Cosmic topology

Boud Roukema
Institute of Astronomy
Nicolaus Copernicus University

https://cosmo.torun.pl/∼boud/Cosmic_topology.pdf

Introduction

- basic cosmology model:
  - GR: curvature = matter-energy content

Introduction

- basic cosmology model:
  - GR: curvature = matter-energy content
  - verbal averaging: homogeneous, isotropic spatial slices

Introduction

- basic cosmology model:
  - GR: curvature = matter-energy content
  - verbal averaging: homogeneous, isotropic spatial slices
  - shape of space (curvature + topology)

Introduction

- basic cosmology model:
  - GR: curvature = matter-energy content
  - verbal averaging: homogeneous, isotropic spatial slices
  - shape of space (curvature + topology)
  - \+ observations of expansion ⇒ hot big bang

Introduction

- basic cosmology model:
  - GR: curvature = matter-energy content
  - verbal averaging: homogeneous, isotropic spatial slices
  - shape of space (curvature + topology)
  - \+ observations of expansion $\Rightarrow$ hot big bang
  - $\Rightarrow$ black body radiation, nucleosynthesis

Introduction

- basic cosmology model:
  - ◆ GR: curvature = matter-energy content
  - ◆ verbal averaging: homogeneous, isotropic spatial slices
  - ◆ shape of space (curvature + topology)
  - \+ observations of expansion $\Rightarrow$ hot big bang
  - $\Rightarrow$ black body radiation, nucleosynthesis
  - ◆ but: $\exists$ galaxies $\Rightarrow$ inhomogeneous, anisotropic spatial slices

Introduction

- basic cosmology model:
  - GR: curvature = matter-energy content
  - verbal averaging: homogeneous, isotropic spatial slices
  - shape of space (curvature + topology)
  - \+ observations of expansion $\Rightarrow$ hot big bang
  - $\Rightarrow$ black body radiation, nucleosynthesis
  - but: $\exists$ galaxies $\Rightarrow$ inhomogeneous, anisotropic spatial slices
  - standard model: density perturbations (anisotropy)

Introduction

- basic cosmology model:
  - GR: curvature = matter-energy content
  - verbal averaging: homogeneous, isotropic spatial slices
  - shape of space (curvature + topology)
  - \+ observations of expansion $\Rightarrow$ hot big bang
  - $\Rightarrow$ black body radiation, nucleosynthesis
  - but: $\exists$ galaxies $\Rightarrow$ inhomogeneous, anisotropic spatial slices
  - standard model: density perturbations (anisotropy)
  - verbal averaging: can we do better?

Introduction

- basic cosmology model:
  - GR: curvature = matter-energy content
  - verbal averaging: homogeneous, isotropic spatial slices
  - shape of space (curvature + topology)
  - \+ observations of expansion $\Rightarrow$ hot big bang
  - $\Rightarrow$ black body radiation, nucleosynthesis
  - but: $\exists$ galaxies $\Rightarrow$ inhomogeneous, anisotropic spatial slices
  - standard model: density perturbations (anisotropy)
  - ~~verbal averaging~~

Introduction

- basic cosmology model:
  - GR: curvature = matter-energy content
  - verbal averaging: homogeneous, isotropic spatial slices
  - shape of space (curvature + topology)
  - \+ observations of expansion $\Rightarrow$ hot big bang
  - $\Rightarrow$ black body radiation, nucleosynthesis
  - but: $\exists$ galaxies $\Rightarrow$ inhomogeneous, anisotropic spatial slices
  - standard model: density perturbations (anisotropy)
  - scalar (GR) averaging: statistically homogeneous spatial slices
- within this model, what is the shape of the Universe?

verbal averaging

- w:Cosmological principle

verbal averaging

- w:Cosmological principle
- practical meaning:

verbal averaging

- w:Cosmological principle
- practical meaning:
  1. assume homogeneity and isotropy

verbal averaging

- w:Cosmological principle
- practical meaning:
  1. assume homogeneity and isotropy
  2. find the (differential 4-pseudo-manifold, metric) pairs $(M, \mathbf{g})$ that solve $\mathbf{G} = 8\pi\mathbf{T}$

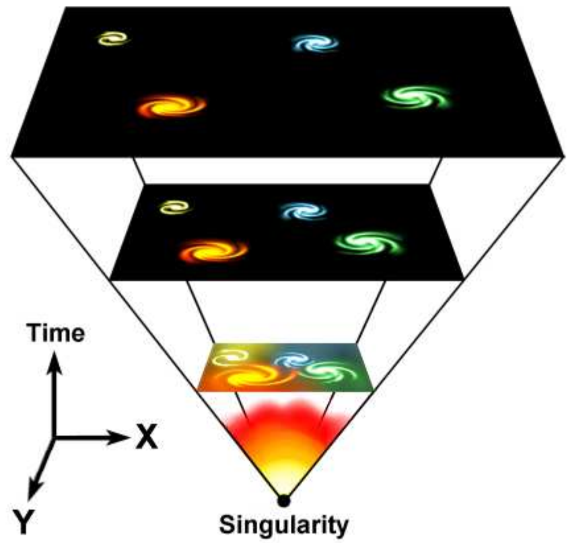
verbal averaging

- w:Cosmological principle
- practical meaning:
  1. assume homogeneity and isotropy
  2. find the (differential 4-pseudo-manifold, metric) pairs $(M, \mathbf{g})$ that solve $\mathbf{G} = 8\pi\mathbf{T}$
  3. assume that $(M, \mathbf{g})$ remains unchanged if we add density perturbations to an early time slice

verbal averaging

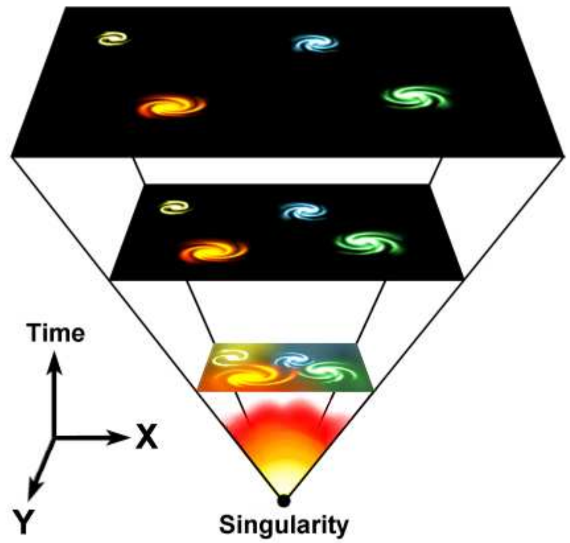
- w:Comoving coordinates

verbal averaging



- w:Comoving coordinates

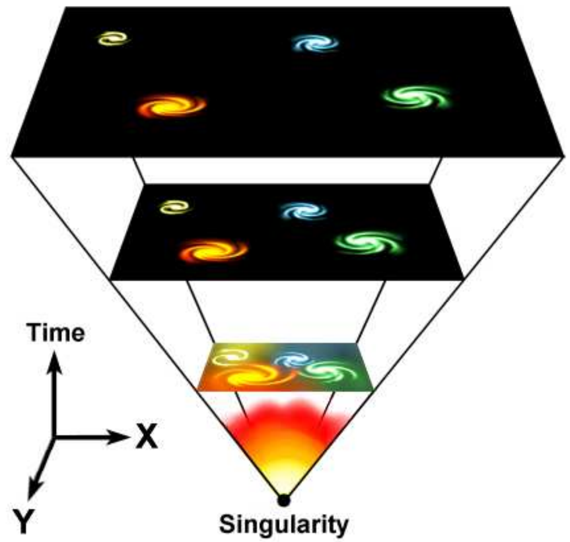
verbal averaging



- w:Comoving coordinates
-

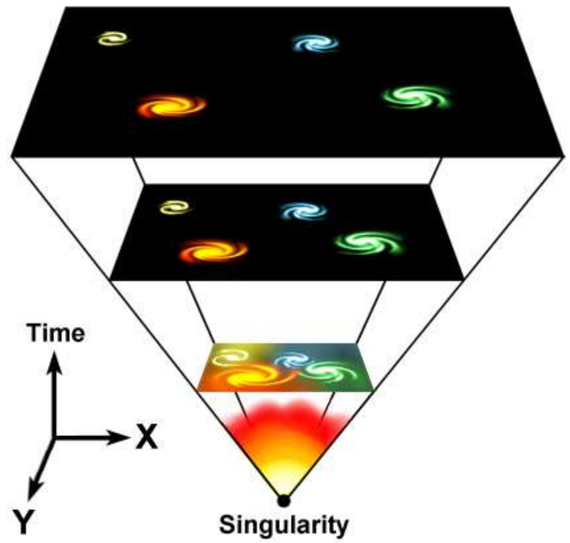
$$\Delta x(t) = a(t)\Delta r$$

verbal averaging



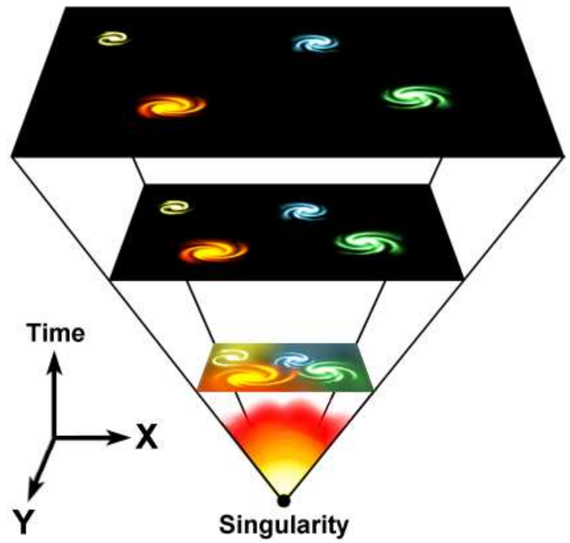
- w:Comoving coordinates
- $$\Delta x(t) = a(t)\Delta r$$
- spherical coordinates for spatial slice

verbal averaging



- w:Comoving coordinates
- $$\int_{(t,r_1,\theta,\phi)}^{(t,r_2,\theta,\phi)} \mathrm{d}s = a(t)\Delta r = a(t)|r_2 - r_1|$$

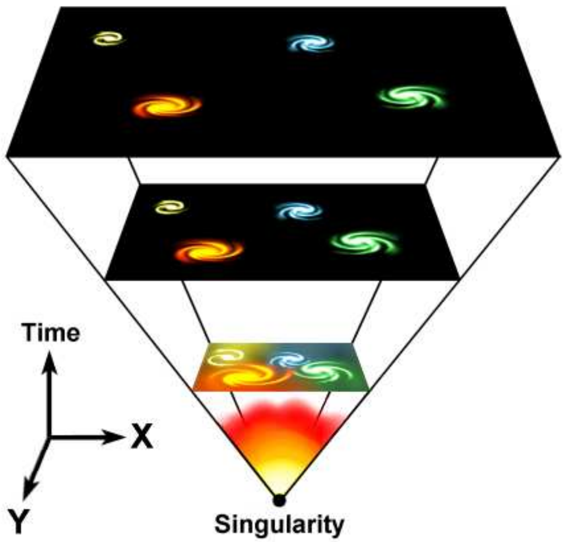
verbal averaging



- w:Comoving coordinates
- $$\int_{(t,r_1,\theta,\phi)}^{(t,r_2,\theta,\phi)} \mathrm{d}s = a(t)\Delta r = a(t)|r_2 - r_1|$$

where all expansion/contraction $\rightarrow$ w:scale factor $a(t)$

verbal averaging



- w:Comoving coordinates
-

$$\int_{(t,r_1,\theta,\phi)}^{(t,r_2,\theta,\phi)} \mathrm{d}s = a(t)\Delta r = a(t)|r_2 - r_1|$$

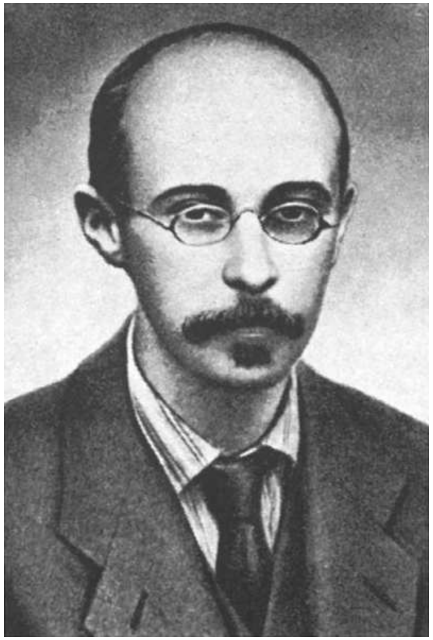
where all expansion/contraction → w:scale factor $a(t)$

- universe is static in comoving coordinates (r, θ, ϕ)

FLRW metric

- w:Friedmann–Lemaître–Robertson–Walker metric

FLRW metric

- w:Friedmann–Lemaître–Robertson–Walker metric
- w: w:
  w:Howard Percy Robertson
  w:Arthur Geoffrey Walker

FLRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \dots$$

FLRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,[\ldots]$$

FLRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + \ldots\right]$$

FLRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r_\perp^2(\ldots)\right]$$

FLRW metric

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r_\perp^2(\mathrm{d}\theta^2 + \cos^2\theta \mathrm{d}\phi^2)\right]}$$

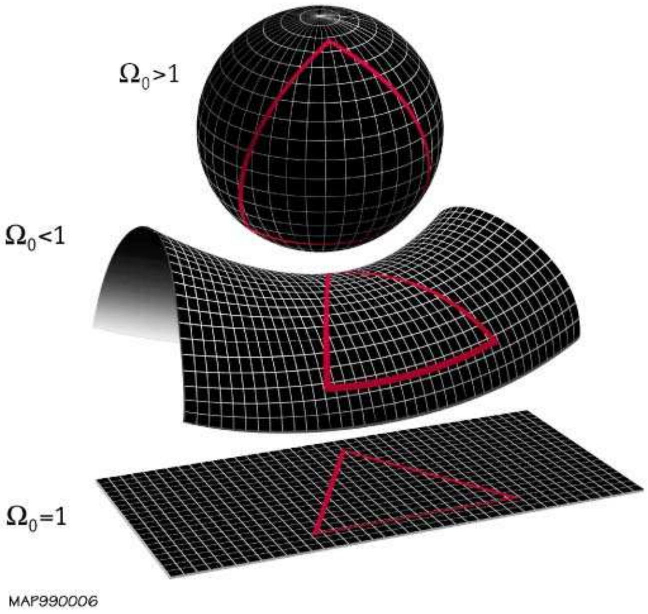
FLRW metric

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r_\perp^2(\mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\phi^2)\right]}$$

where $r_\perp := \begin{cases} R_\mathrm{C}\sinh\frac{r}{R_\mathrm{C}} & k<0 \\ r & k=0 \\ R_\mathrm{C}\sin\frac{r}{R_\mathrm{C}} & k>0 \end{cases}$

FLRW metric

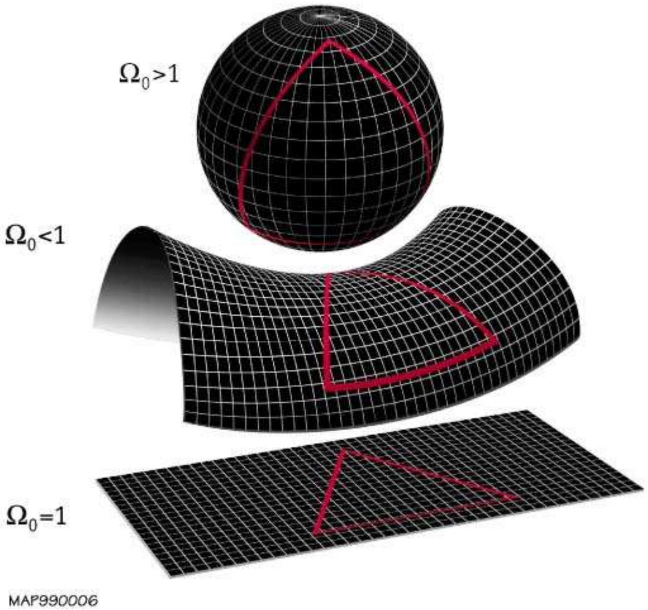
$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r_\perp^2(\mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\phi^2)\right]}$$



where $\quad r_\perp := \begin{cases} R_\mathrm{C}\sinh\frac{r}{R_\mathrm{C}} & k<0 \\ r & k=0 \\ R_\mathrm{C}\sin\frac{r}{R_\mathrm{C}} & k>0 \end{cases}$

FLRW metric

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r_\perp^2(\mathrm{d}\theta^2 + \cos^2\theta \mathrm{d}\phi^2)\right]}$$

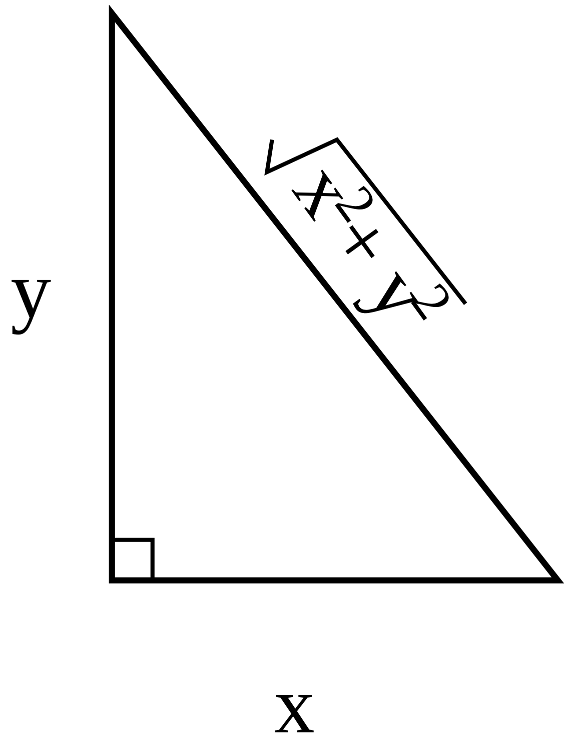


where $\quad r_\perp := \begin{cases} R_\mathrm{C} \sinh \frac{r}{R_\mathrm{C}} & k < 0 \\ r & k = 0 \\ R_\mathrm{C} \sin \frac{r}{R_\mathrm{C}} & k > 0 \end{cases}$

for a comoving radius of curvature R_C and curvature of sign k

curvature

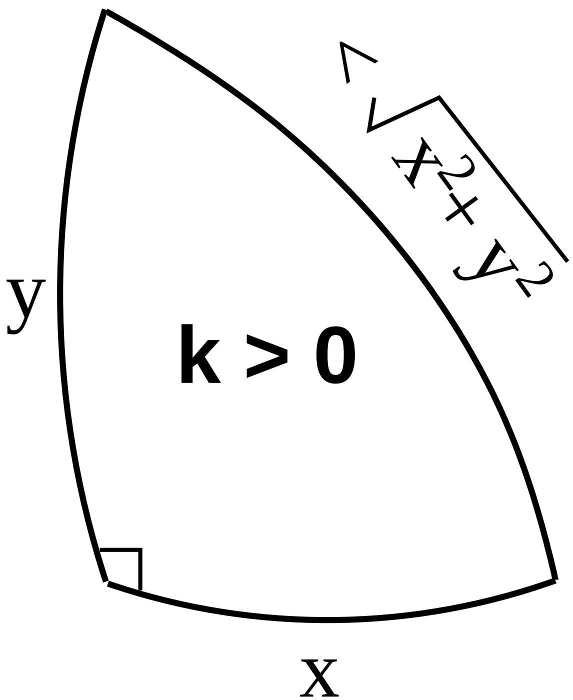
- on a spatial slice (fixed value of t):



$$k = 0$$

curvature

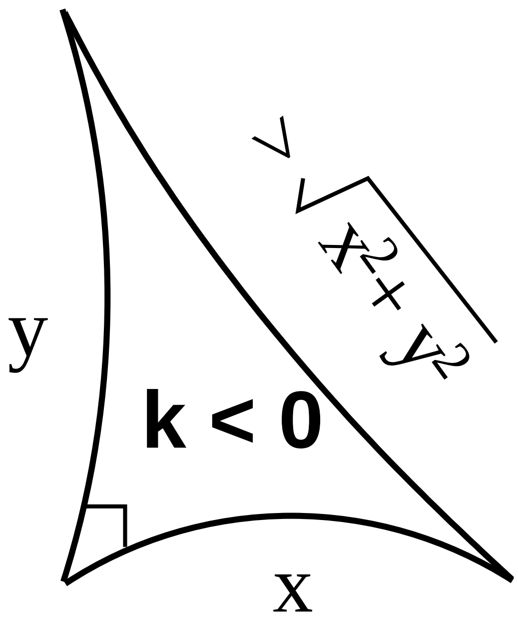
- on a spatial slice (fixed value of t):



$$k > 0$$

curvature

- on a spatial slice (fixed value of t):



$$k < 0$$

2D curvature intuition: $k > 0$

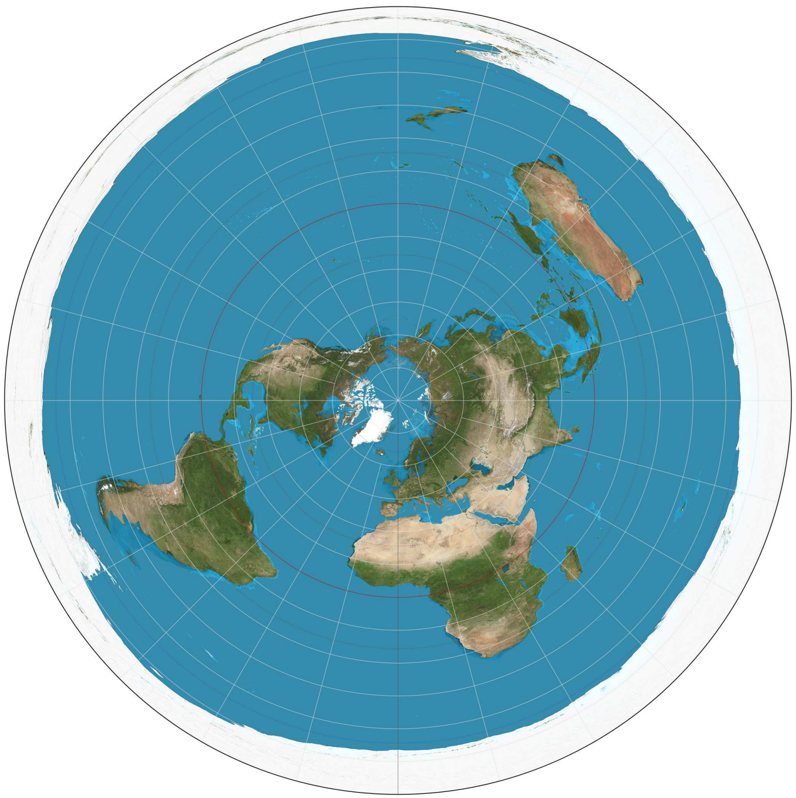
$$\mathrm{d}s^2|_{\phi=\mathrm{const},a=1} = \mathrm{d}r^2 + r_\perp^2 \mathrm{d}\theta^2,$$

2D curvature intuition: $k > 0$

$$\mathrm{d}s^2|_{\phi=\text{const},a=1} = \mathrm{d}r^2 + r_\perp^2 \mathrm{d}\theta^2, \text{ where } r_\perp := R_\mathrm{C} \sin \frac{r}{R_\mathrm{C}}$$

2D curvature intuition: $k > 0$

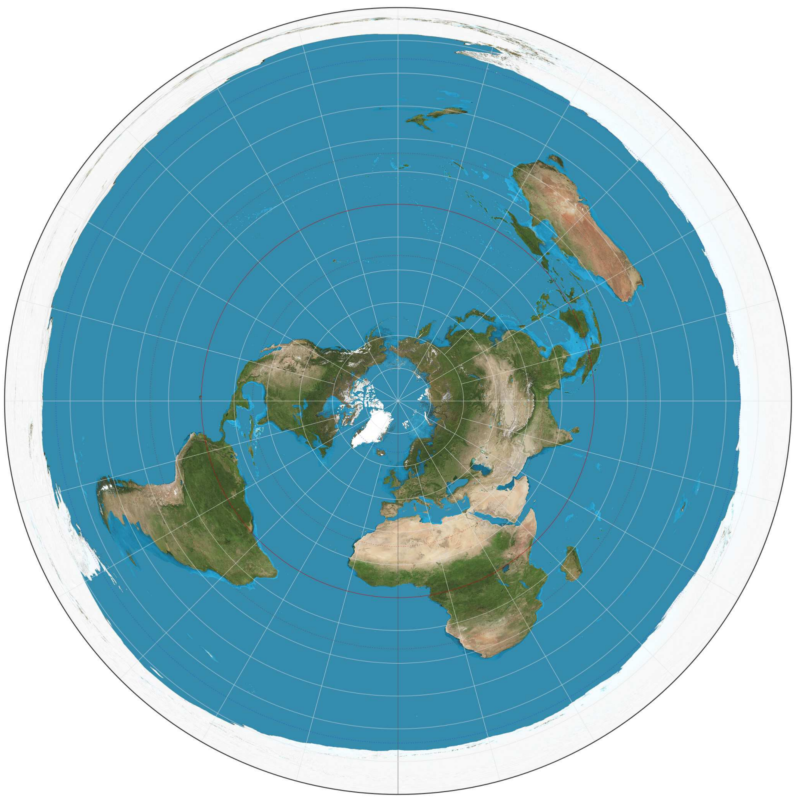
$\mathrm{d}s^2|_{\phi=\mathrm{const},a=1} = \mathrm{d}r^2 + r_\perp^2 \mathrm{d}\theta^2$, where $r_\perp := R_\mathrm{C} \sin \frac{r}{R_\mathrm{C}}$



w: ________________ (al-Biruni, c. 1000 CE)

2D curvature intuition: $k > 0$

$$\mathrm{d}s^2|_{\phi=\mathrm{const},a=1} = \mathrm{d}r^2 + r_\perp^2 \mathrm{d}\theta^2,$$ where $r_\perp := R_\mathrm{C} \sin \frac{r}{R_\mathrm{C}}$



w: (al-Biruni, c. 1000 CE)

- intuition switch: S^2 easier vs S^3 more physical

2D topology intuition ($k = 0$)

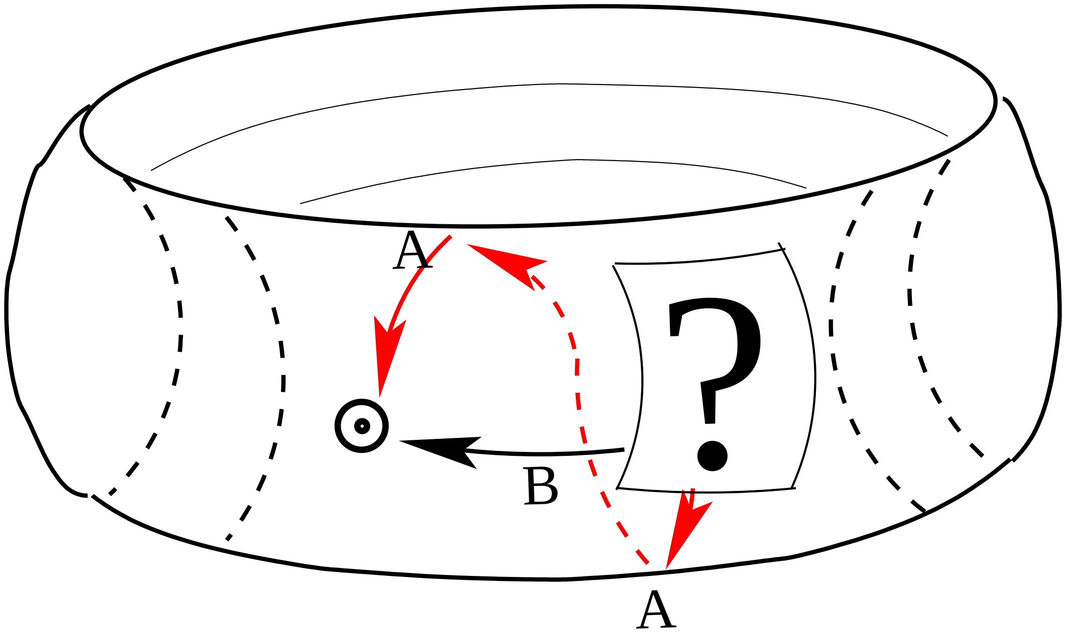
- w:

2D topology intuition ($k = 0$)

- intuition 1: embed in higher dim. space

2D topology intuition ($k = 0$)

- intuition 1: embed in higher dim. space

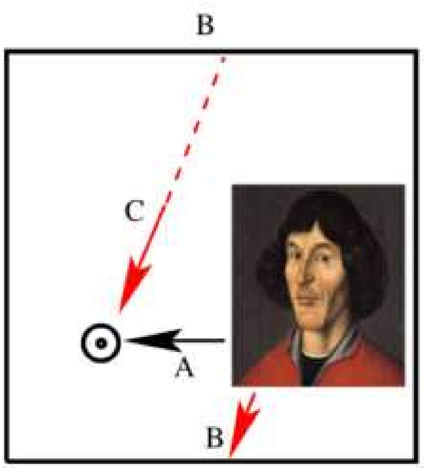


2D topology intuition ($k = 0$)

- intuition 1: embed in higher dim. space
- intuition 2: fundamental domain

2D topology intuition ($k = 0$)

