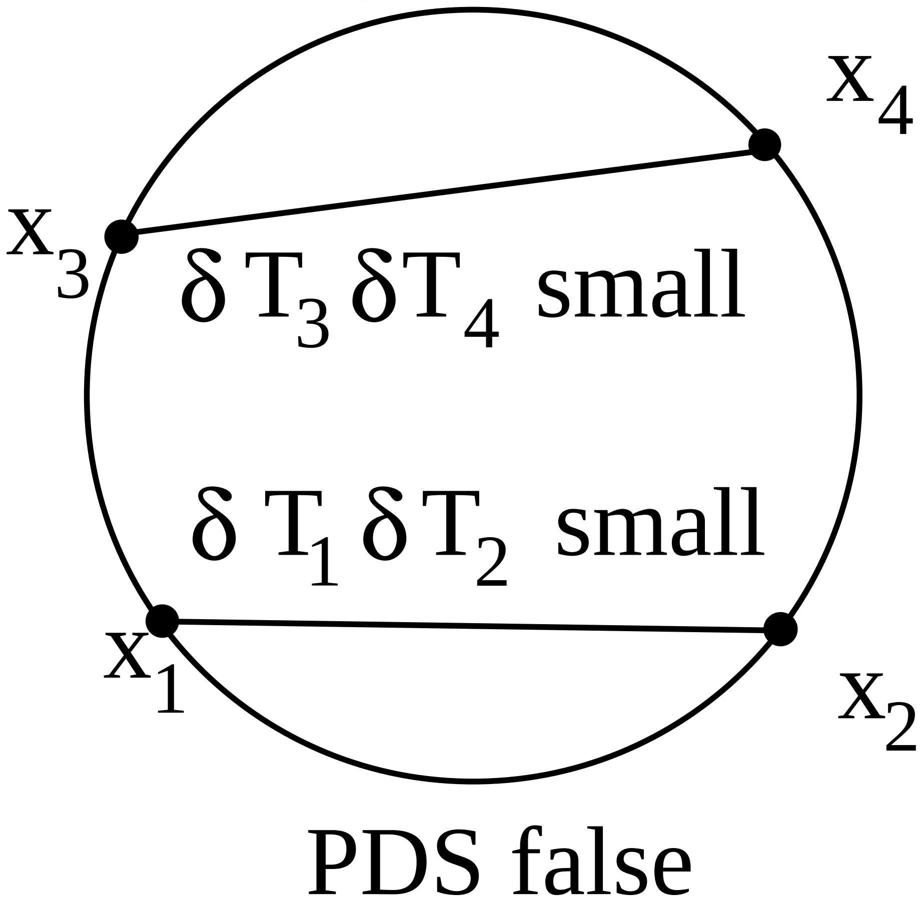

S^3/I^*:

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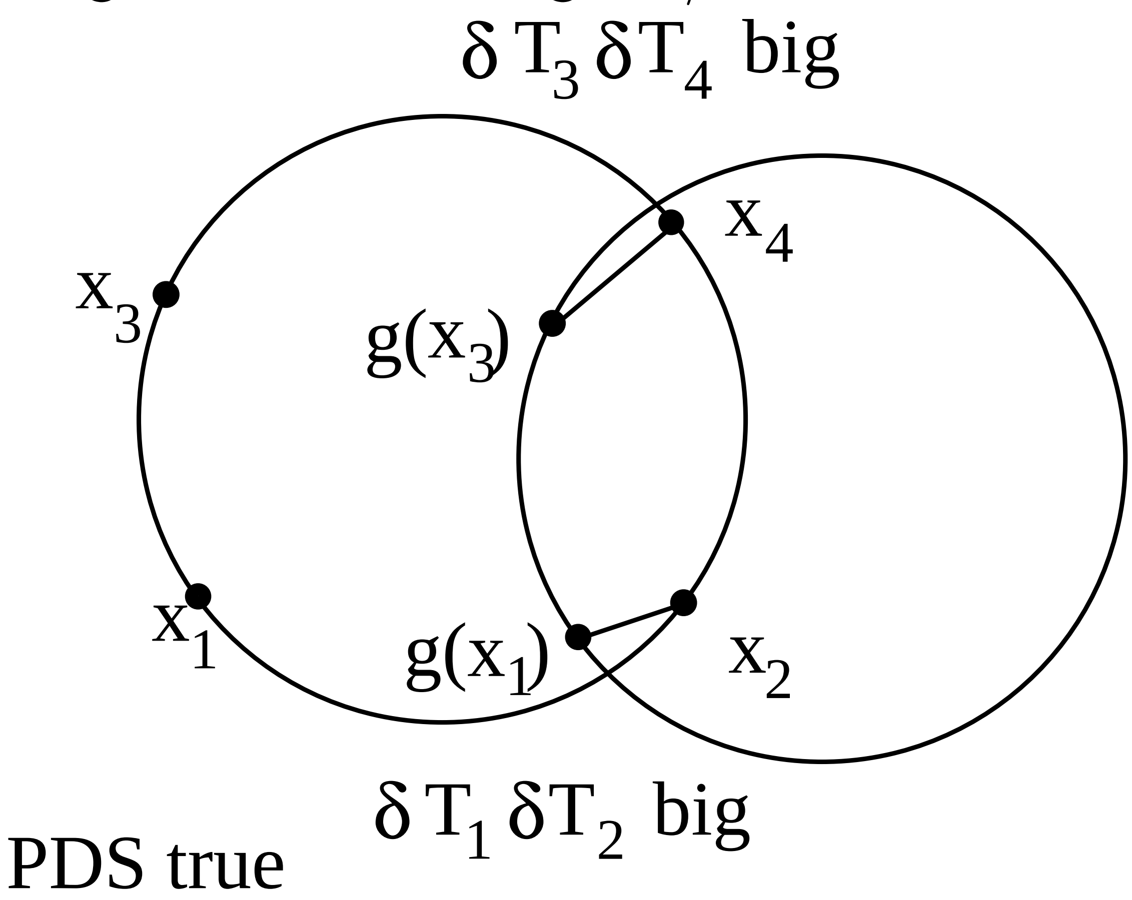