- intuition 1: embed in higher dim. space
- intuition 2: fundamental domain

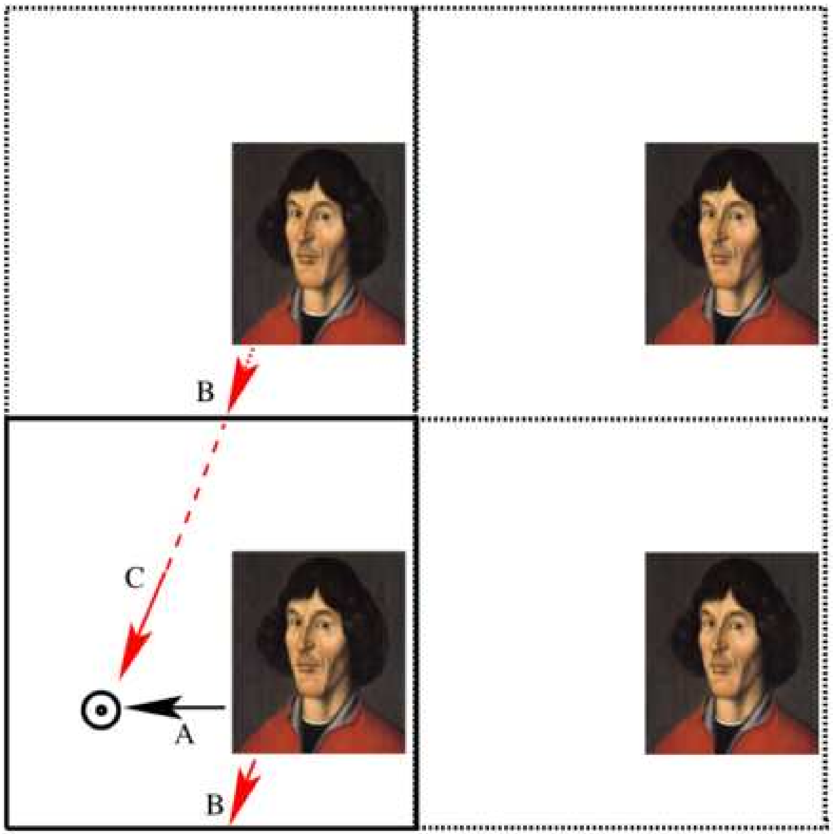


2D topology intuition ($k = 0$)

- intuition 1: embed in higher dim. space
- intuition 2: fundamental domain
- intuition 3: universal covering space

2D topology intuition ($k = 0$)

- intuition 1: embed in higher dim. space
- intuition 2: fundamental domain
- intuition 3: universal covering space



Topology animations

- animations at `http://www.geom.uiuc.edu/docs/forum/sos` (2026-03-13: https only, **not** https)
- by the Geometry Center at University of Minnesota, USA (closed since 1998)

expansion

- Hubble law (Lemaître 1927; ADS:1927ASSB...47...49L):
  $\dot{a}(t_0)/a(t_0) \approx 600$ km/s/Mpc

expansion

- Hubble law (Lemaître 1927; ADS:1927ASSB...47...49L):
  $\dot{a}(t_0)/a(t_0) \approx 600$ km/s/Mpc > 0 (also Hubble 1929)

expansion

- Hubble law (Lemaître 1927; ADS:1927ASSB...47...49L): $\dot{a}(t_0)/a(t_0) \approx 600$ km/s/Mpc > 0 (also Hubble 1929)
- so what is the "Big Bang"?

expansion

- Hubble law (Lemaître 1927; ADS:1927ASSB...47...49L): $\dot{a}(t_0)/a(t_0) \approx 600$ km/s/Mpc > 0 (also Hubble 1929)
- so what is the "Big Bang"?
- it is: $\exists t_{\rm b}$ such that $t \to t_{\rm b}^+ \Rightarrow a(t) \to 0^+$

expansion

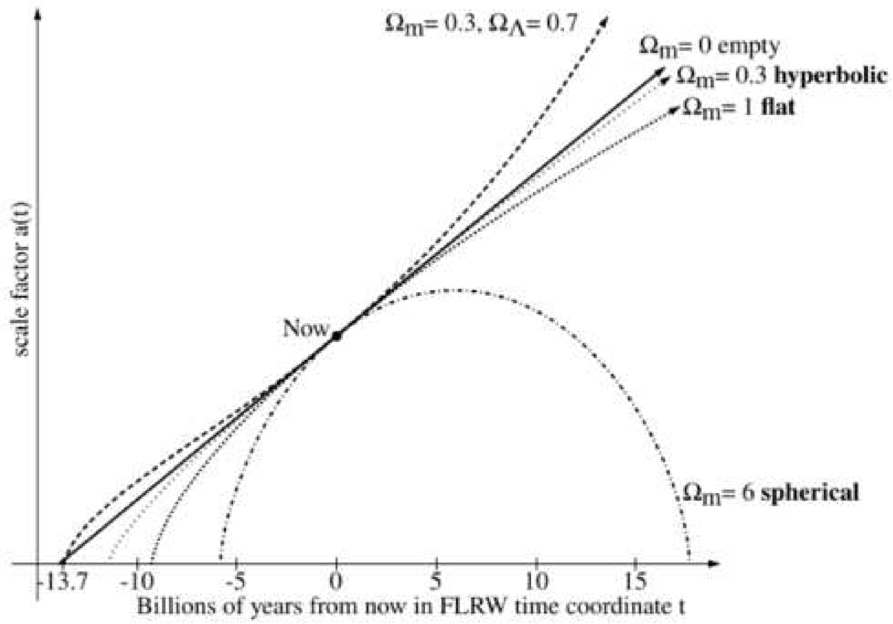
- Hubble law (Lemaître 1927; ADS:1927ASSB...47...49L):
  $\dot{a}(t_0)/a(t_0) \approx 600$ km/s/Mpc > 0 (also Hubble 1929)
- so what is the "Big Bang"?
- convention: $t_\mathrm{b} := 0$, giving $t \to 0^+ \Rightarrow a(t) \to 0^+$

expansion

- Hubble law (Lemaître 1927; ADS:1927ASSB...47...49L): $\dot{a}(t_0)/a(t_0) \approx 600$ km/s/Mpc > 0 (also Hubble 1929)
- so what is the "Big Bang"?
- it is: $\exists t_\mathrm{b}$ such that $t \to t_\mathrm{b}^+ \Rightarrow a(t) \to 0^+$

expansion

- Hubble law (Lemaître 1927; ADS:1927ASSB...47...49L): $\dot{a}(t_0)/a(t_0) \approx 600$ km/s/Mpc > 0 (also Hubble 1929)
- so what is the "Big Bang"?
- it is: $\exists t_\mathrm{b}$ such that $t \to t_\mathrm{b}^+ \Rightarrow a(t) \to 0^+$



W:

expansion

- matter density: $\rho_{\mathrm{m}} \propto a^{-3}$

expansion

- matter density: $\rho_{\mathrm{m}} \propto a^{-3}$
- wavelength: $\lambda \propto a$ [GR—transport four-velocity: Synge (1960); Narlikar (1994; ADS:1994AmJPh..62..903N)]

expansion

- matter density: $\rho_{\mathrm{m}} \propto a^{-3}$
- wavelength: $\lambda \propto a$ [GR—transport four-velocity: Synge (1960); Narlikar (1994; ADS:1994AmJPh..62..903N)]

$\Leftrightarrow$

$$\frac{\lambda_{\mathrm{obs}}}{\lambda_{\mathrm{em}}} = \frac{a_{\mathrm{obs}}}{a_{\mathrm{em}}}$$

expansion

- matter density: $\rho_{\mathrm{m}} \propto a^{-3}$
- wavelength: $\lambda \propto a$ [GR—transport four-velocity: Synge (1960); Narlikar (1994; ADS:1994AmJPh..62..903N)]
- $$\frac{\lambda_{\mathrm{obs}}}{\lambda_{\mathrm{em}}} = \frac{1}{a_{\mathrm{em}}}$$

  (Defn: $a_0 := 1$)

expansion

- matter density: $\rho_{\rm m} \propto a^{-3}$
- wavelength: $\lambda \propto a$ [GR—transport four-velocity: Synge (1960); Narlikar (1994; ADS:1994AmJPh..62..903N)]
-

$$1 + z = \frac{1}{a_{\rm em}}$$

(Defn of redshift z)

expansion

- matter density: $\rho_{\rm m} \propto a^{-3} = (1+z)^3$
- wavelength: $\lambda \propto a$ [GR—transport four-velocity: Synge (1960); Narlikar (1994; ADS:1994AmJPh..62..903N)]
- $$\boxed{1 + z = a^{-1}}$$

  (light-cone convention: a often means $a_{\rm em}$)

expansion

- matter density: $\rho_{\rm m} \propto a^{-3} = (1+z)^3$
- wavelength: $\lambda \propto a$ [GR—transport four-velocity: Synge (1960); Narlikar (1994; ADS:1994AmJPh..62..903N)]
- $$\boxed{1+z = a^{-1}}$$
  (light-cone convention: a often means $a_{\rm em}$)
- radiation density: $E = h\nu \Rightarrow \rho_{\rm r} \propto a^{-4} = (1+z)^4$

Black body: COBE (~ 1992)

- Planck's Law: $I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$

Black body: COBE (~ 1992)

- Planck's Law: $I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$
- wavelength: $\lambda \propto a \Rightarrow$ frequency: $\nu \propto a^{-1} = (1 + z)$

Black body: COBE (~ 1992)

- Planck's Law: $I(\nu, T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}$
- wavelength: $\lambda \propto a \Rightarrow$ frequency: $\nu \propto a^{-1} = (1+z)$
- $\Rightarrow$ temperature: $kT \propto h\nu \propto (1+z)$

Black body: COBE (~ 1992)

- Planck's Law: $I(\nu, T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}$
- wavelength: $\lambda \propto a \Rightarrow$ frequency: $\nu \propto a^{-1} = (1+z)$
- $\Rightarrow$ temperature: $kT \propto h\nu \propto (1+z)$
- $z \gg 1 \Rightarrow$ early Universe dominated by hot, dense plasma = protons, electrons, photons

Black body: COBE (~ 1992)

- Planck's Law: $I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$
- wavelength: $\lambda \propto a \Rightarrow$ frequency: $\nu \propto a^{-1} = (1+z)$
- $\Rightarrow$ temperature: $kT \propto h\nu \propto (1+z)$
- $z \gg 1 \Rightarrow$ early Universe dominated by hot, dense plasma = protons, electrons, photons
- $\Rightarrow$ black body

Black body: COBE (~ 1992)

- Planck's Law: $I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$
- wavelength: $\lambda \propto a \Rightarrow$ frequency: $\nu \propto a^{-1} = (1+z)$
- $\Rightarrow$ temperature: $kT \propto h\nu \propto (1+z)$
- $z \gg 1 \Rightarrow$ early Universe dominated by hot, dense plasma = protons, electrons, photons
- $\Rightarrow$ black body + primordial nucleosynthesis

CMB discovery: McKellar 1941

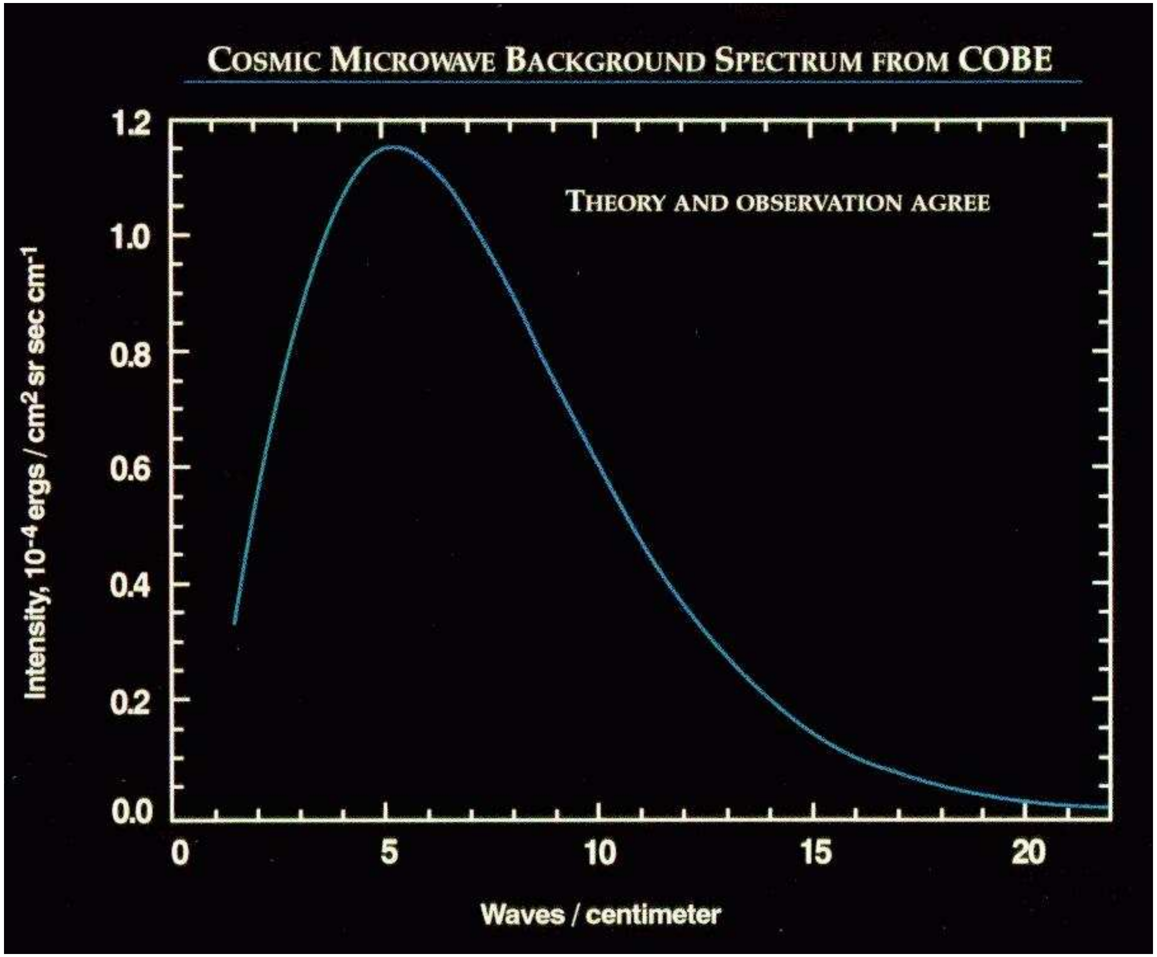
- $T \approx 2.3$ K — Andrew McKellar (1941; ADS:1941PDAO....7..251M) from observations by Walter S. Adams (1941; ADS:1941ApJ....93...11A)
- Penzias & Wilson rediscovery (1965 + Nobel prize)

Black body: COBE (~ 1992)

- COBE /FIRAS (Far Infrared Absolute Spectrophotometer)

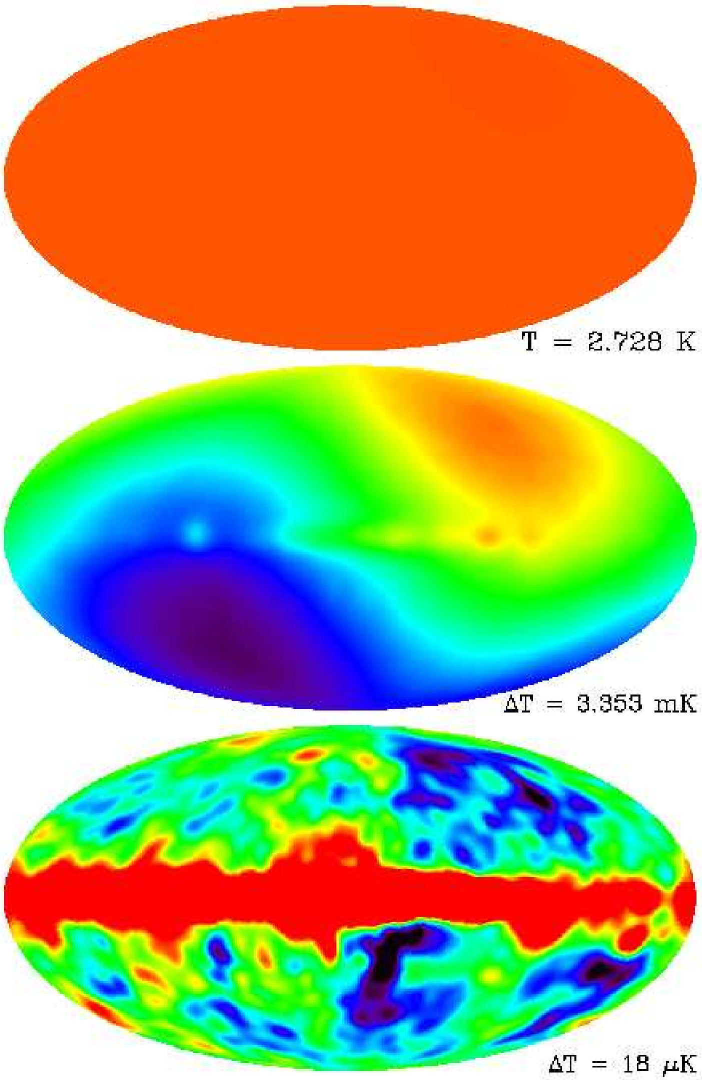
Black body: COBE (~ 1992)

- COBE /FIRAS (Far Infrared Absolute Spectrophotometer)



Black body: COBE (~ 1992)

- COBE /DMR (Differential Microwave Radiometer)



BBN: Big bang nucleosynthesis

- Alpher, Bethe, & Gamow (1948; ADS:1948PhRv...73..803A)

BBN: Big bang nucleosynthesis

- Alpher, Bethe, & Gamow (1948; ADS:1948PhRv...73..803A)

  main reactions of varying probability:

$$
\begin{aligned}
\mathrm{p} + \mathrm{n} &\rightarrow {}^{2}\mathrm{H} + \gamma \\
\mathrm{p} + {}^{2}\mathrm{H} &\rightarrow {}^{3}\mathrm{He} + \gamma \\
{}^{2}\mathrm{H} + {}^{2}\mathrm{H} &\rightarrow {}^{3}\mathrm{He} + \mathrm{n} \\
{}^{2}\mathrm{H} + {}^{2}\mathrm{H} &\rightarrow {}^{3}\mathrm{H} + \mathrm{p} \\
{}^{3}\mathrm{He} + {}^{2}\mathrm{H} &\rightarrow {}^{4}\mathrm{He} + \mathrm{p} \\
{}^{3}\mathrm{H} + {}^{2}\mathrm{H} &\rightarrow {}^{4}\mathrm{He} + \mathrm{n}
\end{aligned}
$$

BBN: Big bang nucleosynthesis

- Alpher, Bethe, & Gamow (1948; ADS:1948PhRv...73..803A)

  main reactions of varying probability:

$$\begin{aligned} \mathrm{p} + \mathrm{n} &\rightarrow {}^{2}\mathrm{H} + \gamma \\ \mathrm{p} + {}^{2}\mathrm{H} &\rightarrow {}^{3}\mathrm{He} + \gamma \\ {}^{2}\mathrm{H} + {}^{2}\mathrm{H} &\rightarrow {}^{3}\mathrm{He} + \mathrm{n} \\ {}^{2}\mathrm{H} + {}^{2}\mathrm{H} &\rightarrow {}^{3}\mathrm{H} + \mathrm{p} \\ {}^{3}\mathrm{He} + {}^{2}\mathrm{H} &\rightarrow {}^{4}\mathrm{He} + \mathrm{p} \\ {}^{3}\mathrm{H} + {}^{2}\mathrm{H} &\rightarrow {}^{4}\mathrm{He} + \mathrm{n} \end{aligned}$$

  w:Big Bang nucleosynthesis

FLRW: $a(t) =?$

- second choice FLRW coord system: $r :=$ w:orthographic projection of radial comoving distance (cf $r :=$ radial comoving distance)

FLRW: $a(t) =?$

- second choice FLRW coord system: $r :=$ w:orthographic projection of radial comoving distance (cf $r :=$ radial comoving distance)

$\Rightarrow$ coord singularity at equator ($:= \pi/2$ from centre) if $k > 0$

FLRW: $a(t) =?$

- second choice FLRW coord system: $r :=$ w:orthographic projection of radial comoving distance (cf $r :=$ radial comoving distance)

⇒ coord singularity at equator ($:= \pi/2$ from centre) if $k > 0$

FLRW: $a(t) =?$

- second choice FLRW coord system: $r :=$ w:orthographic projection of radial comoving distance (cf $r :=$ radial comoving distance)

⇒ coord singularity at equator ($:= \pi/2$ from centre) if $k > 0$

- $$ds^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\phi^2)\right]$$

FLRW: $a(t) =?$

- $$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\phi^2)\right]$$

- universe content: $\mathrm{diag}(\mathbf{T}) = (-\rho, p, p, p)$

FLRW: $a(t) = ?$

- $$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \cos^2\theta\,\mathrm{d}\phi^2)\right]$$
- universe content: $\mathrm{diag}(\mathbf{T}) = (-\rho, p, p, p)$
- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} = 8\pi\mathbf{T}$ and simplify: https://cosmo.torun.pl/Cosmo/FLRWEquationsGR

FLRW: $a(t) =?$

- $$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\phi^2)\right]$$
- universe content: $\mathrm{diag}(\mathbf{T}) = (-\rho, p, p, p)$
- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} = 8\pi\mathbf{T}$ and simplify: https://cosmo.torun.pl/Cosmo/FLRWEquationsGR
- Friedmann Eqn: $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3}$$

FLRW: $a(t) = ?$

- $$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \cos^2\theta\,\mathrm{d}\phi^2)\right]$$

- universe content: $\mathrm{diag}(\mathbf{T}) = (-\rho, p, p, p)$
- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} = 8\pi\mathbf{T}$ and simplify: https://cosmo.torun.pl/Cosmo/FLRWEquationsGR
- Friedmann Eqn: $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3}$$
- acceleration Eqn: $$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G\,(\rho + 3\,p/c^2)}{3}$$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}}$$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}}$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a^3} - \frac{c^2\,k}{a^2}}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\dot{a}^2 \propto a^{-1}$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m}0}}{3\,a} - c^2\,k}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m}0}\,a^{-3}$
- $k = 0$ case: $\dot{a} \propto a^{-1/2}$ for $a > 0$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\mathrm{d}a \propto a^{-1/2}\mathrm{d}t$ for $a > 0$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $a^{1/2}\mathrm{d}a \propto \mathrm{d}t$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $(2/3)a^{3/2} \propto t$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $a \propto t^{2/3}$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

Defn: $\boxed{\text{Hubble parameter } H := \dot{a}/a}$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

Defn: $\boxed{\text{Hubble parameter } H := \dot{a}/a}$

$\Rightarrow$ Hubble constant $H_0 := H(z=0)$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

Defn: $\boxed{\text{Hubble parameter } H := \dot{a}/a}$

$\Rightarrow$ Hubble constant $H_0 := H(z = 0)$

$\Rightarrow$ $k = 0$ case: $\frac{\dot{a}}{a} = \frac{2}{3t}$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m}0}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m}0}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

Defn: $\boxed{\text{Hubble parameter } H := \dot{a}/a}$

$\Rightarrow$ Hubble constant $H_0 := H(z=0)$

$\Rightarrow$ $k = 0$ case: $\frac{\dot{a}}{a} = \frac{2}{3t}$

$\Rightarrow$ $H(t) = \frac{2}{3t}$;

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

Defn: $\boxed{\text{Hubble parameter } H := \dot{a}/a}$

$\Rightarrow$ Hubble constant $H_0 := H(z=0)$

$\Rightarrow$ $k = 0$ case: $\frac{\dot{a}}{a} = \frac{2}{3t}$

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m}0}}{3\,a} - c^2\,k}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m}0}\,a^{-3}$
- $k = 0$ case: $$\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- convenient conversion: 1 km/s ≈ 1.04 kpc/Gyr ≈ 1 kpc/Gyr

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 600 km/s/Mpc

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 600 kpc/Gyr/Mpc

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m}0}}{3\,a} - c^2\,k}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m}0}\,a^{-3}$
- $k = 0$ case: $$\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx 0.6\ \mathrm{Gyr}^{-1}$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 0.6 Gyr^{-1}
- Hubble (1929) ADS:1929PNAS...15..168H: $H_0 \approx$ 500 km/s/Mpc

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 0.6 Gyr^{-1}
- Hubble (1929) ADS:1929PNAS...15..168H: $H_0 \approx$ 500 kpc/Gyr/Mpc

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx 0.6$ Gyr^{-1}
- Hubble (1929) ADS:1929PNAS...15..168H: $H_0 \approx 0.5$ Gyr^{-1}

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 0.6 Gyr^{-1}
- Hubble (1929) ADS:1929PNAS...15..168H: $H_0 \approx$ 0.5 Gyr^{-1}

$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0}$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 0.6 Gyr^{-1}
- Hubble (1929) ADS:1929PNAS...15..168H: $H_0 \approx$ 0.5 Gyr^{-1}

$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0} \approx 1.3$ Gyr

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 0.6 Gyr^{-1}
- Hubble (1929) ADS:1929PNAS...15..168H: $H_0 \approx$ 0.5 Gyr^{-1}

$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0} \approx 1.3$ Gyr $< t_{\mathrm{Earth}} \approx 4.5$ Gyr

FLRW matter-dominated epoch

- Friedmann Eqn: $$\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $$a = \left(\frac{t}{t_0}\right)^{2/3}$$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 0.6 Gyr^{-1}
- Hubble (1929) ADS:1929PNAS...15..168H: $H_0 \approx$ 0.5 Gyr^{-1}

$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0} \approx 1.3$ Gyr $< t_{\mathrm{Earth}} \approx 4.5$ Gyr

- 1980's: $H_0 \approx 50$ or 100 km/s/Mpc

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m}0}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m}0}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 0.6 Gyr^{-1}
- Hubble (1929) ADS:1929PNAS...15..168H: $H_0 \approx$ 0.5 Gyr^{-1}

$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0} \approx 1.3$ Gyr $< t_{\mathrm{Earth}} \approx 4.5$ Gyr

- 1980's: $H_0 \approx 50$ or 100 kpc/Gyr/Mpc

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 0.6 Gyr^{-1}
- Hubble (1929) ADS:1929PNAS...15..168H: $H_0 \approx$ 0.5 Gyr^{-1}

$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0} \approx 1.3$ Gyr $< t_{\mathrm{Earth}} \approx 4.5$ Gyr

- 1980's: $H_0 \approx 0.05$ or 0.1 Gyr^{-1}

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 0.6 Gyr^{-1}
- Hubble (1929) ADS:1929PNAS...15..168H: $H_0 \approx$ 0.5 Gyr^{-1}

$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0} \approx 1.3$ Gyr $< t_{\mathrm{Earth}} \approx 4.5$ Gyr

- 1980's: $H_0 \approx 0.05$ or 0.1 Gyr^{-1} $\Rightarrow t_0(\mathrm{EdS}) \approx 13.0$ or 6.5 Gyr, resp.