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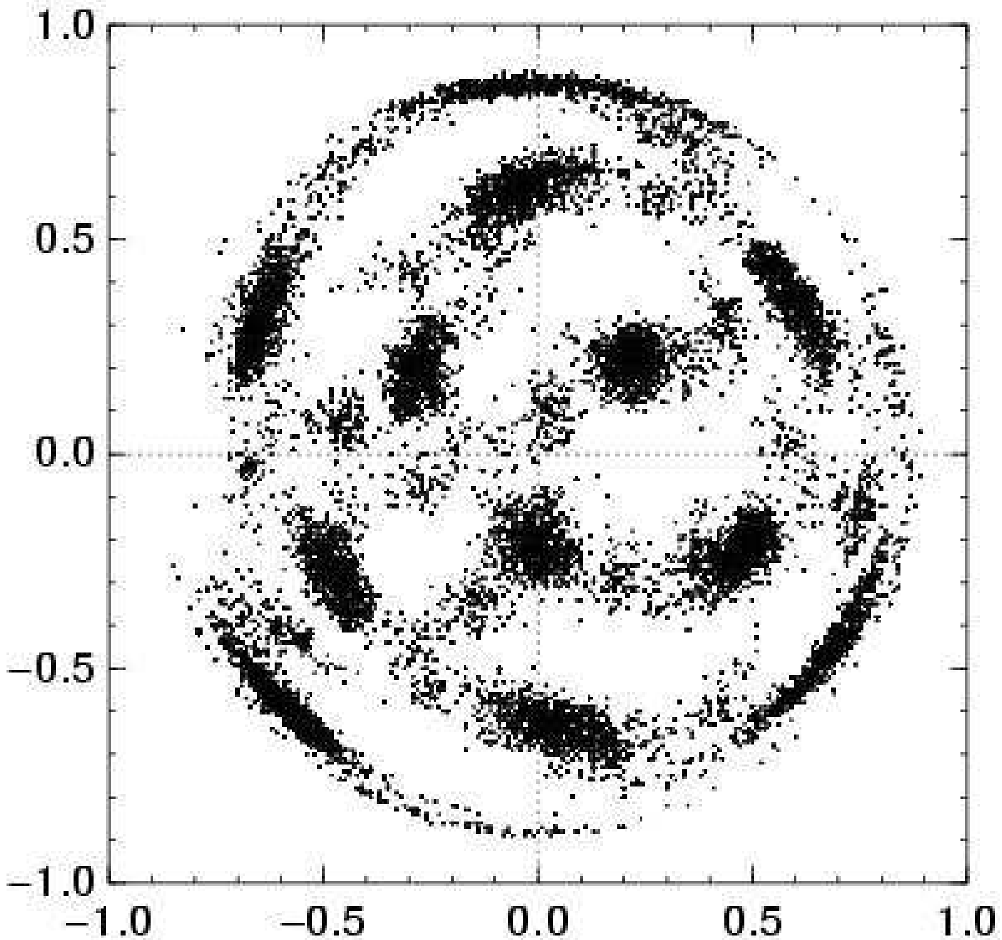
- allow arbitrary ϕ so that accidental correlations are likely to give an invalid value $\phi \neq \pm 36^\circ$

WMAP + Poincaré S^3/I^*



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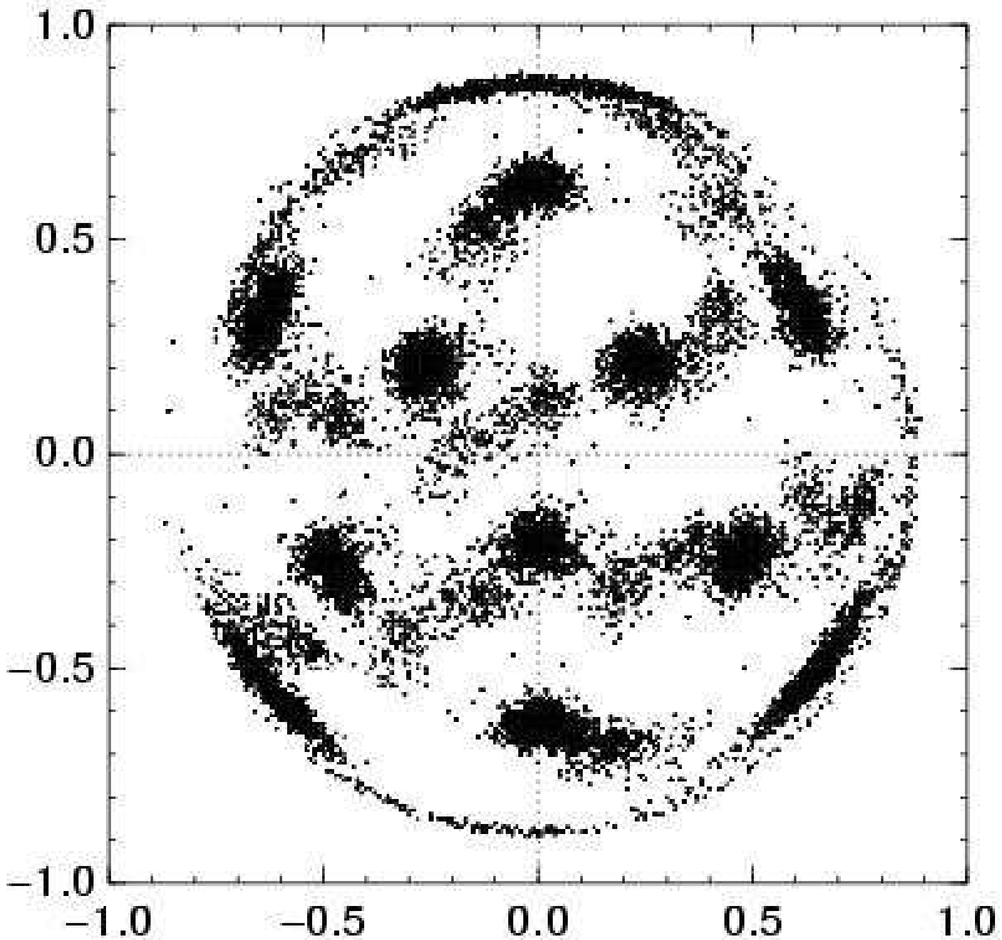