FLRW: ρ_{crit}

- Friedmann Eqn: $\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$

FLRW: ρ_{crit}

- Friedmann Eqn: $$H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$$

FLRW: ρ_{crit}

- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$
- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$
- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$
- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$ flat

FLRW: ρ_{crit}

- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$
- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\rho_{\mathrm{m0}} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$ flat
  - $\rho_{\mathrm{m0}} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0$

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$
- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$ flat
  - $\rho_{\rm m0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0$ spherical

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$
- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$ flat
  - $\rho_{\rm m0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0$ spherical
  - $\rho_{\rm m0} < \frac{3H_0^2}{8\pi G} \Leftrightarrow k < 0$

FLRW: ρ_{crit}

- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$
- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\rho_{\mathrm{m0}} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$ flat
  - $\rho_{\mathrm{m0}} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0$ spherical
  - $\rho_{\mathrm{m0}} < \frac{3H_0^2}{8\pi G} \Leftrightarrow k < 0$ hyperbolic

FLRW: ρ_{crit}

- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $$H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

FLRW: ρ_{crit}

- Friedmann Eqn: $H^2 = \frac{\rho H^2}{\rho_{\mathrm{crit}}} - \frac{c^2\, k}{a^2}$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

FLRW: ρ_{crit}

- Friedmann Eqn: $H^2 = \frac{\rho H^2}{\rho_{\text{crit}}} - \frac{c^2\,k}{a^2}$

Defn: $\boxed{\rho_{\text{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\text{m}} := \frac{\rho}{\rho_{\text{crit}}}}$ matter density parameter

FLRW: ρ_{crit}

- Friedmann Eqn: $H^2 = \Omega_{\mathrm{m}} H^2 - \frac{c^2\,k}{a^2}$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \Omega_{\rm m} H^2 - \frac{c^2\, k}{a^2}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \Omega_{\rm m} H^2 - \frac{c^2\, k}{a^2}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $$H^2 = \Omega_{\rm m} H^2 + \Omega_k H^2$$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\text{m}} + \Omega_k}$

Defn: $\boxed{\rho_{\text{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\text{m}} := \frac{\rho}{\rho_{\text{crit}}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\text{m}} + \Omega_k}$

Defn: $\boxed{\rho_{\text{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\text{m}} := \frac{\rho}{\rho_{\text{crit}}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\mathrm{m}} + \Omega_k}$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\mathrm{m}0} = 1 \Leftrightarrow k = 0$

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm m} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm m} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\text{m}} + \Omega_k}$

Defn: $\boxed{\rho_{\text{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\text{m}} := \frac{\rho}{\rho_{\text{crit}}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\text{m0}} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\text{m0}} > 1 \Leftrightarrow k > 0$ spherical

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm m} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\mathrm{m}} + \Omega_k}$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\mathrm{m}0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\mathrm{m}0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\mathrm{m}0} < 1 \Leftrightarrow k < 0$ hyperbolic

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm tot} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\rm tot} :=$

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\mathrm{tot}} + \Omega_k}$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\mathrm{m}0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\mathrm{m}0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\mathrm{m}0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\mathrm{tot}} := \Omega_{\mathrm{m}} +$

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\mathrm{tot}} + \Omega_k}$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\mathrm{m}0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\mathrm{m}0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\mathrm{m}0} < 1 \Leftrightarrow k < 0$ hyperbolic
- $\Omega_{\mathrm{tot}} := \Omega_{\mathrm{m}} + \Omega_{\mathrm{r}} +$

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\mathrm{tot}} + \Omega_k}$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\mathrm{m}0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\mathrm{m}0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\mathrm{m}0} < 1 \Leftrightarrow k < 0$ hyperbolic
- $\Omega_{\mathrm{tot}} := \Omega_{\mathrm{m}} + \Omega_{\mathrm{r}} + \Omega_\Lambda$

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\mathrm{tot}} + \Omega_k}$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\mathrm{m}0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\mathrm{m}0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\mathrm{m}0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\mathrm{tot}} := \Omega_{\mathrm{b}} + \Omega_{\mathrm{nbDM}} + \Omega_{\mathrm{r}} + \Omega_\Lambda$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_C

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k

- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$

- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; \Omega_{k0} = -\frac{c^2 k}{H_0^2}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; k = -\frac{\Omega_{k\,0} H_0^2}{c^2}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_C
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_\mathrm{C}^2 = 1 \Rightarrow k = 1/R_\mathrm{C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $R_\mathrm{C}^2 = -\frac{c^2}{\Omega_{k0} H_0^2}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; R_{\mathrm{C}}^2 = -\frac{c^2}{H_0^2}\frac{1}{\Omega_{k0}}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; R_{\mathrm{C}}^2 = -\frac{c^2}{H_0^2}\frac{1}{1-\Omega_{\mathrm{tot}\,0}}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot0}} - 1}}}$
- $\Omega_{\mathrm{tot0}} > 1$ *spherical*

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0}\frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_C
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_\mathrm{C}^2 = 1 \Rightarrow k = 1/R_\mathrm{C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_\mathrm{C} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\mathrm{tot}0} = 1$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat*

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: $R_{\rm C}$
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\rm C}^2 = 1 \Rightarrow k = 1/R_{\rm C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\rm C} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\rm tot0} - 1}}}$
- $\Omega_{\rm tot0} > 1$ *spherical* $R_{\rm C}$ real
- $\Omega_{\rm tot0} = 1$ *flat* $R_{\rm C}$ undefined

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined
- $\Omega_{\mathrm{tot}0} < 1$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_C
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_C^2 = 1 \Rightarrow k = 1/R_C^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_C = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_C real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_C undefined
- $\Omega_{\mathrm{tot}0} < 1$ *hyperbolic*

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined
- $\Omega_{\mathrm{tot}0} < 1$ *hyperbolic* R_{C} imaginary (or use $|R_{\mathrm{C}}|$)

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E
- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} + \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
  https://cosmo.torun.pl/Cosmo/FLRWEquationsGR

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E
- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} + \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
  https://cosmo.torun.pl/Cosmo/FLRWEquationsGR
- *hint:* mixed index form of $\mathbf{g}$ is easy

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

  Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Friedmann Eqn ($\Lambda \neq 0$):

$$\frac{c^2\, k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\, G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

acceleration Eqn ($\Lambda \neq 0$):

$$\frac{\ddot{a}}{a} = -\frac{4\,\pi\, G\; (\rho + 3\, p/c^2)}{3} + \frac{c^2\,\Lambda}{3}$$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Friedmann Eqn ($\Lambda \neq 0$):
$$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

acceleration Eqn ($\Lambda \neq 0$):
$$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G\,(\rho + 3\,p/c^2)}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Friedmann Eqn ($\Lambda \neq 0$):
$$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

acceleration Eqn ($\Lambda \neq 0$):
$$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G\,(\rho + 3\,p/c^2)}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G\;(\rho + 3\,p/c^2)}{3} + \Omega_\Lambda\,H^2$$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{H^2}{2}\frac{\rho}{\rho_{\mathrm{crit}}} + \Omega_\Lambda\,H^2$$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{H^2\,\Omega_{\mathrm{m}}}{2} + \Omega_\Lambda\,H^2$$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2} = -\frac{\Omega_{\rm m}}{2} + \Omega_\Lambda$$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\, k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\, G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2} = -\frac{\Omega_\mathrm{m}}{2} + \Omega_\Lambda$$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

acceleration Eqn ($\Lambda \neq 0$): $$q = \frac{\Omega_{\mathrm{m}}}{2} - \Omega_\Lambda$$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ ADS:1917SPAW.......142E

  Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

- $\boxed{q = \frac{\Omega_\mathrm{m}}{2} - \Omega_\Lambda}$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

  Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

- $\boxed{q = \frac{\Omega_\mathrm{m}}{2} - \Omega_\Lambda}$ acceleration equation
- if $\Lambda = 0$ and $\Omega_\mathrm{m} > 0$ then $\frac{\ddot{a}}{a} < 0$, i.e. $q > 0$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

- $\boxed{q = \frac{\Omega_\mathrm{m}}{2} - \Omega_\Lambda}$ acceleration equation
- if $\Lambda = 0$ and $\Omega_\mathrm{m} > 0$ then $\frac{\ddot{a}}{a} < 0$, i.e. $q > 0$ *deceleration*

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

  Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\, k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\, G\, \rho}{3} + \frac{c^2\, \Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\, \Lambda}{3\, H^2}$

Defn: $q := -\frac{\ddot{a} a}{\dot{a}^2}$ "deceleration parameter"

- $\boxed{q = \frac{\Omega_{\mathrm{m}}}{2} - \Omega_\Lambda}$ acceleration equation
- if $\Omega_\Lambda > \Omega_{\mathrm{m}}/2$ then $\frac{\ddot{a}}{a} > 0$, i.e. $q < 0$ *acceleration*

distances in FLRW cosmology

- azimuthal equidistant r:

distances in FLRW cosmology

- azimuthal equidistant r: proper distance at t_0 $\equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$

distances in FLRW cosmology

- azimuthal equidistant r: proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$
- Friedmann Eq: $1 = \Omega_\mathrm{m} + \Omega_k + \Omega_\Lambda$

distances in FLRW cosmology

- azimuthal equidistant r: proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$
- Friedmann Eq: $1 = \Omega_\mathrm{m} + \Omega_k + \Omega_\Lambda$
- $\Omega_\mathrm{m} = \frac{\rho}{\rho_\mathrm{crit}}$

distances in FLRW cosmology

- azimuthal equidistant r: proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$
- Friedmann Eq: $1 = \Omega_\mathrm{m} + \Omega_k + \Omega_\Lambda$
- $\Omega_\mathrm{m} = \frac{\rho}{\rho_\mathrm{crit}} = \frac{\rho_0\, a^{-3}}{\rho_{\mathrm{crit}\,0}\,(H^2/H_0^2)}$

distances in FLRW cosmology

- azimuthal equidistant r: proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$
- Friedmann Eq: $1 = \Omega_\mathrm{m} + \Omega_k + \Omega_\Lambda$
- $\Omega_\mathrm{m} = \frac{\rho}{\rho_\mathrm{crit}} = \frac{\rho_0\, a^{-3}}{\rho_{\mathrm{crit}\,0}\,(H^2/H_0^2)} = \Omega_{\mathrm{m}0}\, H_0^2 a^{-1} \dot{a}^{-2}$

distances in FLRW cosmology

- azimuthal equidistant r: proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$
- Friedmann Eq: $1 = \Omega_\mathrm{m} + \Omega_k + \Omega_\Lambda$
- $\Omega_\mathrm{m} = \frac{\rho}{\rho_\mathrm{crit}} = \frac{\rho_0\, a^{-3}}{\rho_{\mathrm{crit}\,0}\,(H^2/H_0^2)} = \Omega_{\mathrm{m}0}\, H_0^2 a^{-1} \dot{a}^{-2}$
- $\Omega_k = \Omega_{k0}\, \dot{a}^{-2} H_0^2$

distances in FLRW cosmology

- azimuthal equidistant r: proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$
- Friedmann Eq: $1 = \Omega_\mathrm{m} + \Omega_k + \Omega_\Lambda$
- $\Omega_\mathrm{m} = \frac{\rho}{\rho_\mathrm{crit}} = \frac{\rho_0\, a^{-3}}{\rho_{\mathrm{crit}\,0}\,(H^2/H_0^2)} = \Omega_{\mathrm{m}0}\, H_0^2 a^{-1} \dot{a}^{-2}$
- $\Omega_k = \Omega_{k0}\, \dot{a}^{-2} H_0^2$
- $\Omega_\Lambda = \Omega_{\Lambda 0} a^2 \dot{a}^{-2} H_0^2$

distances in FLRW cosmology

- azimuthal equidistant r: proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$
- Friedmann Eq: $1 = \Omega_\mathrm{m} + \Omega_k + \Omega_\Lambda$
- $\Omega_\mathrm{m} = \frac{\rho}{\rho_\mathrm{crit}} = \frac{\rho_0\, a^{-3}}{\rho_{\mathrm{crit}\,0}\,(H^2/H_0^2)} = \Omega_{\mathrm{m}0}\, H_0^2 a^{-1} \dot{a}^{-2}$
- $\Omega_k = \Omega_{k0}\, \dot{a}^{-2} H_0^2$
- $\Omega_\Lambda = \Omega_{\Lambda 0} a^2 \dot{a}^{-2} H_0^2$
- $1 = \Omega_{\mathrm{m}0} H_0^2 a^{-1} \dot{a}^{-2} + \Omega_{k0} \dot{a}^{-2} H_0^2 + \Omega_{\Lambda 0} a^2 \dot{a}^{-2} H_0^2$

distances in FLRW cosmology

- azimuthal equidistant r: proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$
- Friedmann Eq: $1 = \Omega_\mathrm{m} + \Omega_k + \Omega_\Lambda$
- $\Omega_\mathrm{m} = \frac{\rho}{\rho_\mathrm{crit}} = \frac{\rho_0\, a^{-3}}{\rho_{\mathrm{crit}\,0}\,(H^2/H_0^2)} = \Omega_{\mathrm{m}0}\, H_0^2 a^{-1} \dot{a}^{-2}$
- $\Omega_k = \Omega_{k0}\, \dot{a}^{-2} H_0^2$
- $\Omega_\Lambda = \Omega_{\Lambda 0} a^2 \dot{a}^{-2} H_0^2$
- $1 = \Omega_{\mathrm{m}0} H_0^2 a^{-1} \dot{a}^{-2} + \Omega_{k0} \dot{a}^{-2} H_0^2 + \Omega_{\Lambda 0} a^2 \dot{a}^{-2} H_0^2$

$\Rightarrow$ $\dot{a}^2 = H_0^2 \left(\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2\right)$

distances in FLRW cosmology

- azimuthal equidistant r: proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$
- Friedmann Eq: $1 = \Omega_\mathrm{m} + \Omega_k + \Omega_\Lambda$
- $\Omega_\mathrm{m} = \frac{\rho}{\rho_\mathrm{crit}} = \frac{\rho_0\, a^{-3}}{\rho_{\mathrm{crit}\,0}\,(H^2/H_0^2)} = \Omega_{\mathrm{m}0}\, H_0^2 a^{-1} \dot{a}^{-2}$
- $\Omega_k = \Omega_{k0}\, \dot{a}^{-2} H_0^2$
- $\Omega_\Lambda = \Omega_{\Lambda 0} a^2 \dot{a}^{-2} H_0^2$
- $1 = \Omega_{\mathrm{m}0} H_0^2 a^{-1} \dot{a}^{-2} + \Omega_{k0} \dot{a}^{-2} H_0^2 + \Omega_{\Lambda 0} a^2 \dot{a}^{-2} H_0^2$

$\Rightarrow$ $\mathrm{d}a = H_0 \sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}\,\mathrm{d}t$ if $\dot{a}, a, H_0 > 0$

distances in FLRW cosmology

- azimuthal equidistant r: proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$
- Friedmann Eq: $1 = \Omega_\mathrm{m} + \Omega_k + \Omega_\Lambda$
- $\Omega_\mathrm{m} = \frac{\rho}{\rho_\mathrm{crit}} = \frac{\rho_0\, a^{-3}}{\rho_{\mathrm{crit}\,0}\,(H^2/H_0^2)} = \Omega_{\mathrm{m}0}\, H_0^2 a^{-1} \dot{a}^{-2}$
- $\Omega_k = \Omega_{k0}\, \dot{a}^{-2} H_0^2$
- $\Omega_\Lambda = \Omega_{\Lambda 0} a^2 \dot{a}^{-2} H_0^2$
- $1 = \Omega_{\mathrm{m}0} H_0^2 a^{-1} \dot{a}^{-2} + \Omega_{k0} \dot{a}^{-2} H_0^2 + \Omega_{\Lambda 0} a^2 \dot{a}^{-2} H_0^2$

$\Rightarrow$ $\mathrm{d}a = H_0 \sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}\, \mathrm{d}t$ if $\dot{a}, a, H_0 > 0$

$\Rightarrow$ $\mathrm{d}t = \frac{\mathrm{d}a}{H_0 \sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$ if $\dot{a}, a, H_0 > 0$

distances in FLRW cosmology

- azimuthal equidistant r: proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}$
- Friedmann Eq: $1 = \Omega_\mathrm{m} + \Omega_k + \Omega_\Lambda$
- $\Omega_\mathrm{m} = \frac{\rho}{\rho_\mathrm{crit}} = \frac{\rho_0\, a^{-3}}{\rho_{\mathrm{crit}\,0}\,(H^2/H_0^2)} = \Omega_{\mathrm{m}0}\, H_0^2 a^{-1} \dot{a}^{-2}$
- $\Omega_k = \Omega_{k0}\, \dot{a}^{-2} H_0^2$
- $\Omega_\Lambda = \Omega_{\Lambda 0} a^2 \dot{a}^{-2} H_0^2$
- $1 = \Omega_{\mathrm{m}0} H_0^2 a^{-1} \dot{a}^{-2} + \Omega_{k0} \dot{a}^{-2} H_0^2 + \Omega_{\Lambda 0} a^2 \dot{a}^{-2} H_0^2$

$\Rightarrow$ $\mathrm{d}a = H_0 \sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}\, \mathrm{d}t$ if $\dot{a}, a, H_0 > 0$

$\Rightarrow$ $$\boxed{r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}}$$

Low z limit Hubble law

- use r = comoving radial distance

Low z limit Hubble law

- use $r =$ comoving radial distance
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$

Low z limit Hubble law

- use $\underline{r = \text{comoving radial distance}}$
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z}$

Low z limit Hubble law

- use r = comoving radial distance
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z} \Rightarrow \mathrm{d}a = -(1+z)^{-2}\mathrm{d}z$

Low z limit Hubble law

- use $\underline{r = \text{comoving radial distance}}$
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z} \Rightarrow \mathrm{d}a = -(1+z)^{-2}\mathrm{d}z = -a^2\mathrm{d}z$

Low z limit Hubble law

- use $\underline{r = \text{comoving radial distance}}$
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z} \Rightarrow \mathrm{d}a = -(1+z)^{-2}\mathrm{d}z = -a^2\mathrm{d}z$

$$r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0}a^2}}$$

Low z limit Hubble law

- use r = comoving radial distance
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z} \Rightarrow \mathrm{d}a = -(1+z)^{-2}\mathrm{d}z = -a^2\mathrm{d}z$

$$\begin{aligned} r &= \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\sqrt{\Omega_{\mathrm{m}0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0}a^2}} \\ &= -\int_{z}^{0} \frac{c}{H_0} \frac{a^2\,\mathrm{d}z}{a\sqrt{\Omega_{\mathrm{m}0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0}a^2}} \end{aligned}$$

Low z limit Hubble law

- use r = comoving radial distance
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z} \Rightarrow \mathrm{d}a = -(1+z)^{-2}\mathrm{d}z = -a^2\mathrm{d}z$

$$\begin{aligned} r &= \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\sqrt{\Omega_{\mathrm{m}0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0}a^2}} \\ &= -\int_z^0 \frac{c}{H_0} \frac{a\,\mathrm{d}z}{\sqrt{\Omega_{\mathrm{m}0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0}a^2}} \end{aligned}$$

Low z limit Hubble law

- use $\underline{r = \text{comoving radial distance}}$
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z} \Rightarrow \mathrm{d}a = -(1+z)^{-2}\mathrm{d}z = -a^2\mathrm{d}z$

$$r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0}a^2}}$$

$$\approx -\int_{z}^{0} \frac{c}{H_0} \frac{1\,\mathrm{d}z}{\sqrt{\Omega_{\mathrm{m}0} + \Omega_{k0} + \Omega_{\Lambda 0}}}$$

Low z limit Hubble law

- use $\underline{r = \text{comoving radial distance}}$
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z} \Rightarrow \mathrm{d}a = -(1+z)^{-2}\mathrm{d}z = -a^2\mathrm{d}z$

$$\begin{aligned} r &= \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\sqrt{\Omega_{\mathrm{m}0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0}a^2}} \\ &\approx -\int_z^0 \frac{c}{H_0}\frac{1\,\mathrm{d}z}{\sqrt{1}} \end{aligned}$$

Low z limit Hubble law

- use $r =$ comoving radial distance
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z} \Rightarrow \mathrm{d}a = -(1+z)^{-2}\mathrm{d}z = -a^2\mathrm{d}z$

$$
\begin{aligned}
r &= \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0}a^2}} \\
&\approx -\int_z^0 \frac{c}{H_0} \frac{1\,\mathrm{d}z}{\sqrt{1}} \\
&= -\int_z^0 \frac{c}{H_0}\,\mathrm{d}z
\end{aligned}
$$

Low z limit Hubble law

- use $\underline{r = \text{comoving radial distance}}$
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z} \Rightarrow \mathrm{d}a = -(1+z)^{-2}\mathrm{d}z = -a^2\mathrm{d}z$

$$\begin{aligned} r &= \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0}a^2}} \\ &\approx -\int_z^0 \frac{c}{H_0}\frac{1\,\mathrm{d}z}{\sqrt{1}} \\ &= -\int_z^0 \frac{c}{H_0}\,\mathrm{d}z \end{aligned}$$

$\Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}z} \approx \frac{c}{H_0}$

Low z limit Hubble law

- use r = <u>comoving radial distance</u>
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z} \Rightarrow \mathrm{d}a = -(1+z)^{-2}\mathrm{d}z = -a^2\mathrm{d}z$

$$\begin{aligned} r &= \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda0}a^2}} \\ &\approx -\int_z^0 \frac{c}{H_0}\frac{1\,\mathrm{d}z}{\sqrt{1}} \\ &= -\int_z^0 \frac{c}{H_0}\,\mathrm{d}z \end{aligned}$$

$\Rightarrow$ $\frac{\mathrm{d}r}{\mathrm{d}z} \approx \frac{c}{H_0}$

$\Rightarrow$ when $0 < z \ll 1$, $\beta := v/c$, $r \approx \dfrac{z\,c}{H_0}$

Low z limit Hubble law

- use $r =$ comoving radial distance
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z} \Rightarrow \mathrm{d}a = -(1+z)^{-2}\mathrm{d}z = -a^2\mathrm{d}z$

$$\begin{aligned} r &= \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\sqrt{\Omega_{\mathrm{m}0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0}a^2}} \\ &\approx -\int_z^0 \frac{c}{H_0}\frac{1\,\mathrm{d}z}{\sqrt{1}} \\ &= -\int_z^0 \frac{c}{H_0}\,\mathrm{d}z \end{aligned}$$

$\Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}z} \approx \frac{c}{H_0}$

$\Rightarrow$ when $0 < z \ll 1$, $\beta := v/c$, $r \approx \dfrac{z\,c}{H_0} \approx \dfrac{\beta\,c}{H_0}$

Low z limit Hubble law

- use r = <u>comoving radial distance</u>
- consider $0 < z \ll 1$ limit $\Leftrightarrow 1 > a \gg 0.5$
- $a = \frac{1}{1+z} \Rightarrow \mathrm{d}a = -(1+z)^{-2}\mathrm{d}z = -a^2\mathrm{d}z$

$$\begin{aligned} r &= \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\sqrt{\Omega_{\mathrm{m}0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0}a^2}} \\ &\approx -\int_z^0 \frac{c}{H_0}\frac{1\,\mathrm{d}z}{\sqrt{1}} \\ &= -\int_z^0 \frac{c}{H_0}\,\mathrm{d}z \end{aligned}$$

$\Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}z} \approx \frac{c}{H_0}$

$\Rightarrow$ when $0 < z \ll 1$, $\beta := v/c$, $\boxed{r \approx \frac{z\,c}{H_0} \approx \frac{\beta\,c}{H_0} = \frac{v}{H_0}}$

EdS: radial comoving distance

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$

EdS: radial comoving distance

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$

  EdS: $\Omega_{\mathrm{m}0} = 1, \Omega_{k0} = 0, \Omega_{\Lambda 0} = 0$

EdS: radial comoving distance

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$

  EdS: $\Omega_{\mathrm{m}0} = 1, \Omega_{k0} = 0, \Omega_{\Lambda 0} = 0$

$$\Rightarrow \quad r_{\mathrm{EdS}} = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{a^{-1}}}$$

EdS: radial comoving distance

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$

  EdS: $\Omega_{\mathrm{m}0} = 1, \Omega_{k0} = 0, \Omega_{\Lambda 0} = 0$

$\Rightarrow$
$$r_{\mathrm{EdS}} = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{a^{-1}}} = \frac{c}{H_0} \int_{1/(1+z)}^{1} a^{-1/2} \mathrm{d}a$$

EdS: radial comoving distance

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$

  EdS: $\Omega_{\mathrm{m}0} = 1, \Omega_{k0} = 0, \Omega_{\Lambda 0} = 0$

$$\Rightarrow \quad r_{\mathrm{EdS}} = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{a^{-1}}}$$

$$= \frac{c}{H_0} \int_{1/(1+z)}^{1} a^{-1/2} \mathrm{d}a$$

$$= \frac{c}{H_0} \left[2a^{1/2} \right]_{1/(1+z)}^{1}$$

EdS: radial comoving distance

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0}a^{-1}+\Omega_{k0}+\Omega_{\Lambda 0}a^2}}$

EdS: $\Omega_{\mathrm{m}0} = 1, \Omega_{k0} = 0, \Omega_{\Lambda 0} = 0$

$$\Rightarrow \quad r_{\mathrm{EdS}} = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{a^{-1}}} = \frac{c}{H_0} \int_{1/(1+z)}^{1} a^{-1/2}\mathrm{d}a = \frac{c}{H_0} \left[2a^{1/2}\right]_{1/(1+z)}^{1} = \boxed{\frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)}$$

EdS: radial comoving distance

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$

  EdS: $\Omega_{\mathrm{m}0} = 1, \Omega_{k0} = 0, \Omega_{\Lambda 0} = 0$

$$
\Rightarrow \quad
\begin{aligned}
r_{\mathrm{EdS}} &= \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{a^{-1}}} \\
&= \frac{c}{H_0} \int_{1/(1+z)}^{1} a^{-1/2} \mathrm{d}a \\
&= \frac{c}{H_0} \left[2a^{1/2} \right]_{1/(1+z)}^{1} \\
&= \boxed{\frac{2c}{H_0} \left(1 - \frac{1}{\sqrt{1+z}} \right)} \\
&\approx 6h^{-1}\mathrm{Gpc} \left(1 - \frac{1}{\sqrt{1+z}} \right)
\end{aligned}
$$

distances in FLRW cosmology

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$

distances in FLRW cosmology

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$
- GPL numerical package: `cosmdist`
  https://codeberg.org/boud/cosmdist

distances in FLRW cosmology

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$
- GPL numerical package: `cosmdist`
  https://codeberg.org/boud/cosmdist
- standard install to `/usr/local`:
  `./configure && make && make install`

distances in FLRW cosmology

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$
- GPL numerical package: `cosmdist`
  https://codeberg.org/boud/cosmdist
- standard install to `/usr/local`:
  `./configure && make && make install`
- command line pipe:
  `cat myredshifts.lis | cosmdist`

distances in FLRW cosmology

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\,\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$
- GPL numerical package: `cosmdist`
  https://codeberg.org/boud/cosmdist
- standard install to `/usr/local`:
  `./configure && make && make install`
- command line pipe:
  `cat myredshifts.lis | cosmdist`
- command line options: `cosmdist --help`

distances in FLRW cosmology

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$
- GPL numerical package: `cosmdist`
  https://codeberg.org/boud/cosmdist
- standard install to `/usr/local`:
  `./configure && make && make install`
- command line pipe:
  `cat myredshifts.lis | cosmdist`
- command line options: `cosmdist --help`
- static fortran or C library: link to `libcosmdist.a`

distances in FLRW cosmology

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$
- GPL numerical package: `cosmdist`
  https://codeberg.org/boud/cosmdist
- standard install to `/usr/local`:
  `./configure && make && make install`
- command line pipe:
  `cat myredshifts.lis | cosmdist`
- command line options: `cosmdist --help`
- static fortran or C library: link to `libcosmdist.a`
- high-level frontends (e.g. python) should be easy to write

tangential arc-lengths: $r_\perp$ vs d_A

- $r_\perp =:$ "proper motion" distance (Weinberg 1972)

tangential arc-lengths: $r_\perp$ vs d_{A}

- $r_\perp =:$ "proper motion" distance (Weinberg 1972)
- $\equiv$ **comoving** tangential arc-length for 1 radian

tangential arc-lengths: $r_\perp$ vs d_A

- $r_\perp$ =: "proper motion" distance (Weinberg 1972)
- $\equiv$ **comoving** tangential arc-length for 1 radian
- azimuthal equidistant coords: $r_\perp = \begin{cases} R_\mathrm{C} \sinh \frac{r}{R_\mathrm{C}} & k < 0 \\ r & k = 0 \\ R_\mathrm{C} \sin \frac{r}{R_\mathrm{C}} & k > 0 \end{cases}$

tangential arc-lengths: $r_\perp$ vs d_A

- $r_\perp =:$ "proper motion" distance (Weinberg 1972)
- $\equiv$ **comoving** tangential arc-length for 1 radian
- azimuthal equidistant coords: $r_\perp = \begin{cases} R_\mathrm{C} \sinh \frac{r}{R_\mathrm{C}} & k < 0 \\ r & k = 0 \\ R_\mathrm{C} \sin \frac{r}{R_\mathrm{C}} & k > 0 \end{cases}$ where

  $r =$ comoving radial distance
- angular diameter distance d_A

tangential arc-lengths: $r_\perp$ vs d_A

- $r_\perp =:$ "proper motion" distance (Weinberg 1972)
- $\equiv$ **comoving** tangential arc-length for 1 radian
- azimuthal equidistant coords: $r_\perp = \begin{cases} R_\mathrm{C} \sinh \frac{r}{R_\mathrm{C}} & k < 0 \\ r & k = 0 \\ R_\mathrm{C} \sin \frac{r}{R_\mathrm{C}} & k > 0 \end{cases}$ where

  $r =$ comoving radial distance
- angular diameter distance d_A
- $\equiv$ **physical**/

tangential arc-lengths: $r_\perp$ vs d_{A}

- $r_\perp =:$ "proper motion" distance (Weinberg 1972)
- $\equiv$ **comoving** tangential arc-length for 1 radian
- azimuthal equidistant coords: $r_\perp = \begin{cases} R_{\mathrm{C}} \sinh \frac{r}{R_{\mathrm{C}}} & k < 0 \\ r & k = 0 \\ R_{\mathrm{C}} \sin \frac{r}{R_{\mathrm{C}}} & k > 0 \end{cases}$ where

  $r =$ comoving radial distance
- angular diameter distance d_{A}
- $\equiv$ **physical/ metric /**

tangential arc-lengths: $r_\perp$ vs d_A

- $r_\perp =:$ "proper motion" distance (Weinberg 1972)
- $\equiv$ **comoving** tangential arc-length for 1 radian
- azimuthal equidistant coords: $r_\perp = \begin{cases} R_\mathrm{C} \sinh \frac{r}{R_\mathrm{C}} & k < 0 \\ r & k = 0 \\ R_\mathrm{C} \sin \frac{r}{R_\mathrm{C}} & k > 0 \end{cases}$ where

  $r =$ comoving radial distance
- angular diameter distance d_A
- $\equiv$ **physical/ metric / local**

tangential arc-lengths: $r_\perp$ vs d_A

- $r_\perp =:$ "proper motion" distance (Weinberg 1972)
- $\equiv$ **comoving** tangential arc-length for 1 radian
- azimuthal equidistant coords: $r_\perp = \begin{cases} R_\mathrm{C} \sinh \frac{r}{R_\mathrm{C}} & k<0 \\ r & k=0 \\ R_\mathrm{C} \sin \frac{r}{R_\mathrm{C}} & k>0 \end{cases}$ where

  $r =$ comoving radial distance
- angular diameter distance d_A
- $\equiv$ **physical/ metric / local** arc-length for 1 radian

tangential arc-lengths: $r_\perp$ vs d_A

- $r_\perp =:$ "proper motion" distance (Weinberg 1972)
- $\equiv$ **comoving** tangential arc-length for 1 radian
- azimuthal equidistant coords: $r_\perp = \begin{cases} R_\mathrm{C} \sinh \frac{r}{R_\mathrm{C}} & k < 0 \\ r & k = 0 \\ R_\mathrm{C} \sin \frac{r}{R_\mathrm{C}} & k > 0 \end{cases}$ where