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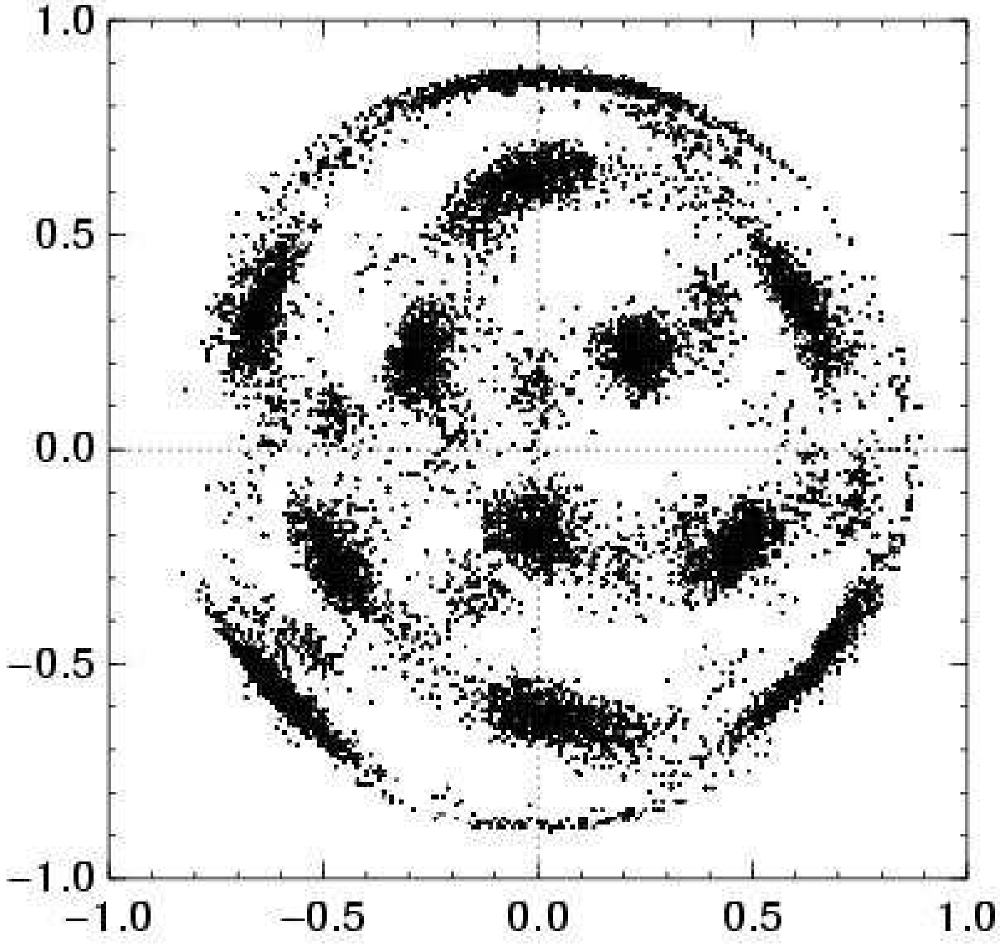

TOH + kp2

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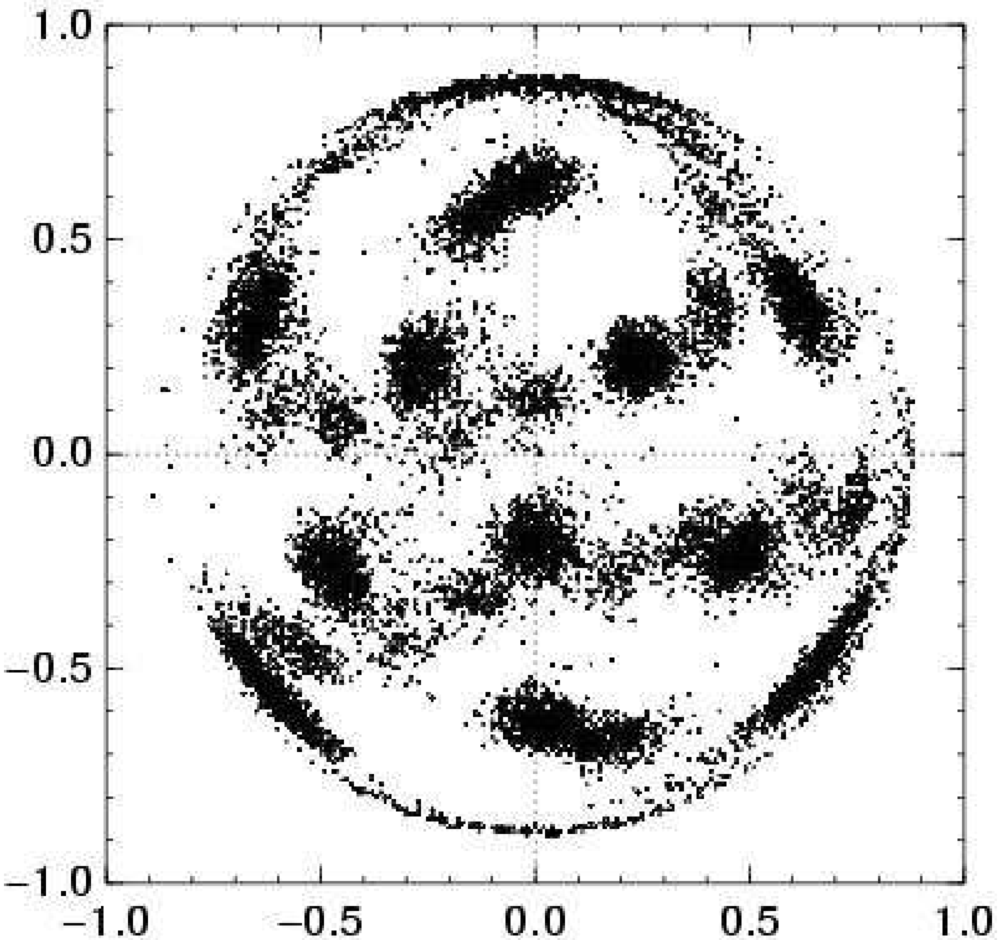