  $r =$ comoving radial distance
- angular diameter distance d_A
- $\equiv$ **physical/ metric / local** arc-length for 1 radian
- $d_\mathrm{A} = r_\perp \, a$ (scaled by the scale factor)

tangential arc-lengths: $r_\perp$ vs d_A

- $r_\perp =:$ "proper motion" distance (Weinberg 1972)
- $\equiv$ **comoving** tangential arc-length for 1 radian
- azimuthal equidistant coords: $r_\perp = \begin{cases} R_\mathrm{C} \sinh \frac{r}{R_\mathrm{C}} & k < 0 \\ r & k = 0 \\ R_\mathrm{C} \sin \frac{r}{R_\mathrm{C}} & k > 0 \end{cases}$ where

  $r =$ comoving radial distance
- angular diameter distance d_A
- $\equiv$ **physical/ metric / local** arc-length for 1 radian
- $$\boxed{d_\mathrm{A} = \frac{r_\perp}{1+z}}$$

luminosity distance

want to define d_L such that flux is $F = \dfrac{L}{4\pi d_L^2}$

luminosity distance

want to define d_L such that flux is $F = \dfrac{L}{4\pi d_L^2}$

- surface area of 2-sphere $= 4\pi r_\perp^2$ (comoving)

luminosity distance

want to define d_L such that flux is $F = \frac{L}{4\pi d_L^2}$

- $E = h\nu$ of photon:

luminosity distance

want to define d_L such that flux is $F = \dfrac{L}{4\pi d_L^2}$

- $E = h\nu$ of photon: emitter frame to observer frame $F \propto 1/\lambda \propto a_{\rm em}$

luminosity distance

want to define d_L such that flux is $F = \frac{L}{4\pi d_L^2}$

- $E = h\nu$ of photon: emitter frame to observer frame $F \propto 1/\lambda \propto a_{\mathrm{em}}$
- erg **per s**: time dilation:

luminosity distance

want to define d_L such that flux is $F = \dfrac{L}{4\pi d_L^2}$

- $E = h\nu$ of photon: emitter frame to observer frame $F \propto 1/\lambda \propto a_{\mathrm{em}}$
- erg **per s**: time dilation: emitter frame to observer frame $F \propto a_{\mathrm{em}}$

luminosity distance

want to define d_L such that flux is $F = \dfrac{L}{4\pi d_L^2}$

- $E = h\nu$ of photon: emitter frame to observer frame $F \propto 1/\lambda \propto a_{\mathrm{em}}$
- erg **per s**: time dilation: emitter frame to observer frame $F \propto a_{\mathrm{em}}$
- $\Rightarrow F = \frac{L}{4\pi r_\perp^2} a^2$

luminosity distance

want to define d_L such that flux is $F = \dfrac{L}{4\pi d_L^2}$

- $E = h\nu$ of photon: emitter frame to observer frame $F \propto 1/\lambda \propto a_{\rm em}$
- erg **per s**: time dilation: emitter frame to observer frame $F \propto a_{\rm em}$
- $\Rightarrow F = \frac{L}{4\pi r_\perp^2} a^2 = \frac{L}{4\pi r_\perp^2 (1+z)^2}$

luminosity distance

want to define d_L such that flux is $F = \frac{L}{4\pi d_L^2}$

- $E = h\nu$ of photon: emitter frame to observer frame $F \propto 1/\lambda \propto a_{\rm em}$
- erg **per** s: time dilation: emitter frame to observer frame $F \propto a_{\rm em}$
- $\Rightarrow F = \frac{L}{4\pi r_\perp^2} a^2 = \frac{L}{4\pi r_\perp^2 (1+z)^2}$
- $\Rightarrow$ luminosity distance $\boxed{d_L := r_\perp (1+z) = d_{\rm A} (1+z)^2}$

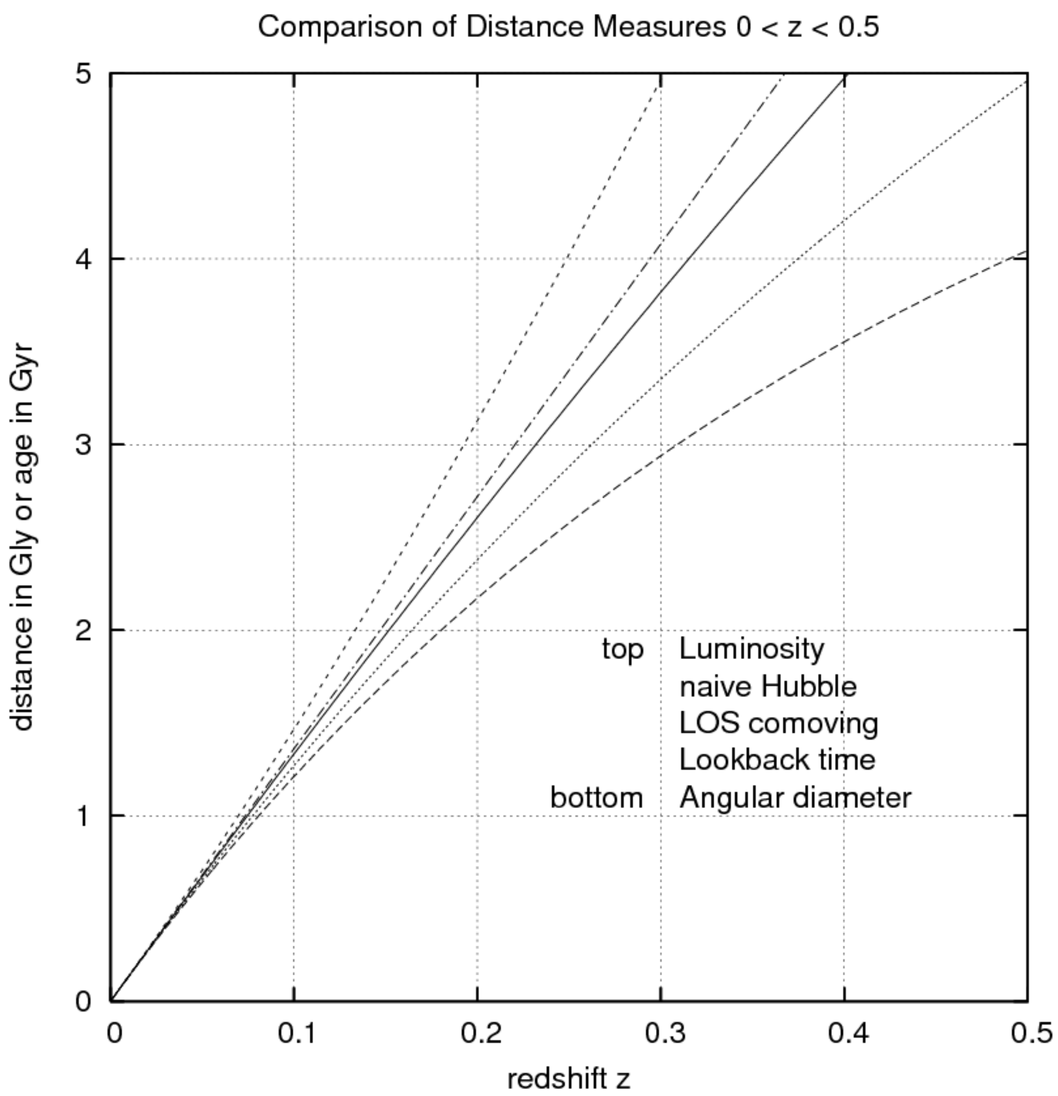
luminosity distance

want to define d_L such that flux is $F = \dfrac{L}{4\pi d_L^2}$

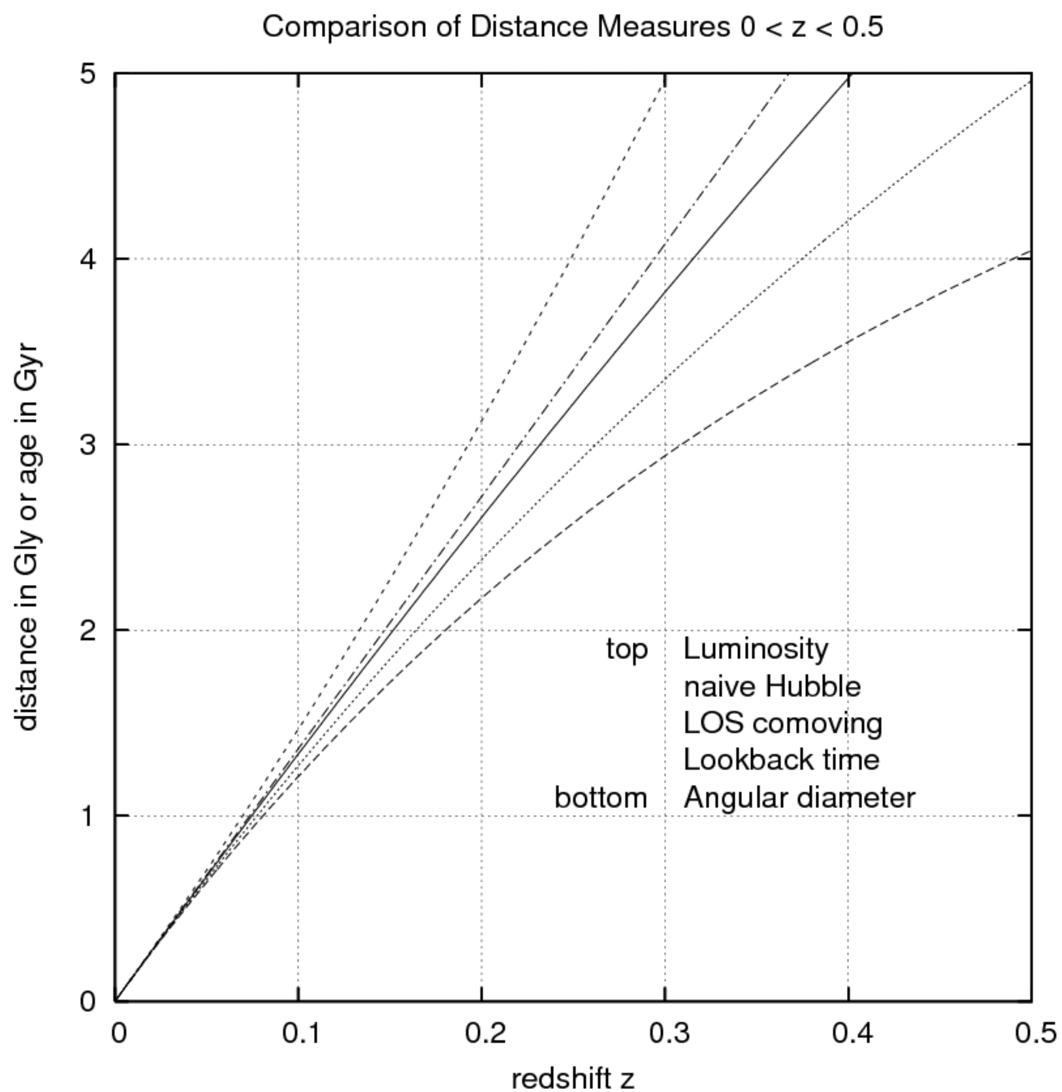
- $E = h\nu$ of photon: emitter frame to observer frame $F \propto 1/\lambda \propto a_{\rm em}$
- erg **per** s: time dilation: emitter frame to observer frame $F \propto a_{\rm em}$
- $\Rightarrow F = \frac{L}{4\pi r_\perp^2} a^2 = \frac{L}{4\pi r_\perp^2 (1+z)^2}$
- $\Rightarrow$ luminosity distance $\boxed{d_L := r_\perp (1+z) = d_{\rm A} (1+z)^2}$

- w:Distance measures (cosmology)

FLRW distances e.g. ΛCDM

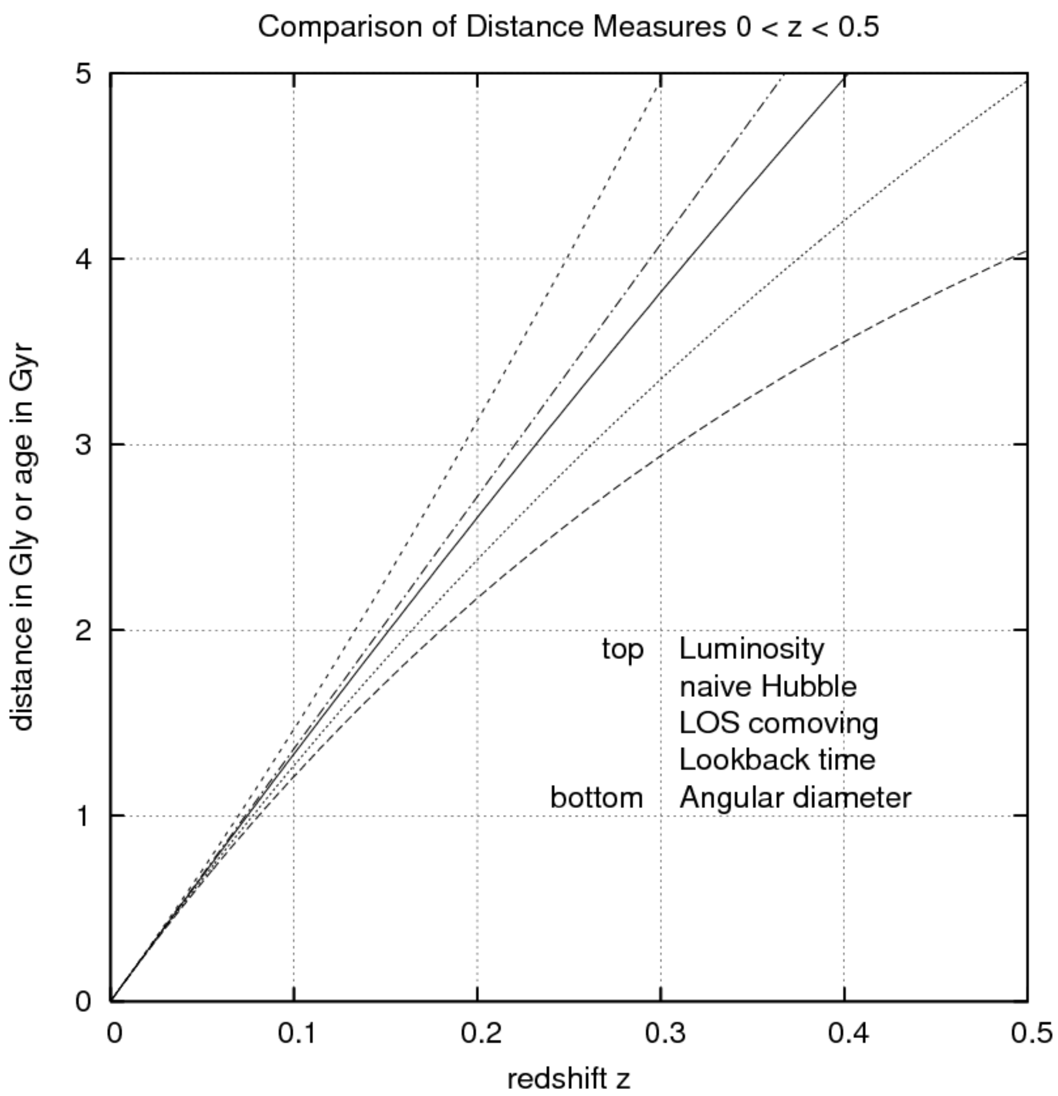


FLRW distances e.g. ΛCDM



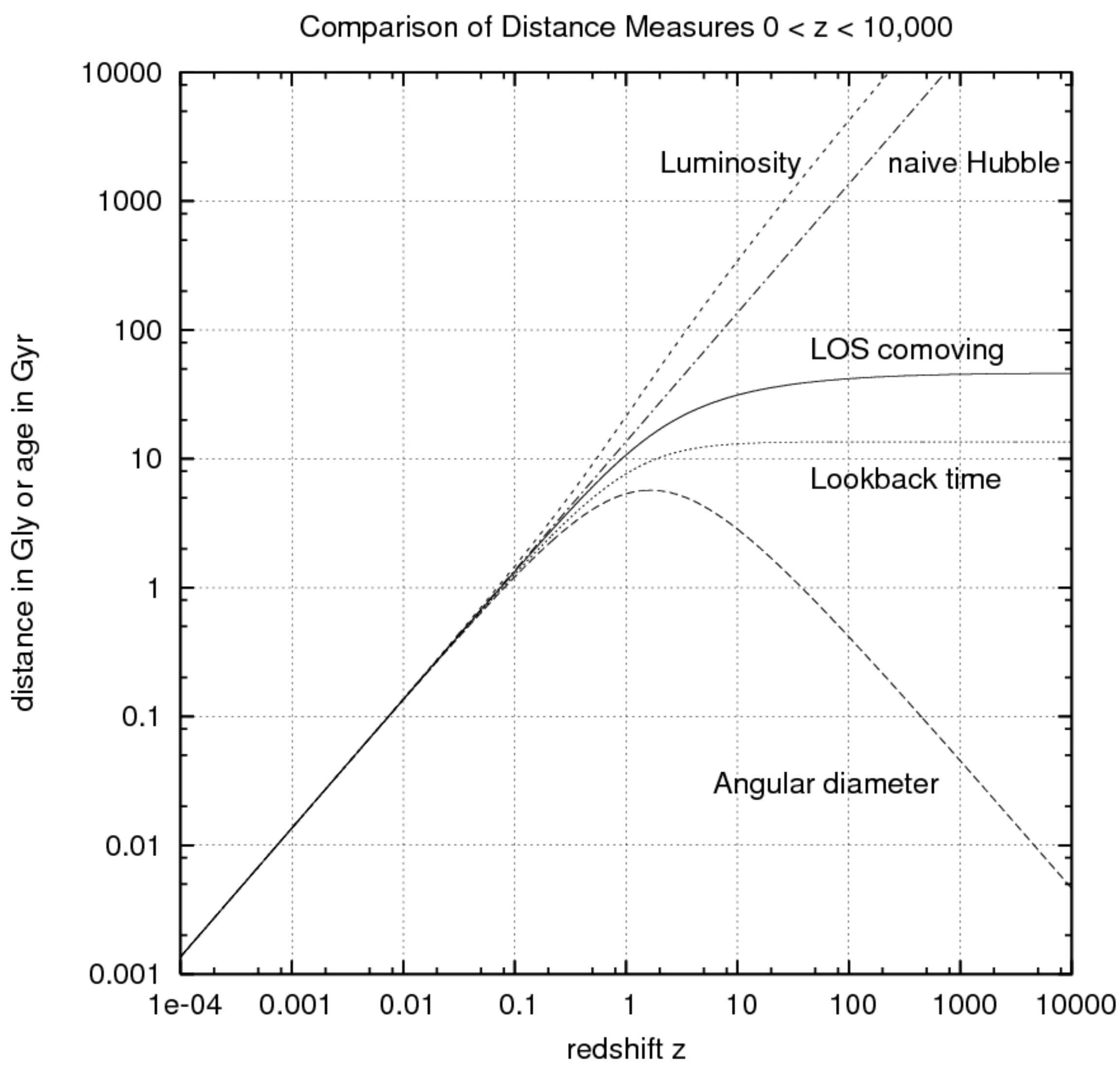
Defn: $\boxed{h := H_0/100 \text{ km/s/Mpc}}$ (without a "0" subscript on h)

FLRW distances e.g. ΛCDM



$$\Omega_{\rm m0} = 0.3,\ \Omega_{\rm r0} = 10^{-4},\ \Omega_{\Lambda 0} = 1.0 - (\Omega_{\rm m0} + \Omega_{\rm r0}),\ h = 0.7,\ \Omega_{k0} = 0$$

FLRW distances e.g. ΛCDM



$$\Omega_{\mathrm{m}0} = 0.3,\ \Omega_{\mathrm{r}0} = 10^{-4},\ \Omega_{\Lambda 0} = 1.0 - (\Omega_{\mathrm{m}0} + \Omega_{\mathrm{r}0}),\ h = 0.7,\ \Omega_{k0} = 0$$

“Velocity” $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?

"Velocity" $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of "velocity":
  1. galaxy as fundamental observer,

“Velocity” $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of “velocity”:
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords,

"Velocity" $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of "velocity":
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. Hubble law assumes $\|$ transport of 4-velocity $\vec{u}_{\mathrm{gal}}$; then compare $\vec{u}_{\mathrm{obs}}, \vec{u}_{\mathrm{gal}}$ *locally* in Minkowski spacetime

“Velocity” $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of “velocity”:
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. Hubble law assumes $\|$ transport of 4-velocity $\vec{u}_{\mathrm{gal}}$; then compare $\vec{u}_{\mathrm{obs}}, \vec{u}_{\mathrm{gal}}$ *locally* in Minkowski spacetime i.e. $1 + z = \sqrt{\frac{1+\beta}{1-\beta}}$ is *assumed* in derivation

“Velocity” $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of “velocity”:
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. Hubble law assumes $\|$ transport of 4-velocity $\vec{u}_{\mathrm{gal}}$; then compare $\vec{u}_{\mathrm{obs}}, \vec{u}_{\mathrm{gal}}$ *locally* in Minkowski spacetime i.e. $1 + z = \sqrt{\frac{1+\beta}{1-\beta}}$ is *assumed* in derivation $v_{\mathrm{gal}} < c$ by assumption ($\|$ transport + Mink.)

"Velocity" $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of "velocity":
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption ($\|$ transport + Mink.) or
  3. define differential "global" comoving velocity $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t}$

“Velocity” $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of “velocity”:
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption ($\|$ transport + Mink.) or
  3. define differential “global” comoving velocity $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t}$

"Velocity" $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of "velocity":
  1. galaxy as fundamental observer, local velocity $v_{\rm gal} = 0$ by defn of comoving coords, or
  2. $v_{\rm gal} < c$ by assumption ($\|$ transport + Mink.) or
  3. define differential "global" comoving velocity $v_{\rm gal}^{\rm glob} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a}$

"Velocity" $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of "velocity":
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption ($\|$ transport + Mink.) or
  3. define differential "global" comoving velocity $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = r\,\frac{\dot{a}}{a}\,a$

"Velocity" $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of "velocity":
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption ($\|$ transport + Mink.) or
  3. define differential "global" comoving velocity $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = r H_0$

"Velocity" $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of "velocity":
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption (∥ transport + Mink.) or
  3. $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = r H_0$ or
  4. define mean "global" comoving velocity $\langle v \rangle_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\int_0^r \mathrm{d}s|_{t,\theta,\phi}}{t_0}$

“Velocity” $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of “velocity”:
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption ($\|$ transport + Mink.) or
  3. $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = r H_0$ or
  4. define mean “global” comoving velocity $\langle v \rangle_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\int_0^r \mathrm{d}s|_{t,\theta,\phi}}{t_0} = \frac{ar}{t_0}$

“Velocity” $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of “velocity”:
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption ($\|$ transport + Mink.) or
  3. $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = r H_0$ or
  4. define mean “global” comoving velocity $\langle v \rangle_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\int_0^r \mathrm{d}s|_{t,\theta,\phi}}{t_0} = \frac{ar}{t_0} = \frac{r}{t_0}$

“Velocity” $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of “velocity”:
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption ($\|$ transport + Mink.) or
  3. $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = r H_0$ or
  4. $\langle v \rangle_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\int_0^r \mathrm{d}s|_{t,\theta,\phi}}{t_0} = \frac{ar}{t_0} = \frac{r}{t_0}$
- $v_{\mathrm{gal}}^{\mathrm{glob}} \geq c$ when

"Velocity" $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of "velocity":
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption ($\|$ transport + Mink.) or
  3. $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = rH_0$ or
  4. $\langle v \rangle_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\int_0^r \mathrm{d}s|_{t,\theta,\phi}}{t_0} = \frac{ar}{t_0} = \frac{r}{t_0}$
- $v_{\mathrm{gal}}^{\mathrm{glob}} \geq c$ when $r \geq c/H_0 \approx 3\mathrm{h}^{-1}$ Gpc

“Velocity” $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of “velocity”:
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption ($\|$ transport + Mink.) or
  3. $v^{\mathrm{glob}}_{\mathrm{gal}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = r H_0$ or
  4. $\langle v \rangle^{\mathrm{glob}}_{\mathrm{gal}} := \frac{\int_0^r \mathrm{d}s|_{t,\theta,\phi}}{t_0} = \frac{ar}{t_0} = \frac{r}{t_0}$
- $v^{\mathrm{glob}}_{\mathrm{gal}} \geq c$ when $r \geq c/H_0 \approx 3\mathrm{h}^{-1}$ Gpc
- $\langle v \rangle^{\mathrm{glob}}_{\mathrm{gal}} \geq c$ when $r/t_0 \geq c$

“Velocity” $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of “velocity”:
  1. galaxy as fundamental observer, local velocity $v_{\rm gal} = 0$ by defn of comoving coords, or
  2. $v_{\rm gal} < c$ by assumption ($\|$ transport + Mink.) or
  3. $v_{\rm gal}^{\rm glob} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = r H_0$ or
  4. $\langle v \rangle_{\rm gal}^{\rm glob} := \frac{\int_0^r \mathrm{d}s|_{t,\theta,\phi}}{t_0} = \frac{ar}{t_0} = \frac{r}{t_0}$
- $v_{\rm gal}^{\rm glob} \geq c$ when $r \geq c/H_0 \approx 3\mathrm{h}^{-1}$ Gpc
- $\langle v \rangle_{\rm gal}^{\rm glob} \geq c$ when $r \geq c t_0 \approx 4.5\mathrm{h}^{-1}$ Gpc

“Velocity” $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r >$ 3h^{-1} Gpc $\Rightarrow \beta > 1$?
- possible definitions of “velocity”:
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption (∥ transport + Mink.) or
  3. $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = r H_0$ or
  4. $\langle v\rangle_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\int_0^r \mathrm{d}s|_{t,\theta,\phi}}{t_0} = \frac{ar}{t_0} = \frac{r}{t_0}$
- $v_{\mathrm{gal}}^{\mathrm{glob}} \geq c$ when $r \geq c/H_0 \approx$ 3h^{-1} Gpc
- $\langle v\rangle_{\mathrm{gal}}^{\mathrm{glob}} \geq c$ when $r \geq ct_0 \approx 4.5$h^{-1} Gpc
- $v_{\mathrm{gal}}^{\mathrm{glob}}$, $\langle v\rangle_{\mathrm{gal}}^{\mathrm{glob}}$ = theoretical, global $\neq$ velocity in tangent spacetime

"Velocity" $v > c$ in FLRW?

- low z Hubble law $r \approx \frac{\beta c}{H_0}$: $r > 3\mathrm{h}^{-1}$ Gpc $\Rightarrow \beta > 1$?
- possible definitions of "velocity":
  1. galaxy as fundamental observer, local velocity $v_{\mathrm{gal}} = 0$ by defn of comoving coords, or
  2. $v_{\mathrm{gal}} < c$ by assumption ($\|$ transport + Mink.) or
  3. $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = r H_0$ or
  4. $\langle v \rangle_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\int_0^r \mathrm{d}s|_{t,\theta,\phi}}{t_0} = \frac{ar}{t_0} = \frac{r}{t_0}$
- $v_{\mathrm{gal}}^{\mathrm{glob}} \geq c$ when $r \geq c/H_0 \approx 3\mathrm{h}^{-1}$ Gpc
- $\langle v \rangle_{\mathrm{gal}}^{\mathrm{glob}} \geq c$ when $r \geq ct_0 \approx 4.5\mathrm{h}^{-1}$ Gpc
- $v_{\mathrm{gal}}^{\mathrm{glob}}$, $\langle v \rangle_{\mathrm{gal}}^{\mathrm{glob}}$ = theoretical, global $\neq$ velocity in tangent spacetime

$\Rightarrow$ no conflict with locally Lorentzian (SR) spacetime

Non-radial spatial geodesics

- What is the comoving distance between two objects at different celestial positions and different redshifts, for an arbitrary curvature $(+, 0, -)$?