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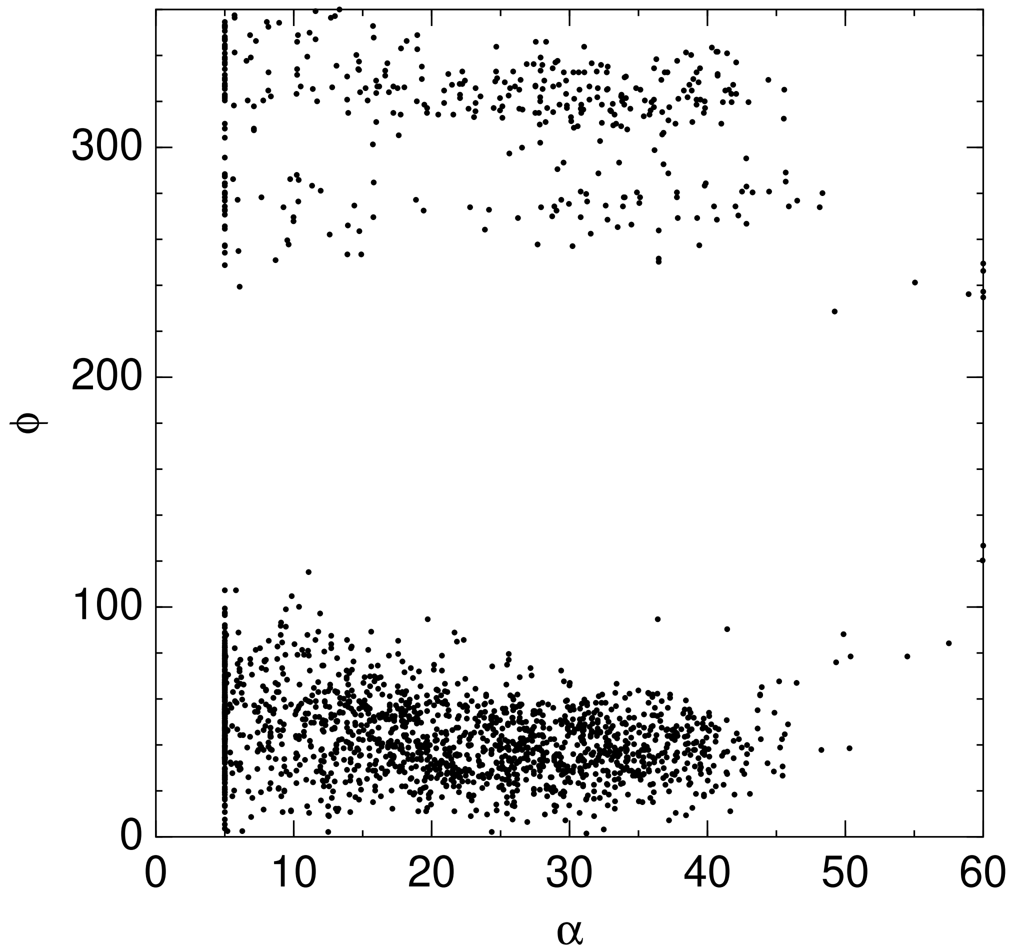
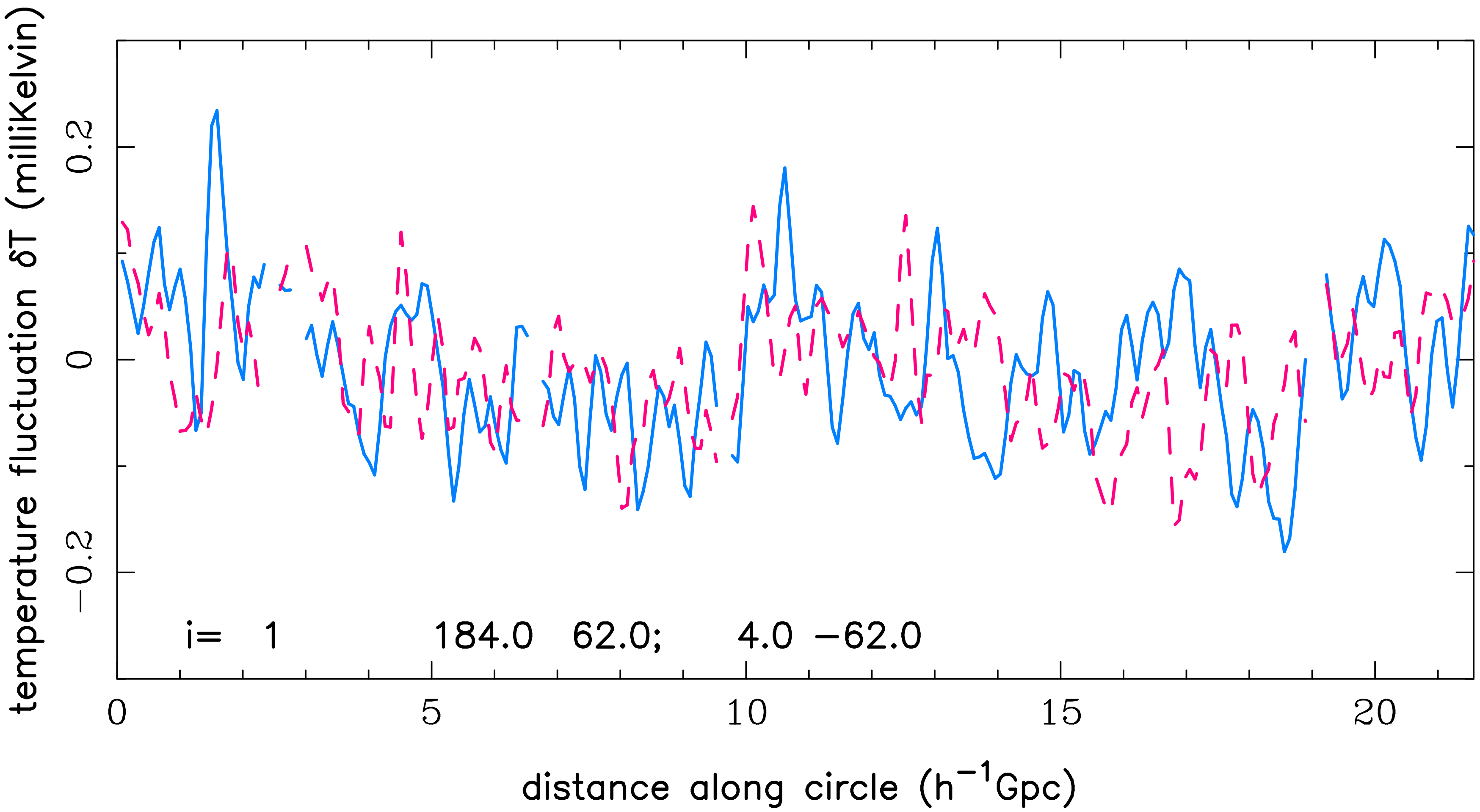