Non-radial spatial geodesics

- distances on the 2-sphere, embedded in $\mathbb{R}^3$

Non-radial spatial geodesics

- distances on the 2-sphere, embedded in $\mathbb{R}^3$

$$\begin{aligned} x_i &= R_\mathrm{C} \cos\delta_i \cos\alpha_i \\ y_i &= R_\mathrm{C} \cos\delta_i \sin\alpha_i \\ w_i &= R_\mathrm{C} \sin\delta_i \end{aligned}$$

Non-radial spatial geodesics

- distances on the 2-sphere, embedded in $\mathbb{R}^3$

$$\begin{aligned} x_i &= R_\mathrm{C} \cos\delta_i \cos\alpha_i \\ y_i &= R_\mathrm{C} \cos\delta_i \sin\alpha_i \\ w_i &= R_\mathrm{C} \sin\delta_i \end{aligned}$$

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = x_1 x_2 + y_1 y_2 + w_1 w_2$$

Non-radial spatial geodesics

- distances on the 2-sphere, embedded in $\mathbb{R}^3$

$$\begin{aligned} x_i &= R_\mathrm{C} \cos\delta_i \cos\alpha_i \\ y_i &= R_\mathrm{C} \cos\delta_i \sin\alpha_i \\ w_i &= R_\mathrm{C} \sin\delta_i \end{aligned}$$

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = x_1 x_2 + y_1 y_2 + w_1 w_2$$

- but also:

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = R^2 \cos\theta_{12}.$$

Non-radial spatial geodesics

- distances on the 2-sphere, embedded in $\mathbb{R}^3$

$$
\begin{aligned}
x_i &= R_\mathrm{C} \cos\delta_i \cos\alpha_i \\
y_i &= R_\mathrm{C} \cos\delta_i \sin\alpha_i \\
w_i &= R_\mathrm{C} \sin\delta_i
\end{aligned}
$$

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = x_1 x_2 + y_1 y_2 + w_1 w_2$$

- but also:

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = R^2 \cos\theta_{12}.$$

- a distance in S^2 = arc-length in $\mathbb{R}^3$:

Non-radial spatial geodesics

- distances on the 2-sphere, embedded in $\mathbb{R}^3$

$$\begin{aligned} x_i &= R_\mathrm{C} \cos\delta_i \cos\alpha_i \\ y_i &= R_\mathrm{C} \cos\delta_i \sin\alpha_i \\ w_i &= R_\mathrm{C} \sin\delta_i \end{aligned}$$

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = x_1 x_2 + y_1 y_2 + w_1 w_2$$

- but also:

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = R^2 \cos\theta_{12}.$$

- a distance in S^2 = arc-length in $\mathbb{R}^3$:

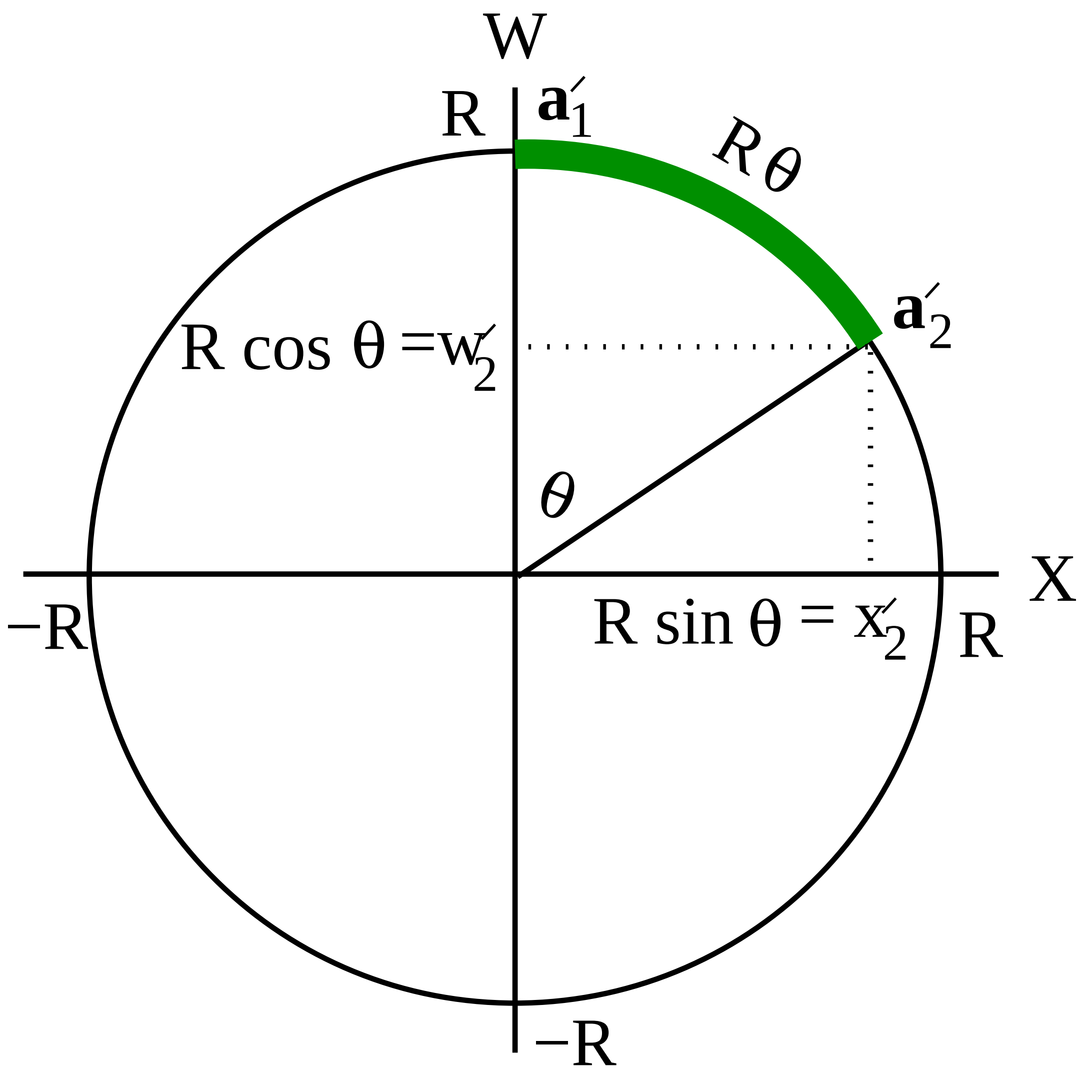
$$\chi_{12} = R_\mathrm{C}\, \theta_{12} = R_\mathrm{C}\, \cos^{-1}\left[\langle \mathbf{a}_1, \mathbf{a}_2 \rangle / R_\mathrm{C}^2\right]$$

Non-radial spatial geodesics

- positive curvature

Non-radial spatial geodesics

- positive curvature

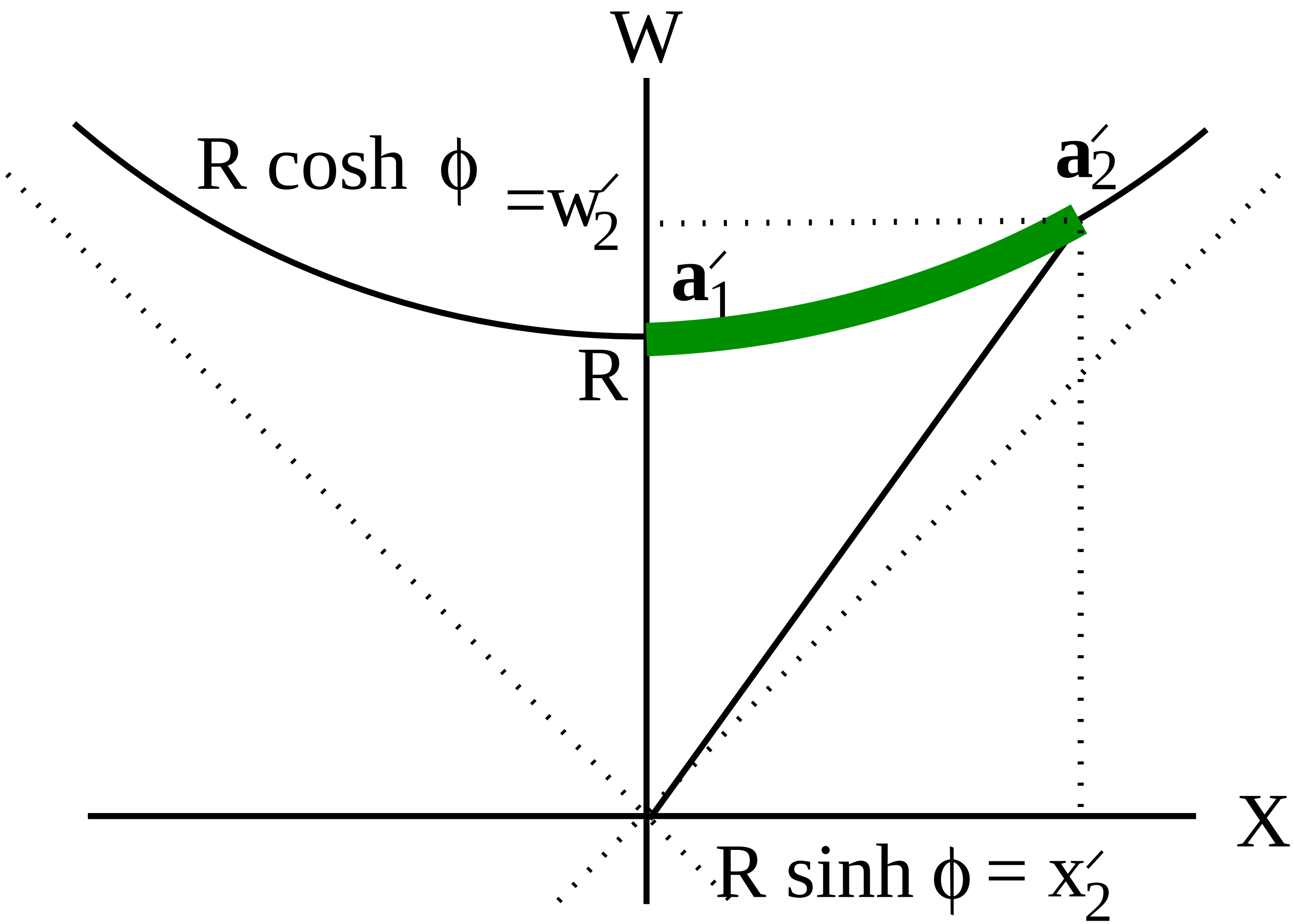


Non-radial spatial geodesics

- negative curvature

Non-radial spatial geodesics

- negative curvature



-

distances on $\mathsf{S}^3 \subset \mathbb{R}^4$ or $\mathsf{H}^3 \subset \mathsf{M}^4$

$$\Sigma(\chi_i) := \begin{cases} R_\mathrm{C}\ \sinh(\chi_i/R_\mathrm{C}) & k < 0 \\ \chi_i & k = 0 \\ R_\mathrm{C}\ \sin(\chi_i/R_\mathrm{C}) & k > 0 \end{cases}$$

distances on $\mathsf{S}^3 \subset \mathbb{R}^4$ or $\mathsf{H}^3 \subset \mathsf{M}^4$

- $$\Sigma(\chi_i) := \begin{cases} R_{\mathrm{C}}\ \sinh(\chi_i/R_{\mathrm{C}}) & k < 0 \\ \chi_i & k = 0 \\ R_{\mathrm{C}}\ \sin(\chi_i/R_{\mathrm{C}}) & k > 0 \end{cases}$$

- $$\begin{aligned} x_i &= \Sigma(\chi_i) \cos\delta_i \cos\alpha_i \\ y_i &= \Sigma(\chi_i) \cos\delta_i \sin\alpha_i \\ z_i &= \Sigma(\chi_i) \sin\delta_i \\ w_i &= \begin{cases} R_{\mathrm{C}}\ \cosh(\chi_i/R_{\mathrm{C}}) & k < 0 \\ 0 & k = 0 \\ R_{\mathrm{C}}\ \cos(\chi_i/R_{\mathrm{C}}) & k > 0 \end{cases} \end{aligned}$$

distances on $\mathsf{S}^3 \subset \mathbb{R}^4$ or $\mathsf{H}^3 \subset \mathsf{M}^4$

- $$\Sigma(\chi_i) := \begin{cases} R_\mathrm{C}\ \sinh(\chi_i/R_\mathrm{C}) & k < 0 \\ \chi_i & k = 0 \\ R_\mathrm{C}\ \sin(\chi_i/R_\mathrm{C}) & k > 0 \end{cases}$$

- $$\begin{aligned} x_i &= \Sigma(\chi_i) \cos\delta_i \cos\alpha_i \\ y_i &= \Sigma(\chi_i) \cos\delta_i \sin\alpha_i \\ z_i &= \Sigma(\chi_i) \sin\delta_i \\ w_i &= \begin{cases} R_\mathrm{C}\ \cosh(\chi_i/R_\mathrm{C}) & k < 0 \\ 0 & k = 0 \\ R_\mathrm{C}\ \cos(\chi_i/R_\mathrm{C}) & k > 0 \end{cases} \end{aligned}$$

- metric on S^3 (or $\mathbb{R}^3$ or H^3):

$$\mathrm{d}s^2 = \begin{cases} (k/|k|)\ (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \mathrm{d}w^2 & k \neq 0 \\ \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 & k = 0 \end{cases}$$

distances on $S^3 \subset \mathbb{R}^4$ or $H^3 \subset M^4$

- $$\Sigma(\chi_i) := \begin{cases} R_C \ \sinh(\chi_i/R_C) & k < 0 \\ \chi_i & k = 0 \\ R_C \ \sin(\chi_i/R_C) & k > 0 \end{cases}$$
- $$\begin{aligned} x_i &= \Sigma(\chi_i)\cos\delta_i\cos\alpha_i \\ y_i &= \Sigma(\chi_i)\cos\delta_i\sin\alpha_i \\ z_i &= \Sigma(\chi_i)\sin\delta_i \\ w_i &= \begin{cases} R_C \ \cosh(\chi_i/R_C) & k < 0 \\ 0 & k = 0 \\ R_C \ \cos(\chi_i/R_C) & k > 0 \end{cases} \end{aligned}$$
- inner product:
$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle \equiv \begin{cases} (k/|k|)\ (x_1x_2 + y_1y_2 + z_1z_2) + w_1w_2 & k \neq 0 \\ x_1x_2 + y_1y_2 + z_1z_2 & k = 0 \end{cases}$$

distances on $\mathsf{S}^3 \subset \mathbb{R}^4$ or $\mathsf{H}^3 \subset \mathsf{M}^4$

- $$\Sigma(\chi_i) := \begin{cases} R_\mathrm{C}\ \sinh(\chi_i/R_\mathrm{C}) & k < 0 \\ \chi_i & k = 0 \\ R_\mathrm{C}\ \sin(\chi_i/R_\mathrm{C}) & k > 0 \end{cases}$$

- $$\begin{aligned} x_i &= \Sigma(\chi_i)\cos\delta_i\cos\alpha_i \\ y_i &= \Sigma(\chi_i)\cos\delta_i\sin\alpha_i \\ z_i &= \Sigma(\chi_i)\sin\delta_i \\ w_i &= \begin{cases} R_\mathrm{C}\ \cosh(\chi_i/R_\mathrm{C}) & k < 0 \\ 0 & k = 0 \\ R_\mathrm{C}\ \cos(\chi_i/R_\mathrm{C}) & k > 0 \end{cases} \end{aligned}$$

$\Rightarrow$

$$\chi_{12} = \begin{cases} R_\mathrm{C}\cosh^{-1}\left[\langle \mathbf{a}_1, \mathbf{a}_2\rangle / R_\mathrm{C}^2\right] & k < 0 \\ \sqrt{\langle \mathbf{a}_1 - \mathbf{a}_2, \mathbf{a}_1 - \mathbf{a}_2\rangle} & k = 0 \\ R_\mathrm{C}\cos^{-1}\left[\langle \mathbf{a}_1, \mathbf{a}_2\rangle / R_\mathrm{C}^2\right] & k > 0 \end{cases}$$

distances on $\mathsf{S}^3 \subset \mathbb{R}^4$ or $\mathsf{H}^3 \subset \mathsf{M}^4$

- $$
\begin{aligned}
x_i &= \Sigma(\chi_i) \cos\delta_i \cos\alpha_i \\
y_i &= \Sigma(\chi_i) \cos\delta_i \sin\alpha_i \\
z_i &= \Sigma(\chi_i) \sin\delta_i \\
w_i &= \begin{cases} R_\mathrm{C} \ \cosh(\chi_i/R_\mathrm{C}) & k < 0 \\ 0 & k = 0 \\ R_\mathrm{C} \ \cos(\chi_i/R_\mathrm{C}) & k > 0 \end{cases}
\end{aligned}
$$

$\Rightarrow$

$$
\chi_{12} = \begin{cases} R_\mathrm{C} \cosh^{-1}\left[\langle \mathbf{a}_1, \mathbf{a}_2 \rangle / R_\mathrm{C}^2\right] & k < 0 \\ \sqrt{\langle \mathbf{a}_1 - \mathbf{a}_2, \mathbf{a}_1 - \mathbf{a}_2 \rangle} & k = 0 \\ R_\mathrm{C} \cos^{-1}\left[\langle \mathbf{a}_1, \mathbf{a}_2 \rangle / R_\mathrm{C}^2\right] & k > 0 \end{cases}
$$

- a distance in S^3 is an arc-length in $\mathbb{R}^4$

distances on $\mathsf{S}^3 \subset \mathbb{R}^4$ or $\mathsf{H}^3 \subset \mathsf{M}^4$

$$
\begin{aligned}
x_i &= \Sigma(\chi_i) \cos\delta_i \cos\alpha_i \\
y_i &= \Sigma(\chi_i) \cos\delta_i \sin\alpha_i \\
z_i &= \Sigma(\chi_i) \sin\delta_i \\
w_i &= \begin{cases} R_\mathrm{C}\ \cosh(\chi_i/R_\mathrm{C}) & k < 0 \\ 0 & k = 0 \\ R_\mathrm{C}\ \cos(\chi_i/R_\mathrm{C}) & k > 0 \end{cases}
\end{aligned}
$$

$\Rightarrow$

$$
\chi_{12} = \begin{cases} R_\mathrm{C} \cosh^{-1}\left[\langle \mathbf{a}_1, \mathbf{a}_2 \rangle / R_\mathrm{C}^2\right] & k < 0 \\ \sqrt{\langle \mathbf{a}_1 - \mathbf{a}_2, \mathbf{a}_1 - \mathbf{a}_2 \rangle} & k = 0 \\ R_\mathrm{C} \cos^{-1}\left[\langle \mathbf{a}_1, \mathbf{a}_2 \rangle / R_\mathrm{C}^2\right] & k > 0 \end{cases}
$$

- a distance in S^3 is an arc-length in $\mathbb{R}^4$
- a distance in H^3 is an arc-length in M^4

distances on $\mathsf{S}^3 \subset \mathbb{R}^4$ or $\mathsf{H}^3 \subset \mathsf{M}^4$

$$
\begin{aligned}
x_i &= \Sigma(\chi_i) \cos\delta_i \cos\alpha_i \\
y_i &= \Sigma(\chi_i) \cos\delta_i \sin\alpha_i \\
z_i &= \Sigma(\chi_i) \sin\delta_i \\
w_i &= \begin{cases} R_C \ \cosh(\chi_i/R_C) & k < 0 \\ 0 & k = 0 \\ R_C \ \cos(\chi_i/R_C) & k > 0 \end{cases}
\end{aligned}
$$

$\Rightarrow$

$$
\chi_{12} = \begin{cases} R_C \cosh^{-1}\left[\langle \mathbf{a}_1, \mathbf{a}_2 \rangle / R_C^2\right] & k < 0 \\ \sqrt{\langle \mathbf{a}_1 - \mathbf{a}_2, \mathbf{a}_1 - \mathbf{a}_2 \rangle} & k = 0 \\ R_C \cos^{-1}\left[\langle \mathbf{a}_1, \mathbf{a}_2 \rangle / R_C^2\right] & k > 0 \end{cases}
$$

- a distance in S^3 is an arc-length in $\mathbb{R}^4$ arXiv:astro-ph/0102099
- a distance in H^3 is an arc-length in M^4

Cosmic topology: definitions

- 3 intuitive methods

Cosmic topology: definitions

- 3 intuitive methods
- $\widetilde{M}$ = universal covering space

Cosmic topology: definitions

- <u>3 intuitive methods</u>
- $\widetilde{M}$ = universal covering space= "apparent space"

Cosmic topology: definitions

- 3 intuitive methods
- $\widetilde{M}$ = universal covering space= "apparent space"
- $\widetilde{M} = \begin{cases} \mathbb{H}^3 & k < 0 \\ & \\ & \end{cases}$

Cosmic topology: definitions

- 3 intuitive methods
- $\widetilde{M}$ = universal covering space= "apparent space"
- $\widetilde{M} = \begin{cases} \mathbb{H}^3 & k < 0 \\ \mathbb{E}^3 & k = 0 \\ & \end{cases}$

Cosmic topology: definitions

- 3 intuitive methods
- $\widetilde{M}$ = universal covering space= “apparent space”
- $\widetilde{M} = \begin{cases} \mathbb{H}^3 & k < 0 \\ \mathbb{E}^3 & k = 0 \\ \mathbb{S}^3 & k > 0 \end{cases}$

Cosmic topology: definitions

- 3 intuitive methods
- $\widetilde{M}$ = universal covering space= “apparent space”
- $\widetilde{M} = \begin{cases} \mathbb{H}^3 & k < 0 \\ \mathbb{E}^3 & k = 0 \\ \mathbb{S}^3 & k > 0 \end{cases}$
- Γ = a group of holonomies (isometries) of $\widetilde{M}$

Cosmic topology: definitions

- 3 intuitive methods
- $\widetilde{M}$ = universal covering space= "apparent space"
- $$\widetilde{M} = \begin{cases} \mathbb{H}^3 & k < 0 \\ \mathbb{E}^3 & k = 0 \\ \mathbb{S}^3 & k > 0 \end{cases}$$
- Γ = a group of holonomies (isometries) of $\widetilde{M}$
- the 3-manifold

Cosmic topology: definitions

- 3 intuitive methods
- $\widetilde{M}$ = universal covering space= “apparent space”
- $\widetilde{M} = \begin{cases} \mathbb{H}^3 & k < 0 \\ \mathbb{E}^3 & k = 0 \\ \mathbb{S}^3 & k > 0 \end{cases}$
- Γ = a group of holonomies (isometries) of $\widetilde{M}$
- the **3-space for FLRW cosmology is $M = \widetilde{M}/\Gamma$**

Cosmic topology: definitions

- 3 intuitive methods
- $\widetilde{M}$ = universal covering space= "apparent space"
- $\widetilde{M} = \begin{cases} \mathbb{H}^3 & k < 0 \\ \mathbb{E}^3 & k = 0 \\ \mathbb{S}^3 & k > 0 \end{cases}$
- Γ = a group of holonomies (isometries) of $\widetilde{M}$
- the 3-space for FLRW cosmology is $M = \widetilde{M}/\Gamma$
- each of the 3 ways of thinking of $\widetilde{M}/\Gamma$ has advantages and disadvantages

Cosmic topology: definitions

- 3 intuitive methods
- $\widetilde{M}$ = universal covering space= "apparent space"
- $\widetilde{M} = \begin{cases} \mathbb{H}^3 & k < 0 \\ \mathbb{E}^3 & k = 0 \\ \mathbb{S}^3 & k > 0 \end{cases}$
- Γ = a group of holonomies (isometries) of $\widetilde{M}$
- the $\boxed{\text{3-space for FLRW cosmology is } M = \widetilde{M}/\Gamma}$
- each of the 3 ways of thinking of $\widetilde{M}/\Gamma$ has advantages and disadvantages
- fundamental domain (FD) is not unique;

Cosmic topology: definitions

- 3 intuitive methods
- $\widetilde{M}$ = universal covering space= "apparent space"
- $\widetilde{M} = \begin{cases} \mathbb{H}^3 & k < 0 \\ \mathbb{E}^3 & k = 0 \\ \mathbb{S}^3 & k > 0 \end{cases}$
- Γ = a group of holonomies (isometries) of $\widetilde{M}$
- the $\boxed{\text{3-space for FLRW cosmology is } M = \widetilde{M}/\Gamma}$
- each of the 3 ways of thinking of $\widetilde{M}/\Gamma$ has advantages and disadvantages
- fundamental domain (FD) is not unique; shape of FD may be non-unique

Cosmic topology: definitions

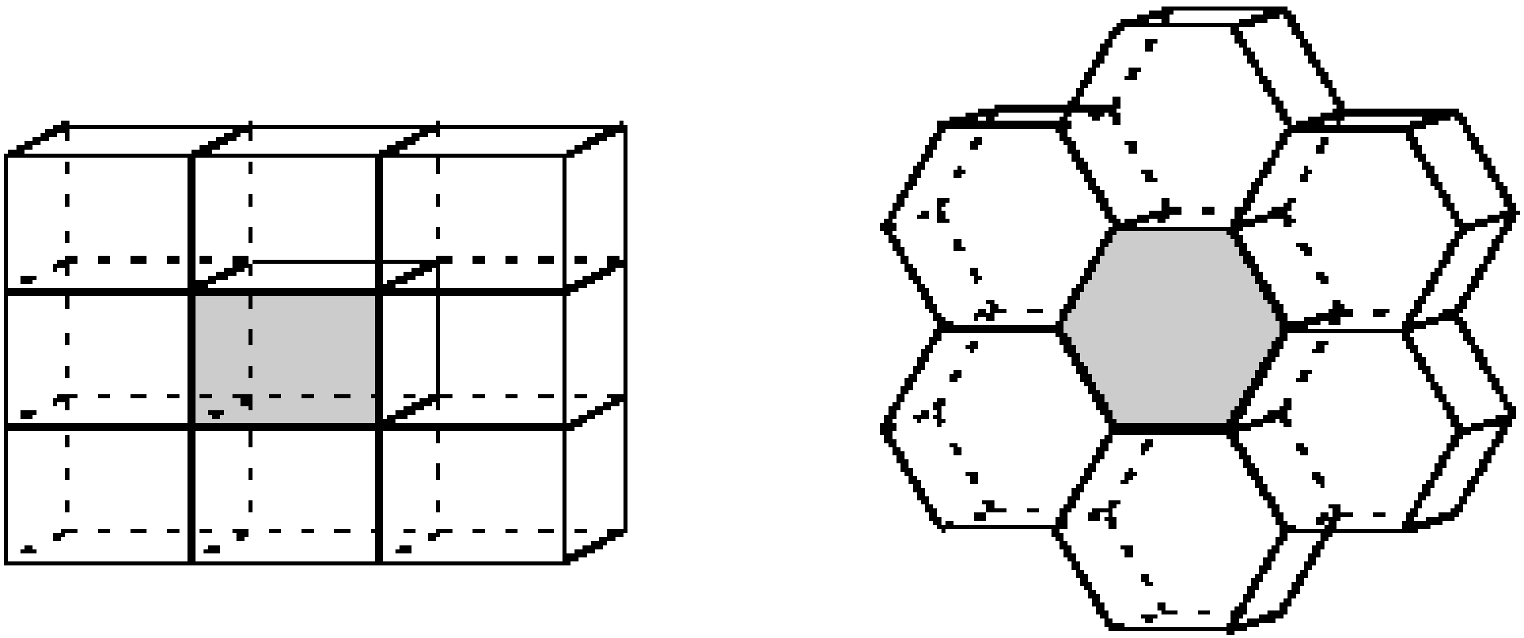
- <u>3 intuitive methods</u>
- $\widetilde{M}$ = universal covering space= "apparent space"
- $\widetilde{M} = \begin{cases} \mathbb{H}^3 & k < 0 \\ \mathbb{E}^3 & k = 0 \\ \mathbb{S}^3 & k > 0 \end{cases}$
- Γ = a group of holonomies (isometries) of $\widetilde{M}$
- the $\boxed{\text{3-space for FLRW cosmology is } M = \widetilde{M}/\Gamma}$
- each of the <u>3 ways</u> of thinking of $\widetilde{M}/\Gamma$ has advantages and disadvantages
- fundamental domain (FD) is not unique; shape of FD may be non-unique
- 2D example: 2-torus: $\widetilde{M} = \mathbb{E}^2$; $\Gamma = \mathbb{Z} \times \mathbb{Z}$;
  $M = \boxed{\mathbb{E}^2/\left(\mathbb{Z} \times \mathbb{Z}\right) =: \mathbb{T}^2}$

Cosmic topology: definitions

- 3 intuitive methods
- $\widetilde{M}$ = universal covering space= "apparent space"
- $\widetilde{M} = \begin{cases} \mathbb{H}^3 & k < 0 \\ \mathbb{E}^3 & k = 0 \\ \mathbb{S}^3 & k > 0 \end{cases}$
- Γ = a group of holonomies (isometries) of $\widetilde{M}$
- the $\boxed{\text{3-space for FLRW cosmology is } M = \widetilde{M}/\Gamma}$
- each of the 3 ways of thinking of $\widetilde{M}/\Gamma$ has advantages and disadvantages
- fundamental domain (FD) is not unique; shape of FD may be non-unique

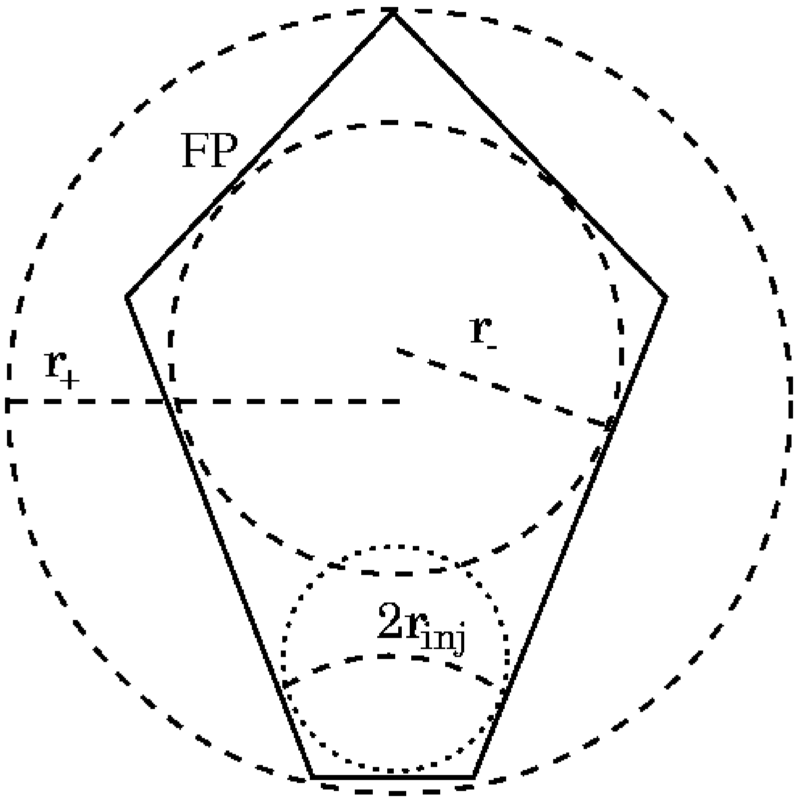
• 2D example: 2-torus: $\widetilde{M} = \mathbb{E}^2$; $\Gamma = \mathbb{Z} \times \mathbb{Z}$;
$M = \boxed{\mathbb{E}^2/\left(\mathbb{Z} \times \mathbb{Z}\right) =: \mathbb{T}^2} \equiv \mathbb{S}^1 \times \mathbb{S}^1$