ϕ vs matched circle size α

WMAP + Poincaré S^3/I^*

$P_{\min}$	n	α°	$\sigma^\circ_{\langle\alpha\rangle}$	ϕ°	$\sigma^\circ_{\langle\phi\rangle}$
0.4	12589.0	20.6	0.6	**39.0**	**2.4**
0.5	6537.5	20.8	0.7	**38.7**	**2.2**
0.6	2961.0	22.1	0.5	**37.4**	**2.1**

RBSG08 matched circles



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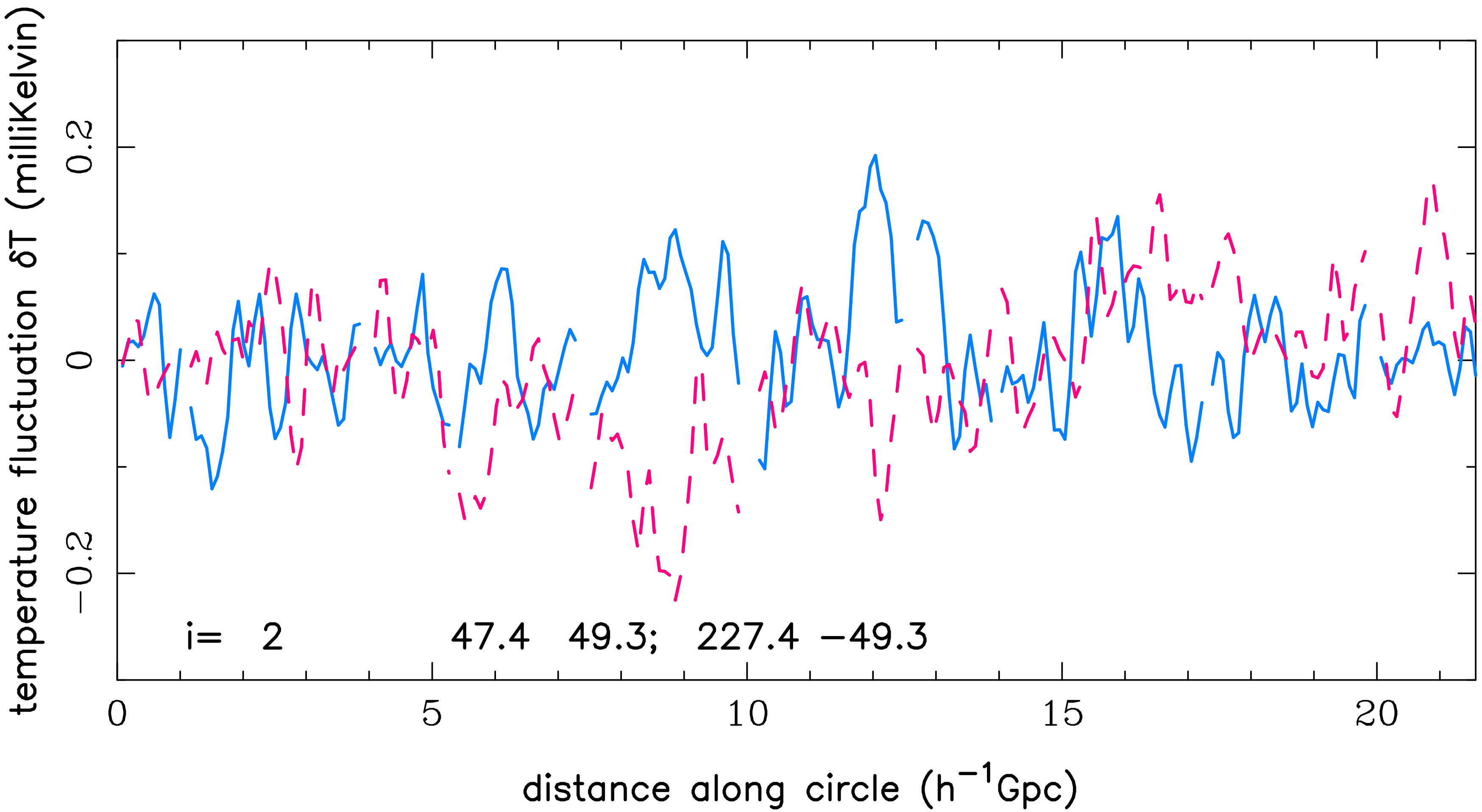