Cosmic topology: definitions

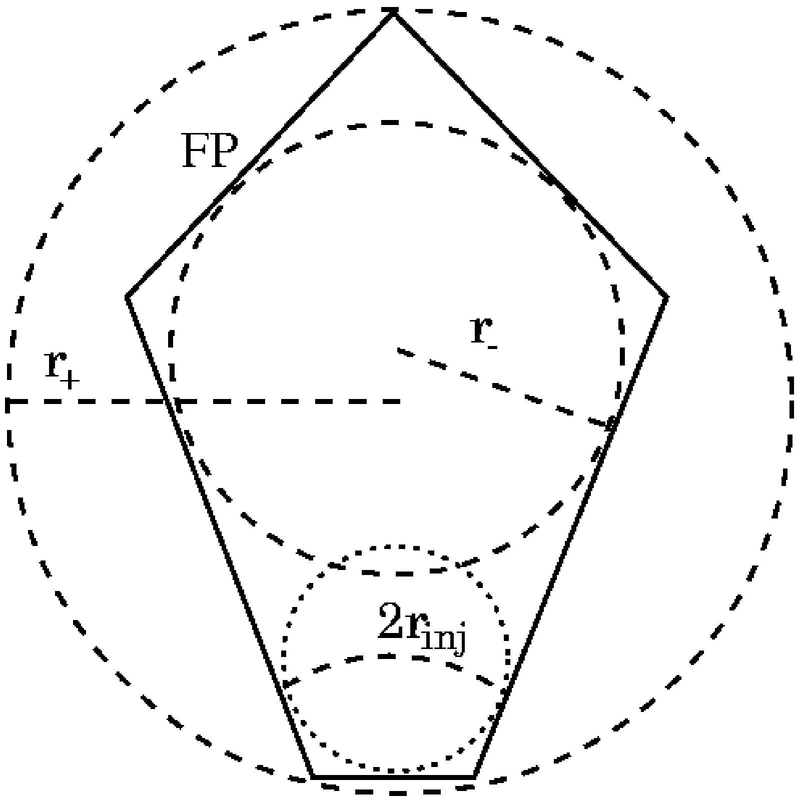


3D flat examples arXiv:astro-ph/9901364

Cosmic topology: definitions



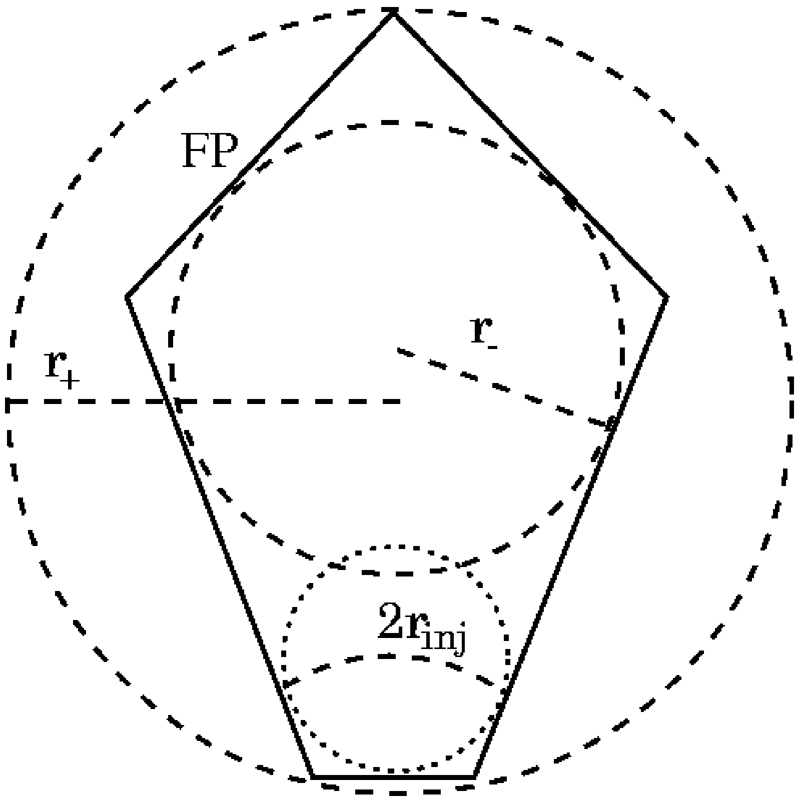
Cosmic topology: definitions



size of universe:

- r_- : biggest sphere *inside* FD

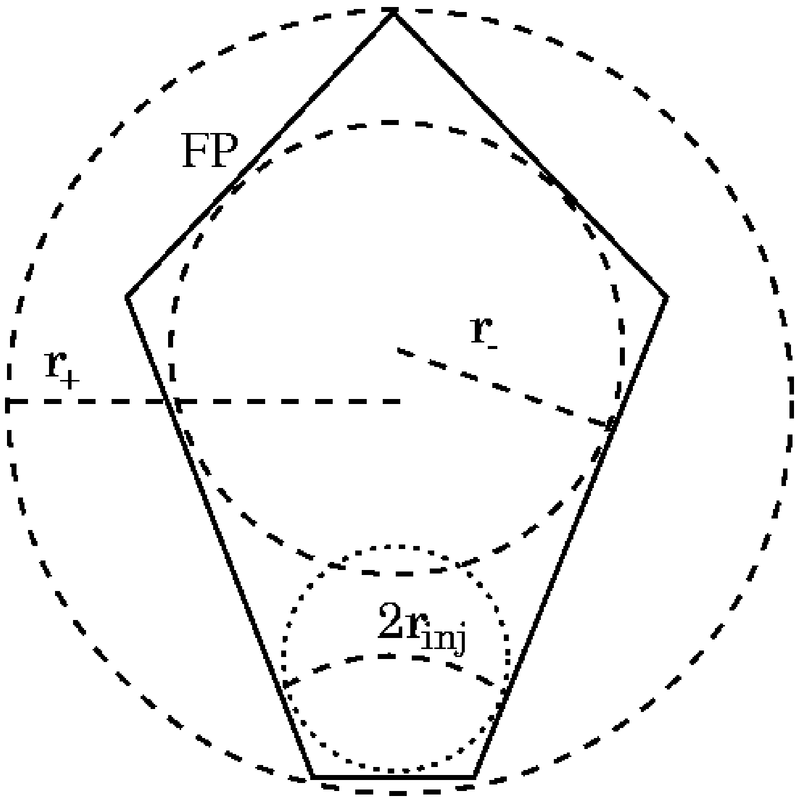
Cosmic topology: definitions



size of universe:

- r_- : biggest sphere *inside* FD
- r_+ : smallest sphere *containing* FD

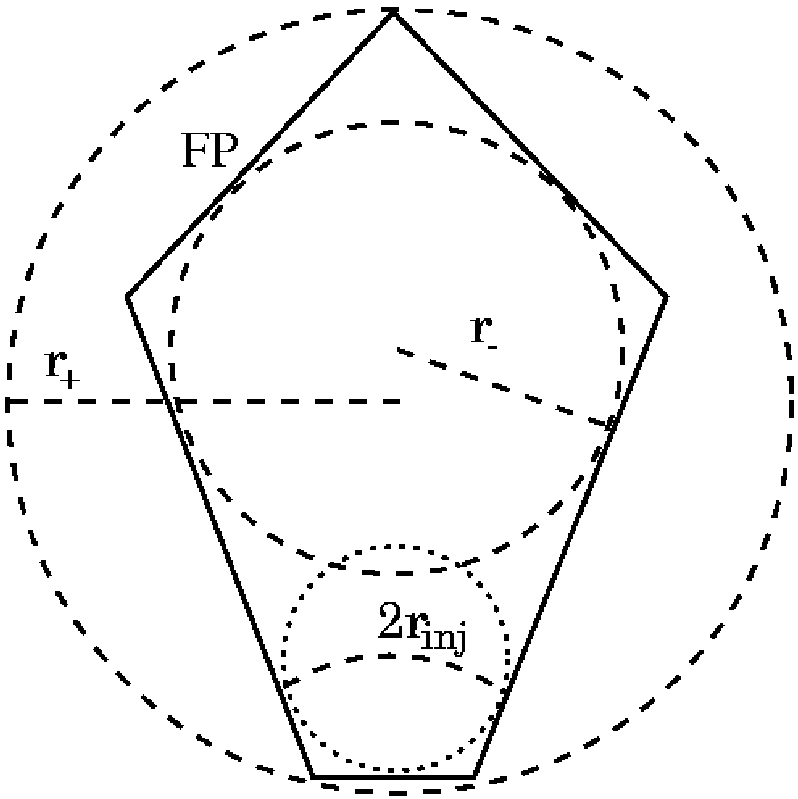
Cosmic topology: definitions



size of universe:

- r_- : biggest sphere *inside* FD
- r_+ : smallest sphere *containing* FD
- $2r_{\rm inj}$: smallest closed spatial geodesic

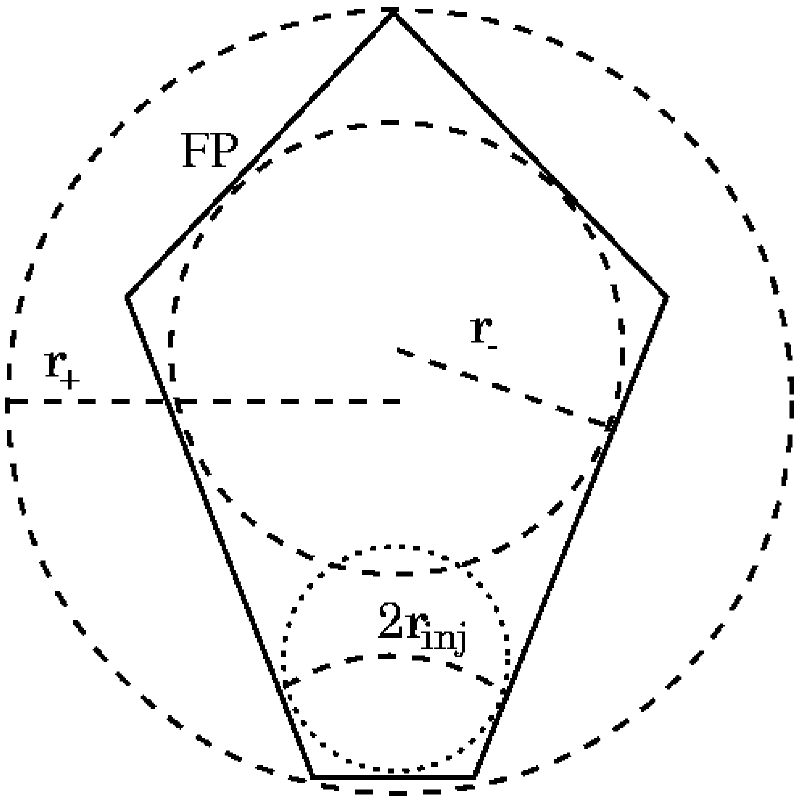
Cosmic topology: definitions



size of universe:

- r_- : biggest sphere *inside* FD
- r_+ : smallest sphere *containing* FD
- $2r_{\rm inj}$: smallest closed spatial geodesic
- $V_{\rm FD}^{1/3}$ volume cube root

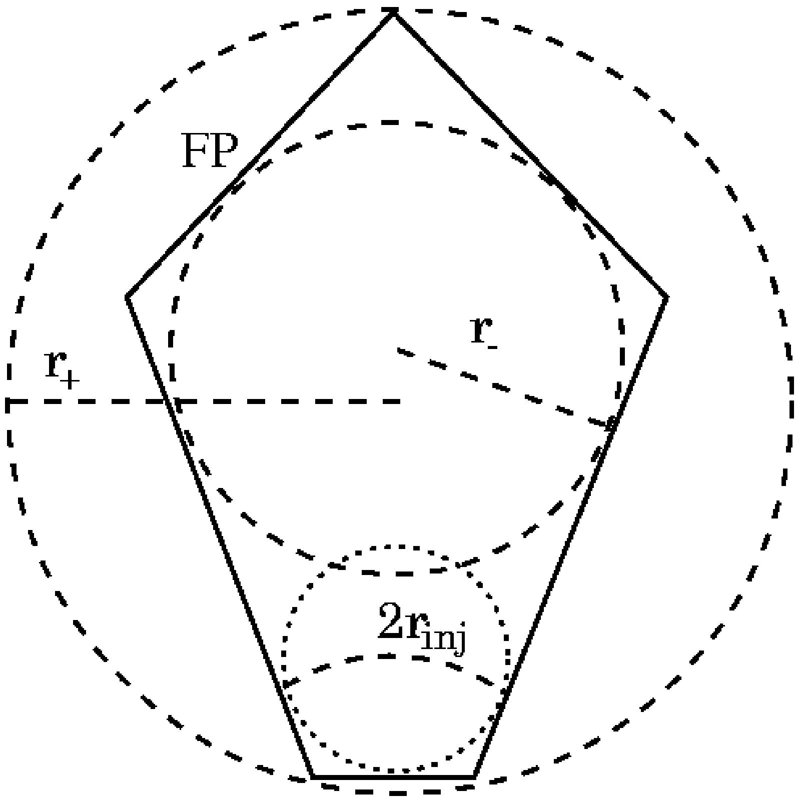
Cosmic topology: definitions



size of universe:

- r_- : biggest sphere *inside* FD
- r_+ : smallest sphere *containing* FD
- $2r_{\rm inj}$: smallest closed spatial geodesic
- $V_{\rm FD}^{1/3}$ volume cube root
- $r_- \le r_+$ always

Cosmic topology: definitions



size of universe:

- r_- : biggest sphere *inside* FD
- r_+ : smallest sphere *containing* FD
- $2r_{\rm inj}$: smallest closed spatial geodesic
- $V_{\rm FD}^{1/3}$ volume cube root
- $r_- \le r_+$ always
- $r_{\rm inj} < r_-$ or $r_{\rm inj} \ll V_{\rm FD}^{1/3}$ possible

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite,

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3 / (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$,

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3 / (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$, some infinite,

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3 / (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$, some infinite, e.g. $\mathbb{E}^3$

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3/\left(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\right)$, some infinite, e.g. $\mathbb{E}^3$
- spherical: arXiv:gr-qc/0106033 Gausmann et al.

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3/\left(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\right)$, some infinite, e.g. $\mathbb{E}^3$
- spherical: arXiv:gr-qc/0106033 Gausmann et al.
  - $\mathbb{S}^3/\Gamma$—countably infinite set of 3-spaces

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3/\left(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\right)$, some infinite, e.g. $\mathbb{E}^3$
- spherical: arXiv:gr-qc/0106033 Gausmann et al.
  - $\mathbb{S}^3/\Gamma$—countably infinite set of 3-spaces
  - all finite ("compact")

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3 / (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$, some infinite, e.g. $\mathbb{E}^3$
- spherical: arXiv:gr-qc/0106033 Gausmann et al.
  - $\mathbb{S}^3/\Gamma$—countably infinite set of 3-spaces
  - all finite ("compact") $\Rightarrow \Gamma =$ finite group

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3/\left(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\right)$, some infinite, e.g. $\mathbb{E}^3$
- spherical: arXiv:gr-qc/0106033 Gausmann et al.
  - $\mathbb{S}^3/\Gamma$—countably infinite set of 3-spaces
  - all finite (“compact”) $\Rightarrow \Gamma =$ finite group
  - completely classified (Threlfall & Seifert 1930)

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3 / (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$, some infinite, e.g. $\mathbb{E}^3$
- spherical: arXiv:gr-qc/0106033 Gausmann et al.
  - $\mathbb{S}^3/\Gamma$—countably infinite set of 3-spaces
  - all finite ("compact") $\Rightarrow \Gamma =$ finite group
  - completely classified (Threlfall & Seifert 1930)
  - S^3, $S^3/\mathbb{Z}_2$,

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3/\left(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\right)$, some infinite, e.g. $\mathbb{E}^3$
- spherical: arXiv:gr-qc/0106033 Gausmann et al.
  - $\mathbb{S}^3/\Gamma$—countably infinite set of 3-spaces
  - all finite ("compact") $\Rightarrow \Gamma =$ finite group
  - completely classified (Threlfall & Seifert 1930)
  - S^3, $S^3/\mathbb{Z}_2$, lens spaces ,

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3 / (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$, some infinite, e.g. $\mathbb{E}^3$
- spherical: arXiv:gr-qc/0106033 Gausmann et al.
  - $\mathbb{S}^3/\Gamma$—countably infinite set of 3-spaces
  - all finite (“compact”) $\Rightarrow \Gamma =$ finite group
  - completely classified (Threlfall & Seifert 1930)
  - S^3, $S^3/\mathbb{Z}_2$, lens spaces ,“well-proportioned” spaces, . . .

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3 / (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$, some infinite, e.g. $\mathbb{E}^3$
- spherical: arXiv:gr-qc/0106033 Gausmann et al.
  - $\mathbb{S}^3/\Gamma$—countably infinite set of 3-spaces
  - all finite (“compact”) $\Rightarrow \Gamma =$ finite group
  - completely classified (Threlfall & Seifert 1930)
  - S^3, $S^3/\mathbb{Z}_2$, lens spaces ,“well-proportioned” spaces, . . .
  - w:Poincaré Conjecture *“Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.”* w:Grigori Perelman, arXiv:math/0211159 + arXiv:math/0303109 + arXiv:math/0307245

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3/ (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$, some infinite, e.g. $\mathbb{E}^3$
- spherical: arXiv:gr-qc/0106033 Gausmann et al.
  - $\mathbb{S}^3/\Gamma$—countably infinite set of 3-spaces
  - all finite (“compact”) $\Rightarrow \Gamma =$ finite group
  - completely classified (Threlfall & Seifert 1930)
  - S^3, $S^3/\mathbb{Z}_2$, lens spaces ,“well-proportioned” spaces, . . .
- hyperbolic:
  - countably infinite superset known

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3/\left(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\right)$, some infinite, e.g. $\mathbb{E}^3$
- spherical: arXiv:gr-qc/0106033 Gausmann et al.
  - $\mathbb{S}^3/\Gamma$—countably infinite set of 3-spaces
  - all finite ("compact") $\Rightarrow \Gamma =$ finite group
  - completely classified (Threlfall & Seifert 1930)
  - S^3, $S^3/\mathbb{Z}_2$, lens spaces ,"well-proportioned" spaces, ...
- hyperbolic:
  - countably infinite superset known
  - some finite, some infinite

Families of const k 3-spaces

- flat 3-spaces: 18 arXiv:1103.1466 Fujii&Yoshii 2011, Tab1+Fig1
  - some finite, e.g. $\mathbb{T}^3 := \mathbb{E}^3/\left(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\right)$, some infinite, e.g. $\mathbb{E}^3$
- spherical: arXiv:gr-qc/0106033 Gausmann et al.
  - $\mathbb{S}^3/\Gamma$—countably infinite set of 3-spaces
  - all finite ("compact") $\Rightarrow \Gamma =$ finite group
  - completely classified (Threlfall & Seifert 1930)
  - S^3, $S^3/\mathbb{Z}_2$, lens spaces ,"well-proportioned" spaces, . . .
- hyperbolic:
  - countably infinite superset known
  - some finite, some infinite
  - active research area, e.g. arXiv:0705.4325 min. vol.
- 5 other Thurston classes – w:Geometrization conjecture

Cosmic topol: top. accel.

cosmic topology theory:

Cosmic topol: top. accel.

cosmic topology theory:

- (quantum gravity arguments)

Cosmic topol: top. accel.

cosmic topology theory:

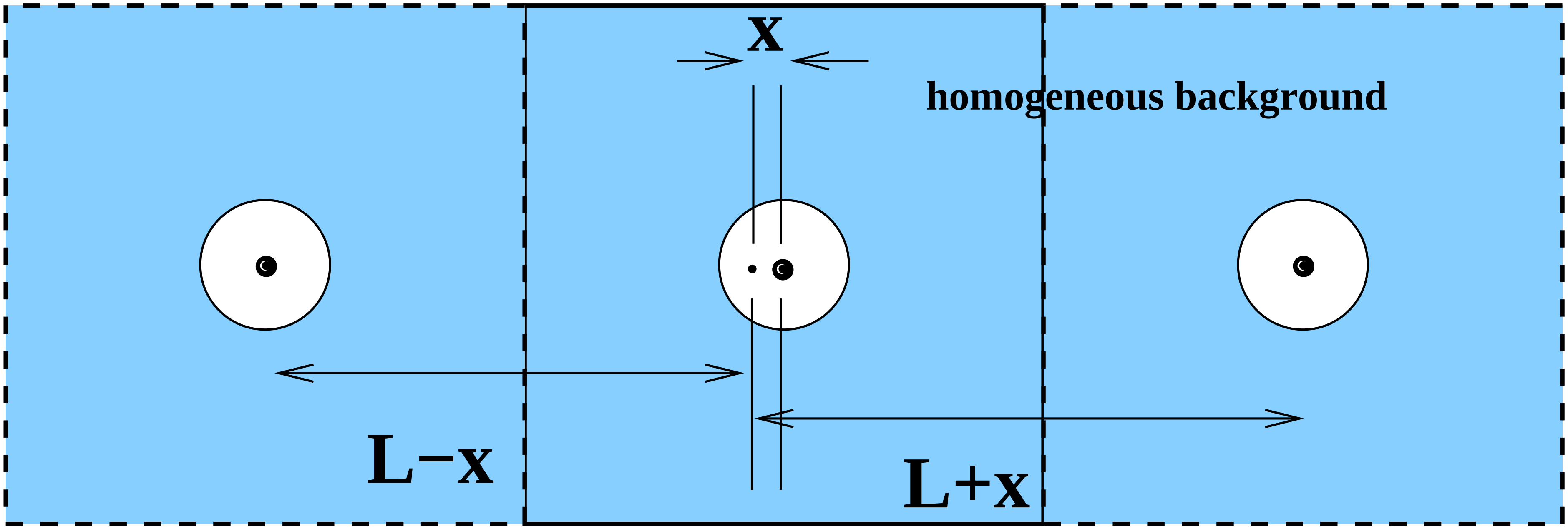
- (quantum gravity arguments)
- global spatial topology: topological acceleration (patching away black holes): Roukema+2007 (A&A 2007) arXiv:astro-ph/0602159

Cosmic topol: top. accel.

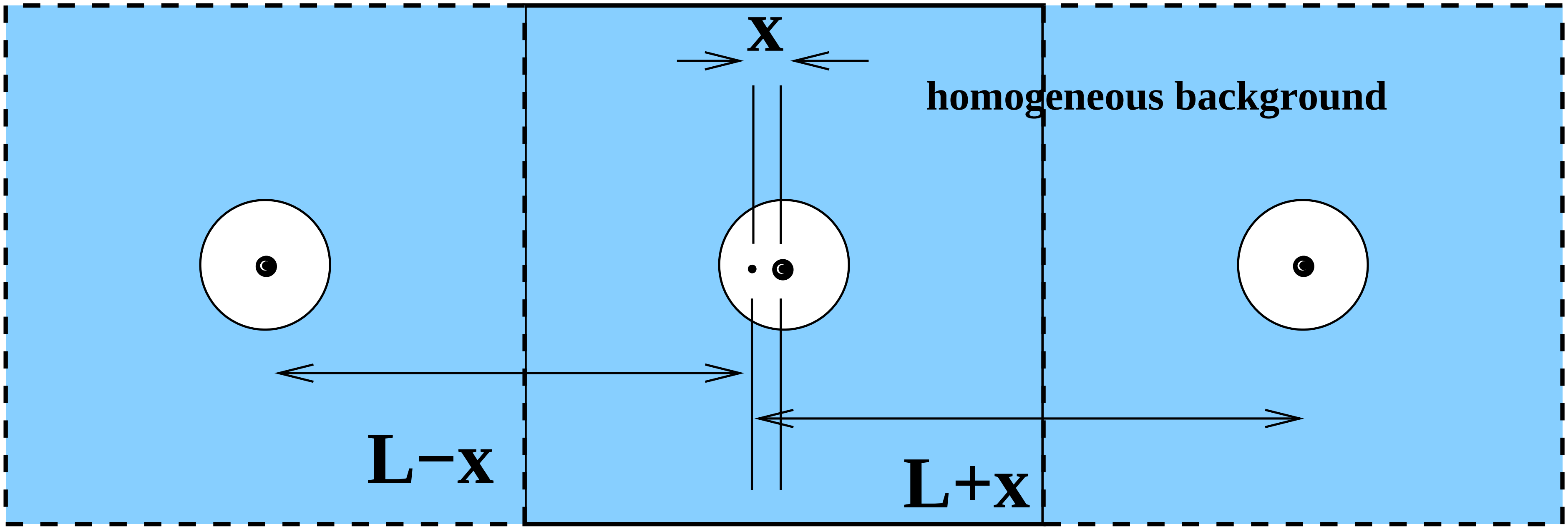
cosmic topology theory:

- (quantum gravity arguments)
- global spatial topology: topological acceleration (patching away black holes): Roukema+2007 (A&A 2007) arXiv:astro-ph/0602159
- scalar averaging and dynamical topology change (e.g. black holes): Brunswic & Buchert (CQG, 2020) arXiv:2002.08336

Cosmic topol: top. accel.

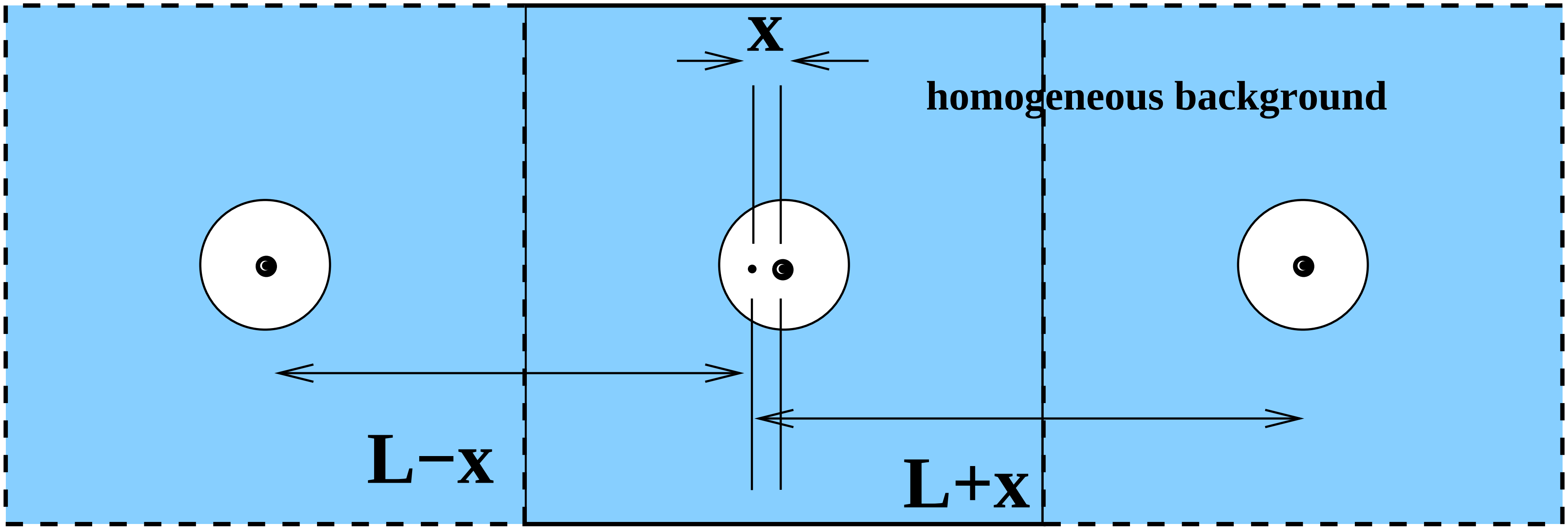


Cosmic topol: top. accel.



$$\ddot{x} \;\approx\; -G\frac{m}{x^2} + Gm\left[\frac{1}{(L-x)^2} - \frac{1}{(L+x)^2}\right]$$

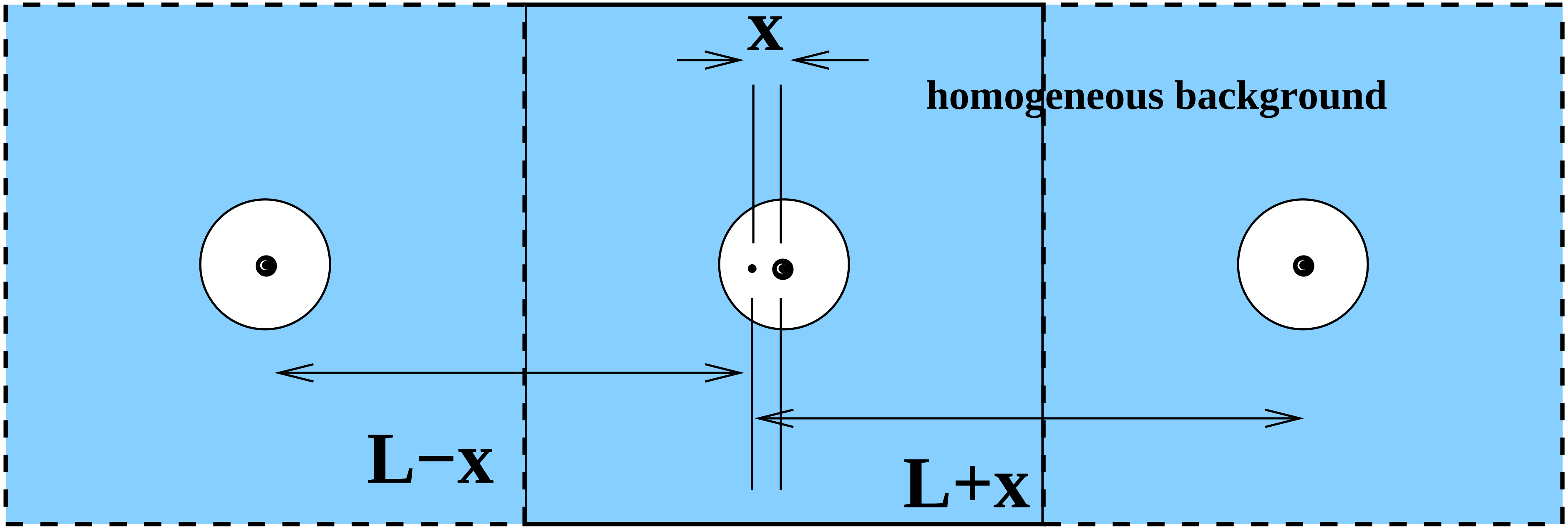
Cosmic topol: top. accel.



$$\ddot{x} \approx -G\frac{m}{x^2} + Gm\left[\frac{1}{(L-x)^2} - \frac{1}{(L+x)^2}\right]$$

$$\approx -G\frac{m}{x^2} + \frac{4Gm}{L^2}\,\frac{x}{L}$$

Cosmic topol: top. accel.