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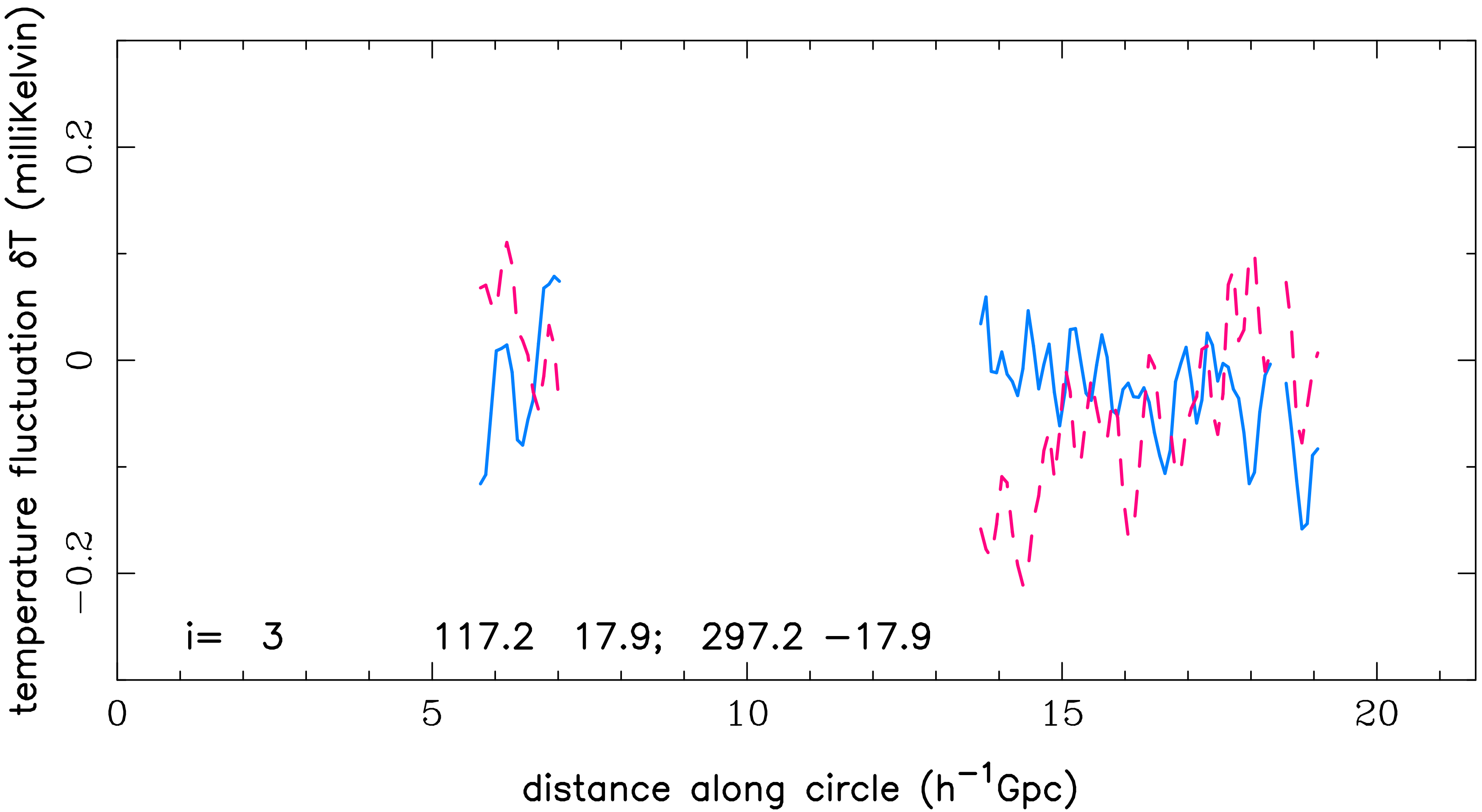

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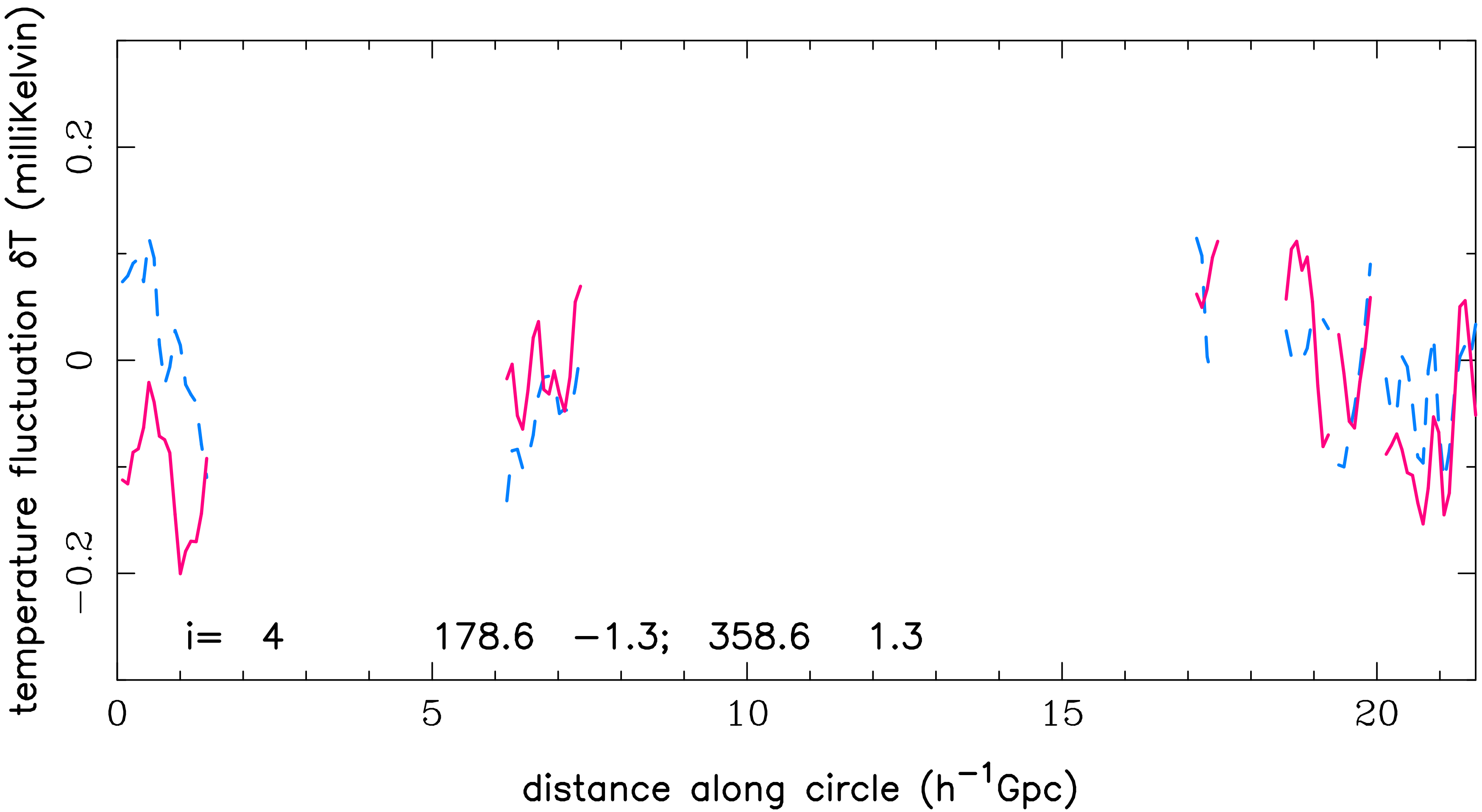

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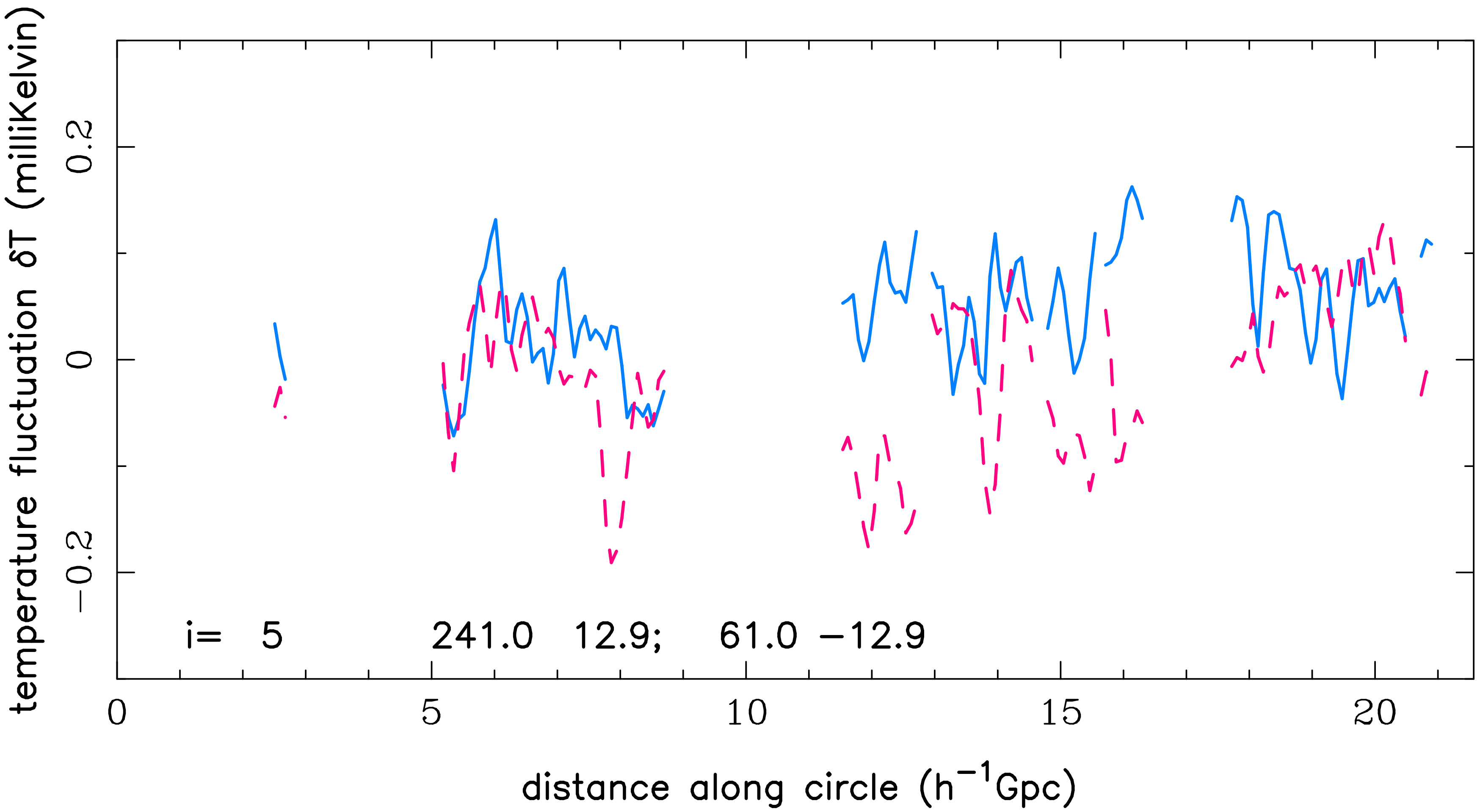

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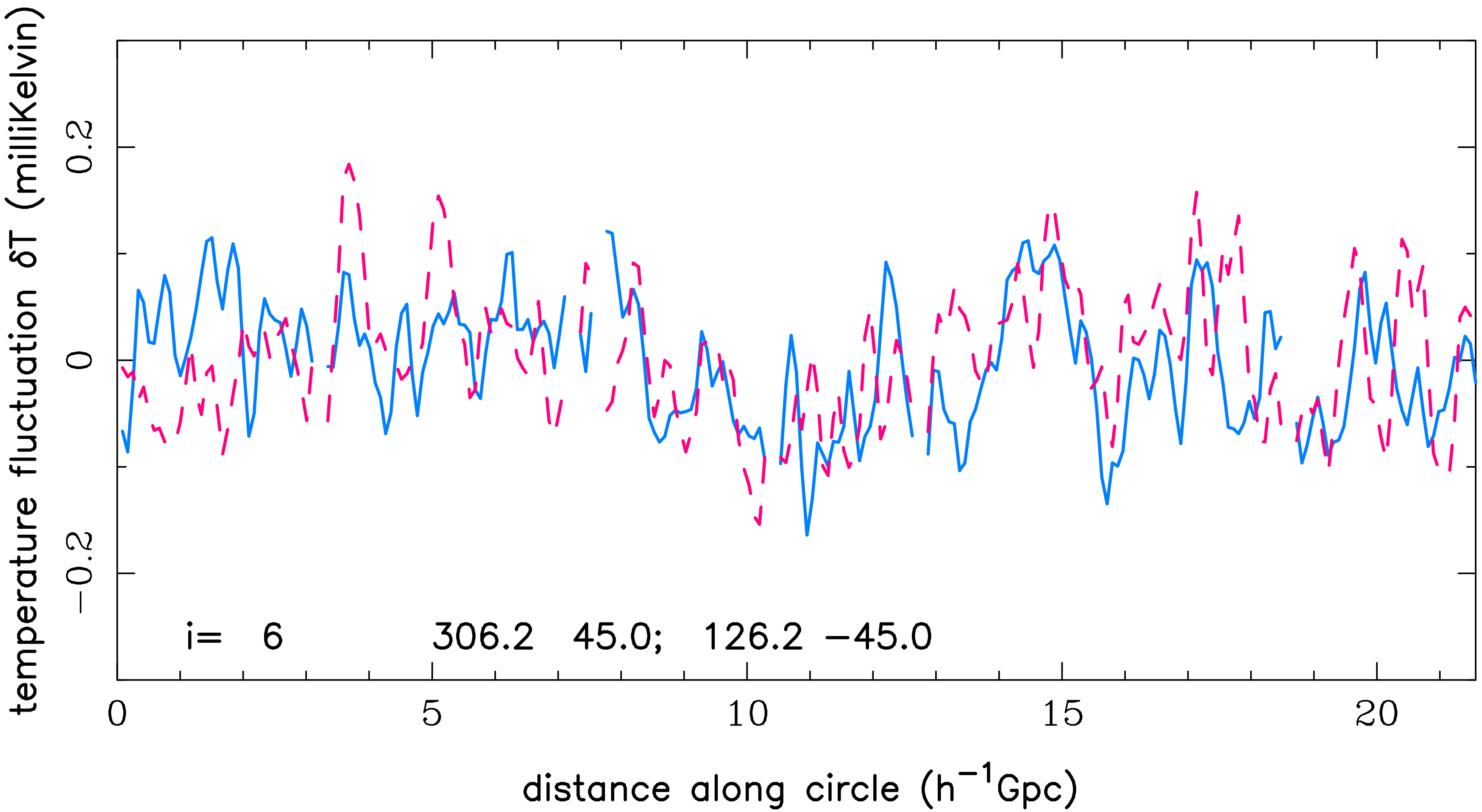

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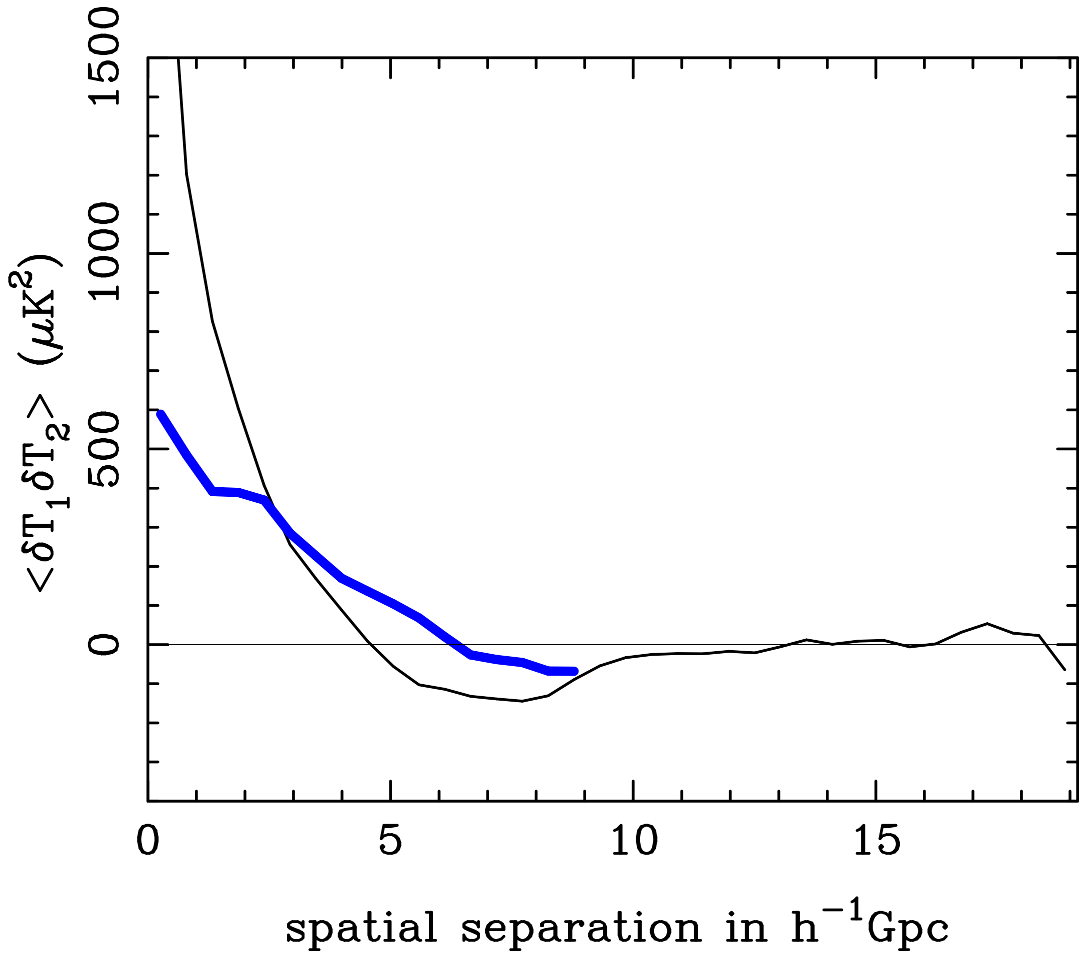

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- Planck (2013): (i) perturbation statististics assumption method; + (ii) identified circles: small correlation signal from S^3/I^* and other well-proportioned spaces, but consistent with noise

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