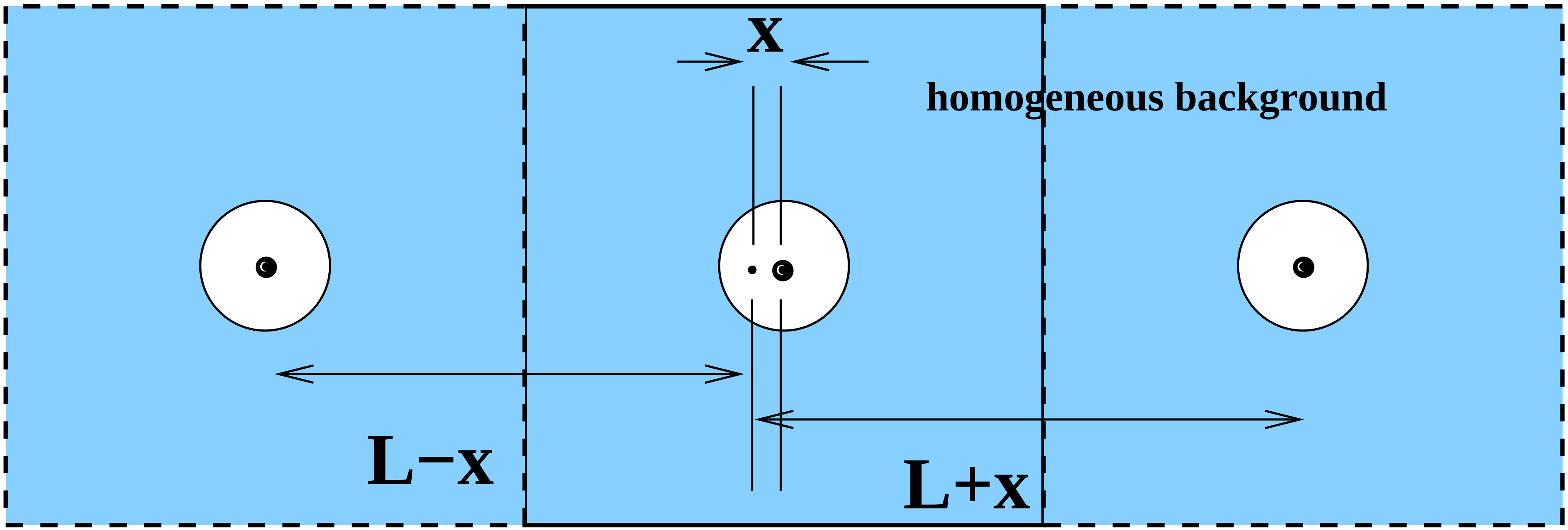


$$\ddot{x} \approx -G\frac{m}{x^2} + Gm\left[\frac{1}{(L-x)^2} - \frac{1}{(L+x)^2}\right]$$

$$\approx -G\frac{m}{x^2} + \frac{4Gm}{L^2}\,\frac{x}{L}$$

$$\boxed{\ddot{x}_{\rm resid} \propto (x/L)^1 + \ldots}$$

Cosmic topol: top. accel.



$$\ddot{x} \approx -G\frac{m}{x^2} + Gm\left[\frac{1}{(L-x)^2} - \frac{1}{(L+x)^2}\right]$$

$$\approx -G\frac{m}{x^2} + \frac{4Gm}{L^2}\,\frac{x}{L}$$

$$\boxed{\ddot{x}_{\mathrm{resid}} \propto (x/L)^1 + \ldots}$$

topological acceleration— arXiv:astro-ph/0602159

Heuristic top. accel.

original heuristic — Roukema+2007 A&A arXiv:astro-ph/0602159

- weak-field gravity of distant, multiple images
- covering space $\mathbb{E}^3$ or $\mathbb{S}^3$
- calculations made in covering space
- consider only first layer of topological images (e.g. particle horizon)

Heuristic top. accel.

original heuristic — Roukema+2007 A&A arXiv:astro-ph/0602159

- weak-field gravity of distant, multiple images
- covering space $\mathbb{E}^3$ or $\mathbb{S}^3$
- calculations made in covering space
- consider only first layer of topological images (e.g. particle horizon)
- $\mathbb{T}^3 = \mathbb{E}^3/\mathbb{Z}^3 \;\Rightarrow\; \ddot{x}_{\mathrm{resid}} \propto (x/L)^3 + \ldots$
- $\mathbb{S}^3/T^*$ (octahedral space) $\;\Rightarrow\; \ddot{x}_{\mathrm{resid}} \propto (x/R_C)^3 + \ldots$
- $\mathbb{S}^3/O^*$ (truncated cube space) $\;\Rightarrow\; \ddot{x}_{\mathrm{resid}} \propto (x/R_C)^3 + \ldots$
- $\mathbb{S}^3/I^*$ (Poincaré dodecahedral space) $\;\Rightarrow\; \ddot{x}_{\mathrm{resid}} \propto (x/R_C)^5 + \ldots$
- *topological acceleration is manifold-dependent*
  Roukema & Różański arXiv:0902.3402, A&A, 502, 27

Newt. non-Euclid. top.accel.

NEN: $\Phi_{\mathbb{S}}(\xi) \propto -\cot\xi\,(1-\xi/\pi) + A$

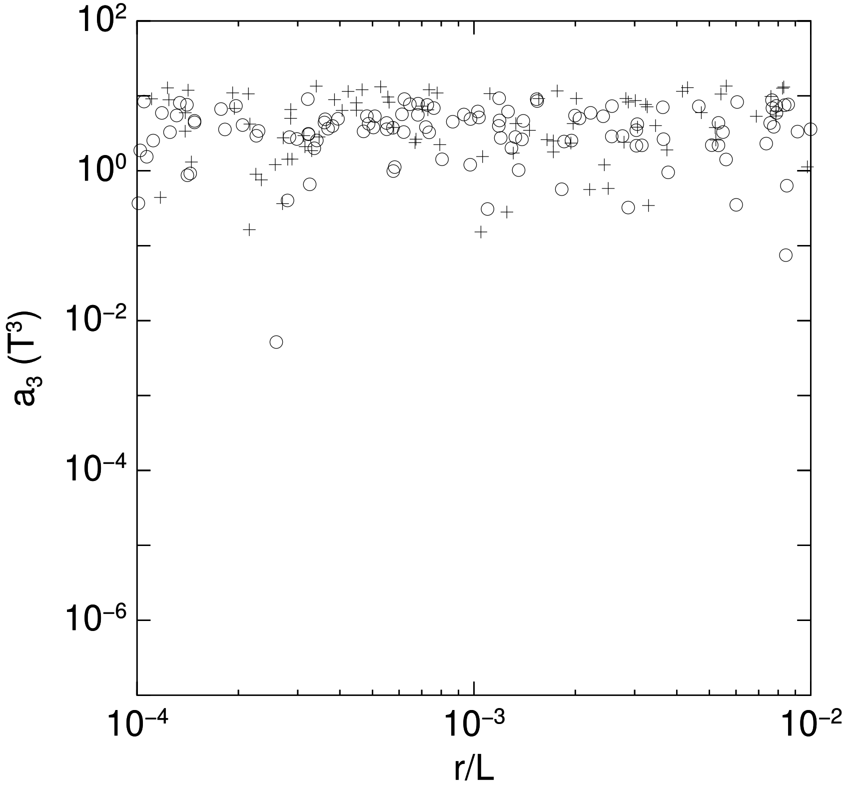
Topology	N_Σ	Φ_{-1}	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5
Euclidean (infinite or Thurston-type)							
$\mathbb{E}^3$		-1	0	0	0	0	0
$\mathbb{T}^3$		-1	0	$-\frac{2\pi}{3}\frac{1}{V_\Sigma}$	0	–	0
Spherical							
$\mathbb{S}^3$	1	-1	$\frac{1}{3}\frac{\mathcal{R}}{6}$	$-\frac{2\pi}{3}\frac{1}{V_\Sigma}$	$\frac{1}{45}\left(\frac{\mathcal{R}}{6}\right)^2$	$-\frac{2\pi}{45}\frac{\mathcal{R}/6}{V_\Sigma}$	$\frac{2}{945}\left(\frac{\mathcal{R}}{6}\right)^3$
$M_3 \equiv \mathbb{S}^3/D_2^*$	8	-1	$\frac{1}{3}\frac{\mathcal{R}}{6}$	$-\frac{2\pi}{3}\frac{1}{V_\Sigma}$	$\frac{1}{45}\left(\frac{\mathcal{R}}{6}\right)^2$	–	$\frac{2}{945}\left(\frac{\mathcal{R}}{6}\right)^3$
$M_6 \equiv \mathbb{S}^3/T^*$	24	-1	$\frac{1}{3}\frac{\mathcal{R}}{6}$	$-\frac{2\pi}{3}\frac{1}{V_\Sigma}$	$\frac{1}{45}\left(\frac{\mathcal{R}}{6}\right)^2$	–	$\frac{2}{945}\left(\frac{\mathcal{R}}{6}\right)^3$
$M_7 \equiv \mathbb{S}^3/I^*$	120	-1	$\frac{1}{3}\frac{\mathcal{R}}{6}$	$-\frac{2\pi}{3}\frac{1}{V_\Sigma}$	$\frac{1}{45}\left(\frac{\mathcal{R}}{6}\right)^2$	$-\frac{2\pi}{45}\frac{\mathcal{R}/6}{V_\Sigma}$	$\frac{2}{945}\left(\frac{\mathcal{R}}{6}\right)^3$
Hyperbolic (infinite)							
$\mathbb{H}^3$		-1	$\frac{1}{3}\frac{\mathcal{R}}{6}$	0	$\frac{1}{45}\left(\frac{\mathcal{R}}{6}\right)^2$	0	$\frac{2}{945}\left(\frac{\mathcal{R}}{6}\right)^3$

even terms $\Rightarrow$ closed; odd terms $\Rightarrow$ curved

Vigneron & Roukema (2022) arXiv:2201.09102

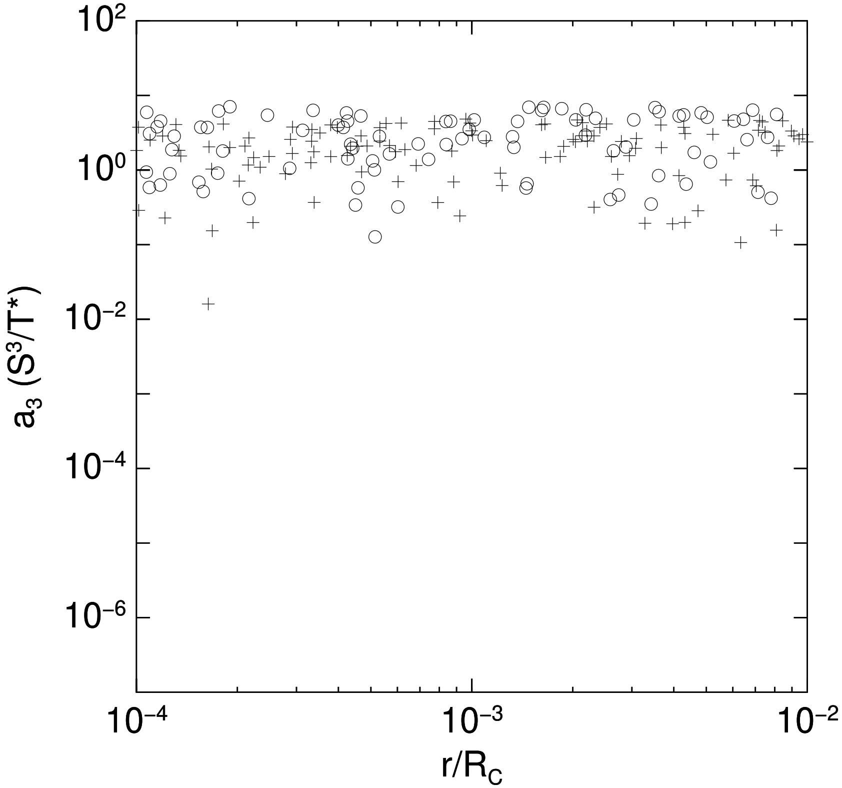
quaternionic space: $\mathbb{S}^3/D_2^*$

$T^3, S^3/\Gamma$



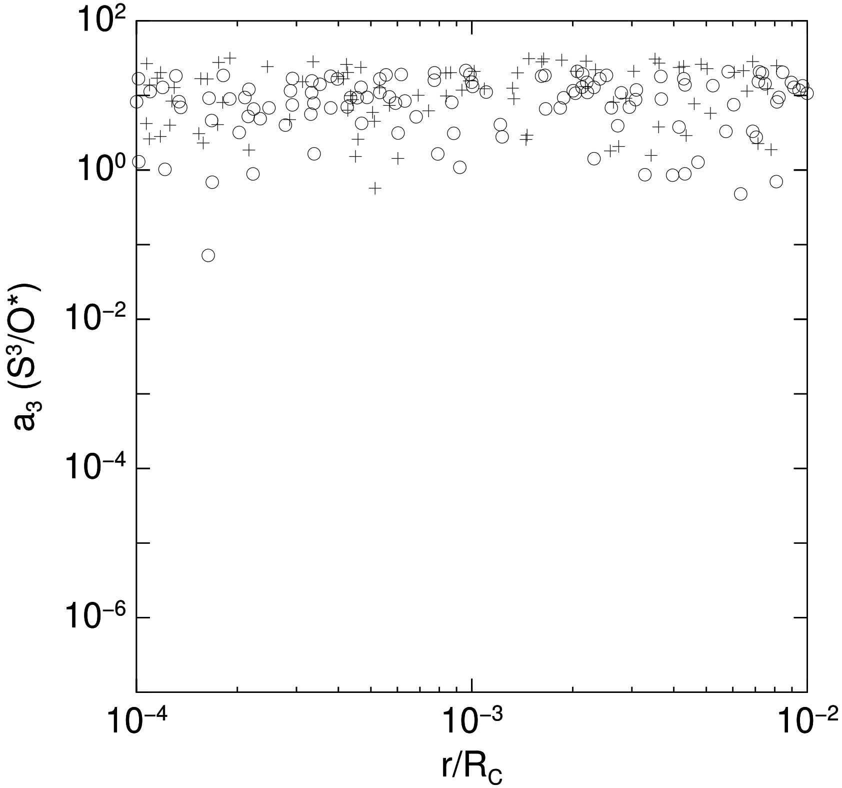
$$T^3 = \mathbb{R}^3/\mathbb{Z}^3 \quad \Rightarrow \quad \ddot{x}_{\text{resid}} \propto (x/L)^3 + \ldots$$

$T^3, S^3/\Gamma$



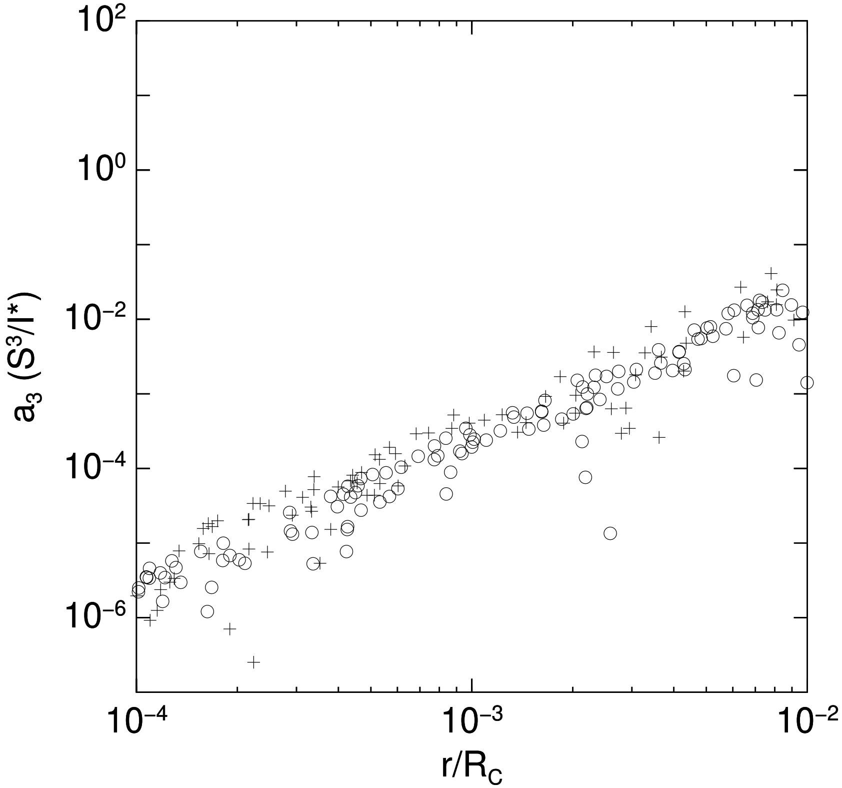
S^3/T^* (octahedral space) $\Rightarrow$ $\ddot{x}_{\mathrm{resid}} \propto (x/R_C)^3 + \ldots$

$T^3, S^3/\Gamma$



S^3/O^* (truncated cube space) $\Rightarrow$ $\ddot{x}_{\mathrm{resid}} \propto (x/R_C)^3 + \ldots$

$T^3, S^3/\Gamma$



S^3/I^* (Poincaré dodecahedral space) $\Rightarrow$ $\ddot{x}_{\text{resid}} \propto (x/R_C)^5 + \ldots$

$T^3, S^3/\Gamma$

- Some spaces are more equal than others.
- Roukema & Różański arXiv:0902.3402, A&A, 502, 27
- Newton–Cartan approach for preparing for full GR approach: Vigneron (2020, PRD) arXiv:2010.10247; Vigneron (2021, PRD) arXiv:2012.10213; Vigneron (2022a, PRD) arXiv:2109.10336; Vigneron (2022b, CQG) arXiv:2201.02112; Vigneron & Roukema (2022) arXiv:2201.09102

Top. accel – standard GR?

- Korotkin & Nikolai (1994) arXiv:gr-qc/9403029 solution: Schwarzschild-like BH in slab space $M = \mathbb{E}^3/\mathbb{Z} \equiv \mathbb{E}^2 \times \mathbb{S}^1$
- outside event horizon, inside topology scale:

$$\ddot{x} = 4\zeta(3)G\,\frac{M}{L^3}x \propto x$$

Ostrowski, Roukema & Buliński (2012) arXiv:1109.1596

$\Rightarrow$ Yes.

Cosmic topol: obs. strategies

empirical strategies: arXiv:astro-ph/0010189

Cosmic topol: obs. strategies

empirical strategies: arXiv:astro-ph/0010189

A. multiple topological images:

Cosmic topol: obs. strategies

empirical strategies: arXiv:astro-ph/0010189

A. multiple topological images:

A.i 3D (grav collapsed objects):

Cosmic topol: obs. strategies

empirical strategies: arXiv:astro-ph/0010189

A. multiple topological images:

A.i 3D (grav collapsed objects):

A.i.1 local isometries—collect "type I pairs" or "local pairs" (pair types)

Cosmic topol: obs. strategies

empirical strategies: arXiv:astro-ph/0010189

A. multiple topological images:

   A.i 3D (grav collapsed objects):

      A.i.1 local isometries—collect "type I pairs" or "local pairs" (pair types)

      A.i.2 cosmic crystallography—collect "type II pairs" or "holonomy pairs"

Cosmic topol: obs. strategies

empirical strategies: arXiv:astro-ph/0010189

A. multiple topological images:

- A.i 3D (grav collapsed objects):
  - A.i.1 local isometries—collect "type I pairs" or "local pairs" (pair types)
  - A.i.2 cosmic crystallography—collect "type II pairs" or "holonomy pairs"
  - A.i.3 successive filters (obs)
  - A.i.4 characteristics of individual objects

Cosmic topol: obs. strategies

A. multiple topological images:

Cosmic topol: obs. strategies

A. multiple topological images:

   A.ii 2D (e.g. microwave background = CMB):

Cosmic topol: obs. strategies

A. multiple topological images:

A.ii 2D (e.g. microwave background = CMB):

A.ii.1 cutoff of large-scale power [also 3D] (obs)

Cosmic topol: obs. strategies

A. multiple topological images:

- A.ii 2D (e.g. microwave background = CMB):
  - A.ii.1 cutoff of large-scale power [also 3D] (obs)
  - A.ii.2 identified circles principle, (obs)

Cosmic topol: obs. strategies

A. multiple topological images:

A.ii 2D (e.g. microwave background = CMB):

A.ii.1 cutoff of large-scale power [also 3D] (obs)

A.ii.2 identified circles principle, (obs)

A.ii.3 matched discs corollary

Cosmic topol: obs. strategies

A. multiple topological images:

A.ii 2D (e.g. microwave background = CMB):

A.ii.1 cutoff of large-scale power [also 3D] (obs)

A.ii.2 identified circles principle, (obs)

A.ii.3 matched discs corollary

A.ii.4 patterns of spots

Cosmic topol: obs. strategies

A. multiple topological images:

  A.ii 2D (e.g. microwave background = CMB):

    A.ii.1 cutoff of large-scale power [also 3D] (obs)

    A.ii.2 identified circles principle, (obs)

    A.ii.3 matched discs corollary

    A.ii.4 patterns of spots

    A.ii.5 perturbation statistics assumptions

Cosmic topol: obs. strategies

A. multiple topological images:

A.ii 2D (e.g. microwave background = CMB):

A.ii.1 cutoff of large-scale power [also 3D] (obs)

A.ii.2 identified circles principle, (obs)

A.ii.3 matched discs corollary

A.ii.4 patterns of spots

A.ii.5 perturbation statistics assumptions

B. other:

Cosmic topol: obs. strategies

A. multiple topological images:

A.ii 2D (e.g. microwave background = CMB):

A.ii.1 cutoff of large-scale power [also 3D] (obs)

A.ii.2 identified circles principle, (obs)

A.ii.3 matched discs corollary

A.ii.4 patterns of spots

A.ii.5 perturbation statistics assumptions

B. other:

B.i cosmic strings

Cosmic topol: obs. strategies

A. multiple topological images:

   A.ii 2D (e.g. microwave background = CMB):

      A.ii.1 cutoff of large-scale power [also 3D] (obs)

      A.ii.2 identified circles principle, (obs)

      A.ii.3 matched discs corollary

      A.ii.4 patterns of spots

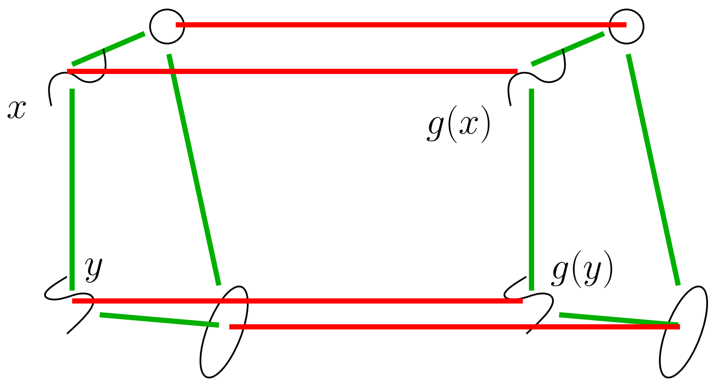
      A.ii.5 perturbation statistics assumptions

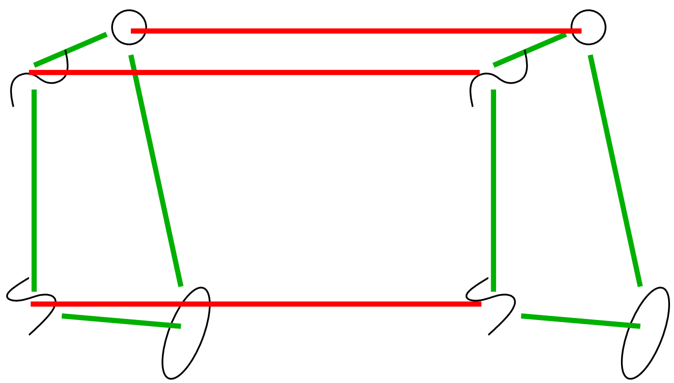
B. other:

   B.i cosmic strings

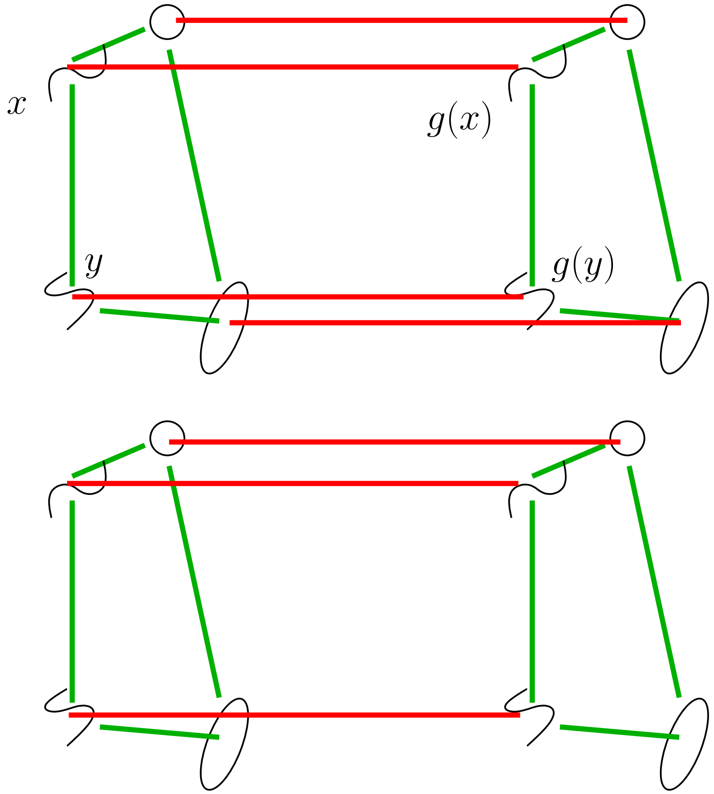
   B.ii topological acceleration

3D strategies—pair types



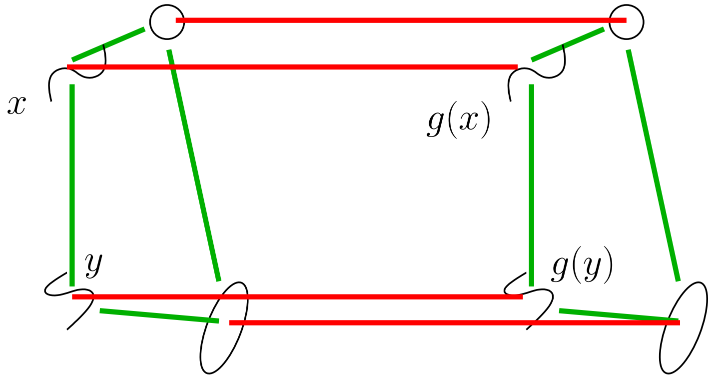


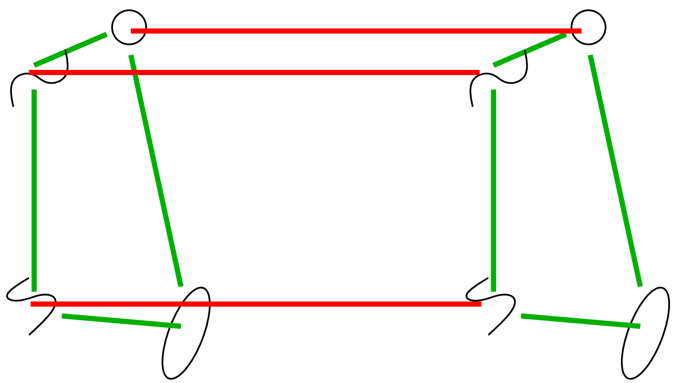
3D strategies—pair types



Type I pairs = local pairs or n-tuples

3D strategies—pair types

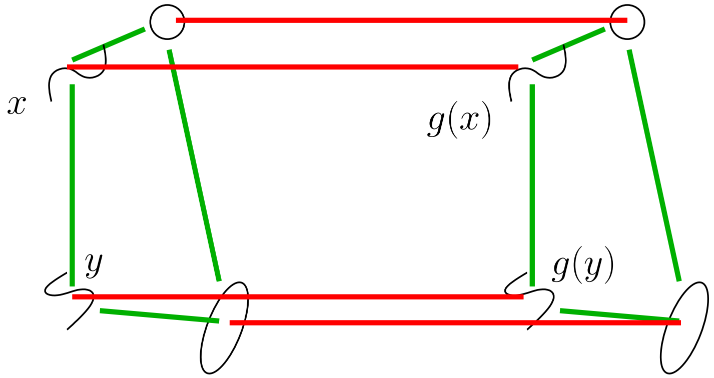


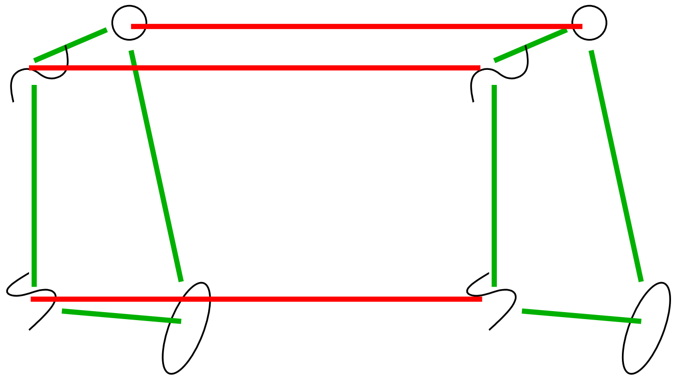


Type I pairs = local pairs or n-tuples

Type II pairs = fundamental length pairs

3D strategies—pair types



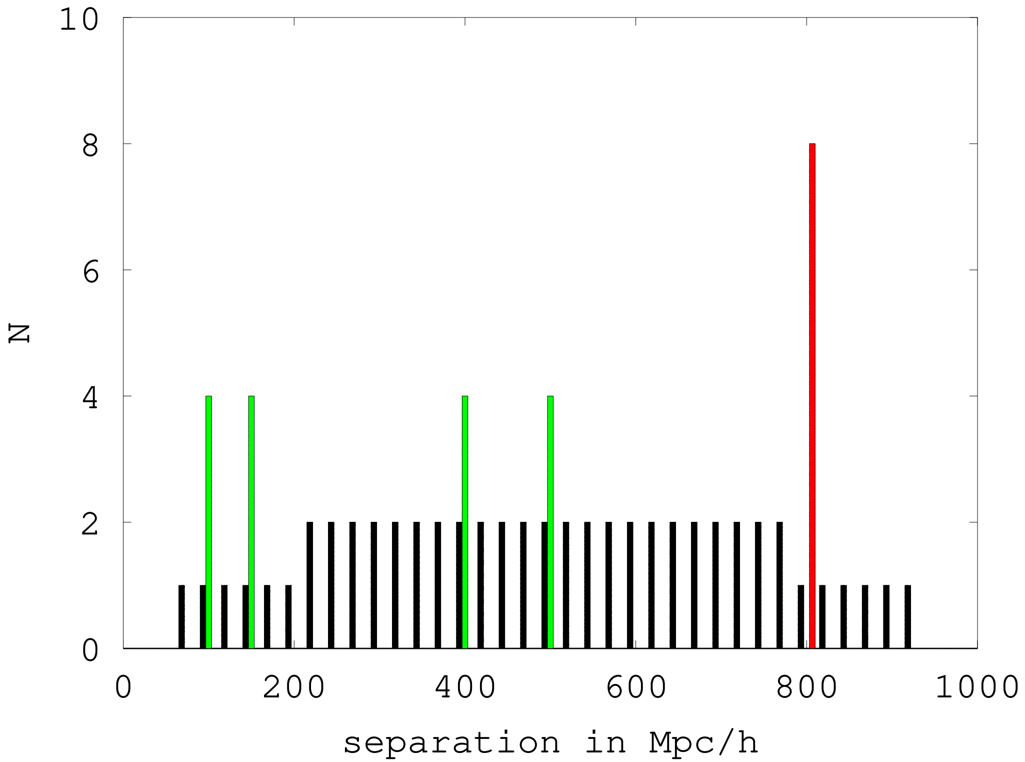


Type I pairs = local pairs or n-tuples

Type II pairs = fundamental length pairs

PSH = pair separation histogram

3D strategies—pair types

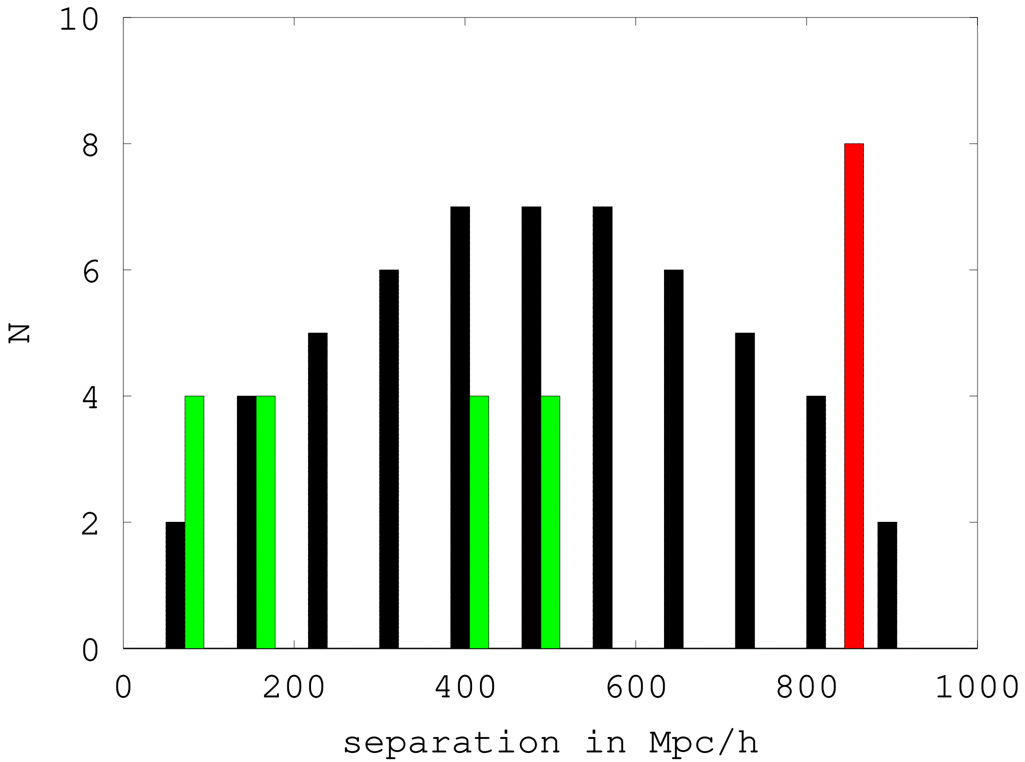


Type I pairs = local pairs or n-tuples

Type II pairs = fundamental length pairs

PSH = pair separation histogram

3D strategies—pair types



Type I pairs = local pairs or n-tuples

Type II pairs = fundamental length pairs

PSH = pair separation histogram

3D strategies—history

- quasar–galaxy alignments, Fagundes (1985) ADS:1985ApJ...291..450F;

3D strategies—history

- quasar–galaxy alignments, Fagundes (1985) ADS:1985ApJ...291..450F;
- opposite QSO pairs: Demiański & Lapucha (1987); Fagundes & Wichoski (1987)

3D strategies—history

- quasar–galaxy alignments, Fagundes (1985) ADS:1985ApJ...291..450F;
- opposite QSO pairs: Demiański & Lapucha (1987); Fagundes & Wichoski (1987)
- type II pair collection: "cosmic crystallography"—Lehoucq, Lachièze-Rey, Luminet (1996) arXiv:gr-qc/9604050

3D strategies—history

- quasar–galaxy alignments, Fagundes (1985) ADS:1985ApJ...291..450F;
- opposite QSO pairs: Demiański & Lapucha (1987); Fagundes & Wichoski (1987)
- type II pair collection: "cosmic crystallography"—Lehoucq, Lachièze-Rey, Luminet (1996) arXiv:gr-qc/9604050
- type I pair or n-tuple collection: Roukema (1996) arXiv:astro-ph/9603052

3D strategies—history

- quasar–galaxy alignments, Fagundes (1985) ADS:1985ApJ...291..450F;
- opposite QSO pairs: Demiański & Lapucha (1987); Fagundes & Wichoski (1987)
- type II pair collection: "cosmic crystallography"—Lehoucq, Lachièze-Rey, Luminet (1996) arXiv:gr-qc/9604050
- type I pair or n-tuple collection: Roukema (1996) arXiv:astro-ph/9603052
- "type I, type II" terminology: Lehoucq, Luminet, Uzan (1999) arXiv:astro-ph/9811107

3D strategies—history

- quasar–galaxy alignments, Fagundes (1985) ADS:1985ApJ...291..450F;
- opposite QSO pairs: Demiański & Lapucha (1987); Fagundes & Wichoski (1987)
- type II pair collection: "cosmic crystallography"—Lehoucq, Lachièze-Rey, Luminet (1996) arXiv:gr-qc/9604050
- type I pair or n-tuple collection: Roukema (1996) arXiv:astro-ph/9603052
- "type I, type II" terminology: Lehoucq, Luminet, Uzan (1999) arXiv:astro-ph/9811107
- successive filters: Marecki, Roukema, Bajtlik (2005)

3D strategies—history

- quasar–galaxy alignments, Fagundes (1985) ADS:1985ApJ...291..450F;
- opposite QSO pairs: Demiański & Lapucha (1987); Fagundes & Wichoski (1987)
- type II pair collection: "cosmic crystallography"—Lehoucq, Lachièze-Rey, Luminet (1996) arXiv:gr-qc/9604050
- type I pair or n-tuple collection: Roukema (1996) arXiv:astro-ph/9603052
- "type I, type II" terminology: Lehoucq, Luminet, Uzan (1999) arXiv:astro-ph/9811107
- successive filters: Marecki, Roukema, Bajtlik (2005)
- quadruples + successive filters + collect membership s of quadruples Fujii & Yoshii (2013)

3D strategies—pair types

- Type I pairs = local pairs or n-tuples

3D strategies—pair types

- Type I pairs = local pairs or n-tuples

  occur for any curvature

3D strategies—pair types

- Type I pairs = local pairs or n-tuples

  occur for any curvature

- Type II pairs = fundamental length pairs

3D strategies—pair types

- Type I pairs = local pairs or n-tuples

  occur for any curvature

- Type II pairs = fundamental length pairs

  occur for some flat and spherical cases:

3D strategies—pair types

- Type I pairs = local pairs or n-tuples

  occur for any curvature

- Type II pairs = fundamental length pairs

  occur for some flat and spherical cases:

  require a holonomy (mapping) $g : x \to g(x)$ for which

$$\forall x, y, \;\; d(x, g(x)) = d(y, g(y))$$

  where d = comoving distance

3D strategies—pair types

- Type I pairs = local pairs or n-tuples

  occur for any curvature

- Type II pairs = fundamental length pairs

  occur for some flat and spherical cases:

  require a holonomy (mapping) $g : x \to g(x)$ for which

  $$\forall x, y, \;\; d(x, g(x)) = d(y, g(y))$$

  where d = comoving distance
  this defines a: *Clifford translation*

- Clifford translation examples: T^3: yes;

3D strategies—pair types

- Type I pairs = local pairs or n-tuples

  occur for any curvature

- Type II pairs = fundamental length pairs

  occur for some flat and spherical cases:

  require a holonomy (mapping) $g : x \to g(x)$ for which

$$\forall x, y, \;\; d(x, g(x)) = d(y, g(y))$$

  where d = comoving distance
  this defines a: *Clifford translation*

- Clifford translation examples: T^3: yes; S^2: rotations = no;

3D strategies—pair types

- Type I pairs = local pairs or n-tuples

  occur for any curvature

- Type II pairs = fundamental length pairs

  occur for some flat and spherical cases:

  require a holonomy (mapping) $g : x \to g(x)$ for which

$$\forall x, y, \;\; d(x, g(x)) = d(y, g(y))$$

  where d = comoving distance
  this defines a: *Clifford translation*

- Clifford translation examples: T^3: yes; S^2: rotations = no; S^3: pair of orthogonal rotations = possible

AGNs—successive filters

Marecki, Roukema, Bajtlik (2005) arXiv:astro-ph/0412181

- method valid for T^3:
- Type II pairs +

AGNs—successive filters

Marecki, Roukema, Bajtlik (2005) arXiv:astro-ph/0412181

- method valid for T^3:
- Type II pairs +
- vectorial filter ($\equiv$ require type I pairs) +

AGNs—successive filters

Marecki, Roukema, Bajtlik (2005) arXiv:astro-ph/0412181

- method valid for T^3:
- Type II pairs +
- vectorial filter ($\equiv$ require type I pairs) +
- δt filter—short QSO lifetimes +

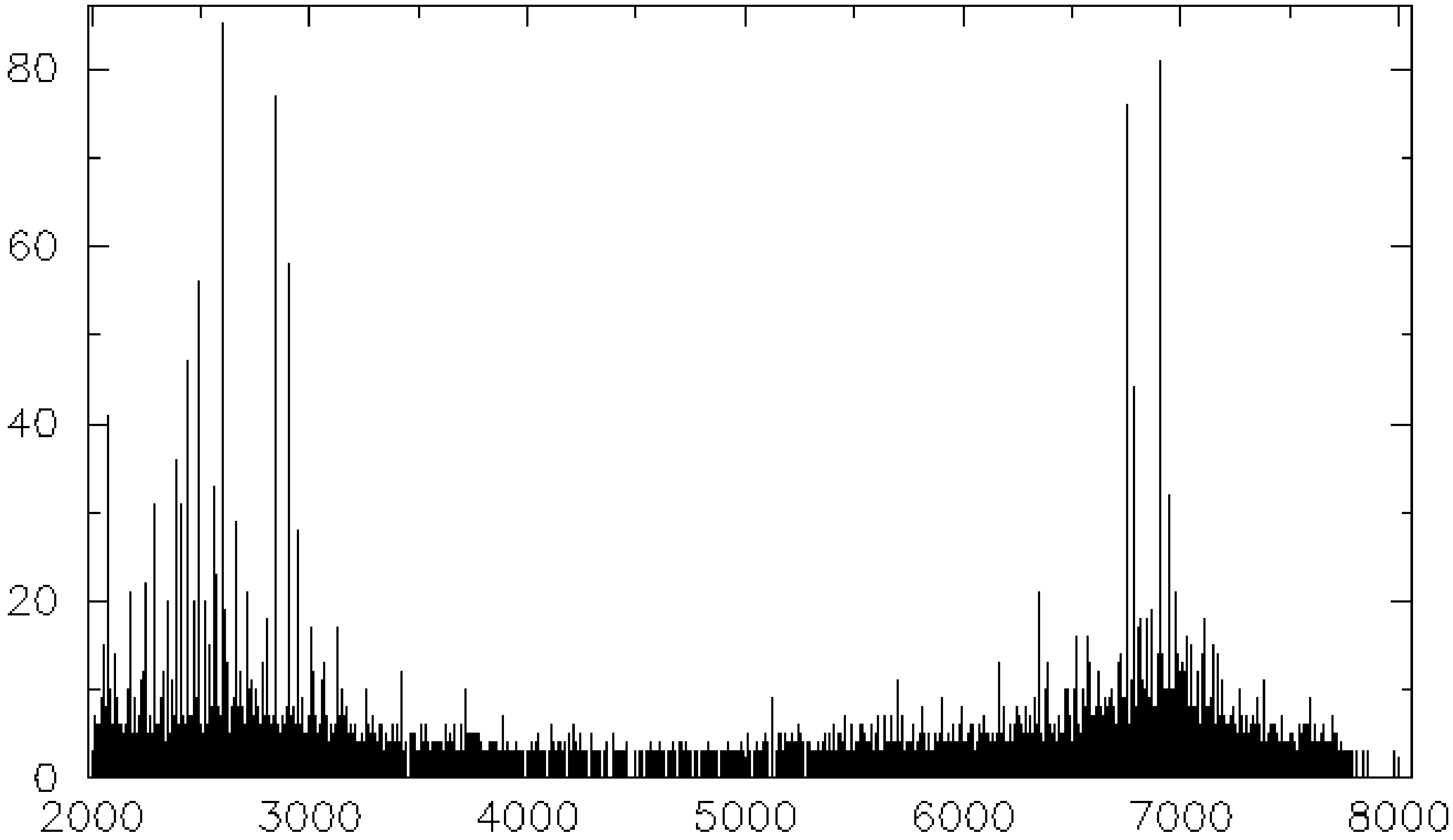
AGNs—successive filters

Marecki, Roukema, Bajtlik (2005) arXiv:astro-ph/0412181

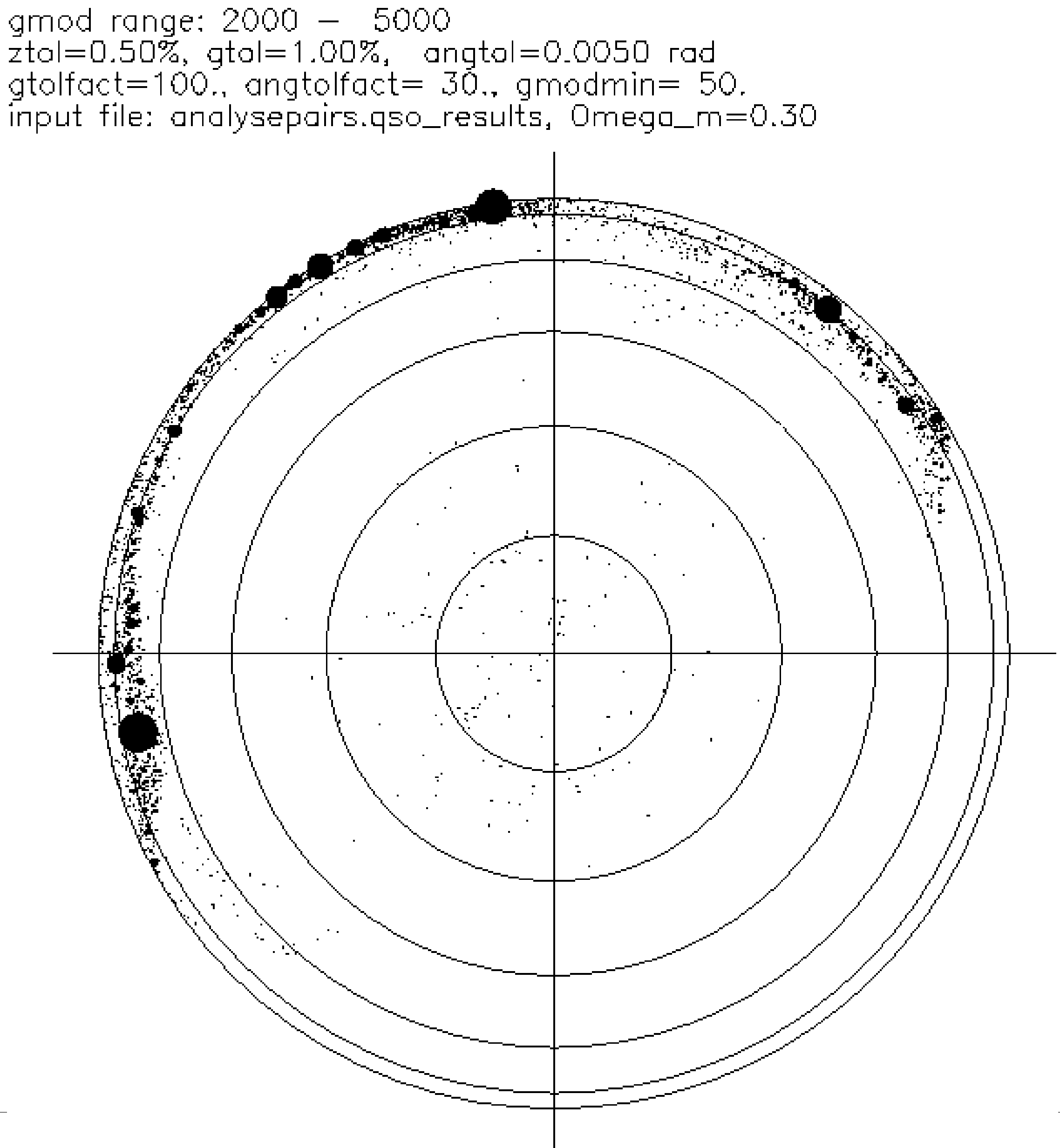
- method valid for T^3:
- Type II pairs +
- vectorial filter ($\equiv$ require type I pairs) +
- δt filter—short QSO lifetimes +
- anti-selection effect filter

AGNs—successive filters

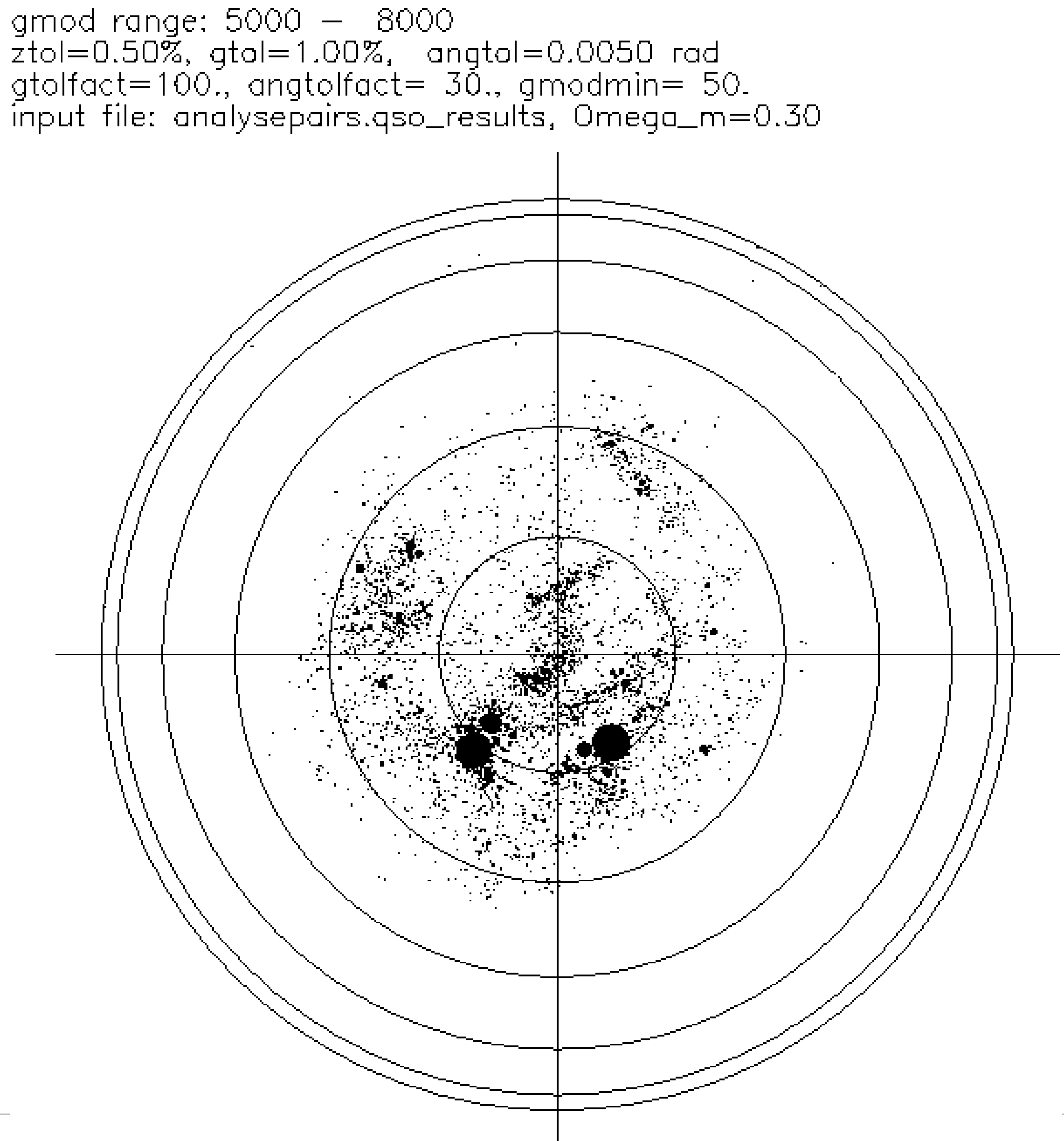
Marecki, Roukema, Bajtlik (2005) arXiv:astro-ph/0412181



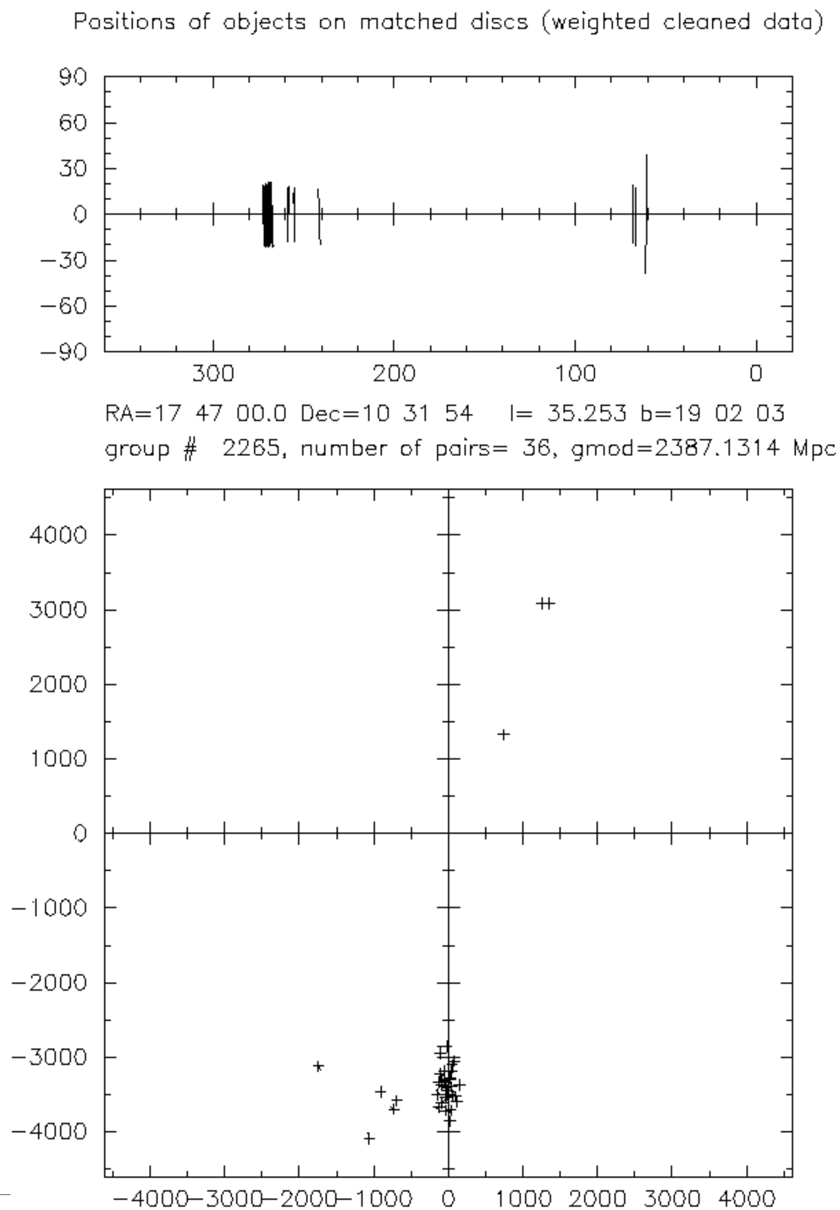
AGN Catalogues



AGN Catalogues



AGN Catalogues



AGN Catalogues

Marecki et al. (2005) main results:

- AGN short lifetimes $\Rightarrow$ redshift filter to improve S/N

AGN Catalogues

Marecki et al. (2005) main results:

- AGN short lifetimes $\Rightarrow$ redshift filter to improve S/N
- application to large AGN catalogue compilation revealed apparent signals

AGN Catalogues

Marecki et al. (2005) main results:

- AGN short lifetimes $\Rightarrow$ redshift filter to improve S/N
- application to large AGN catalogue compilation revealed apparent signals
- closer analysis $\Rightarrow$ these are selection effects

AGN Catalogues

Marecki et al. (2005) main results:

- AGN short lifetimes $\Rightarrow$ redshift filter to improve S/N
- application to large AGN catalogue compilation revealed apparent signals
- closer analysis $\Rightarrow$ these are selection effects
- no signal found in compilation of radio-loud AGNs (RLAGNs)

AGNs—successive filters

Fujii & Yoshii (2013) arXiv:1103.1466

- method valid for compact flat spaces:

AGNs—successive filters

Fujii & Yoshii (2013) arXiv:1103.1466

- method valid for compact flat spaces:
- pairs of Type II pairs = quadruples +

AGNs—successive filters

Fujii & Yoshii (2013) arXiv:1103.1466

- method valid for compact flat spaces:
- pairs of Type II pairs = quadruples +
- require type I pairs , i.e. require:

AGNs—successive filters

Fujii & Yoshii (2013) arXiv:1103.1466

- method valid for compact flat spaces:
- pairs of Type II pairs = quadruples +
- require type I pairs , i.e. require:
  - translations (= vectorial filter), or

AGNs—successive filters

Fujii & Yoshii (2013) arXiv:1103.1466

- method valid for compact flat spaces:
- pairs of Type II pairs = quadruples +
- require type I pairs , i.e. require:
  - translations (= vectorial filter), or
  - half-turn corkscrew motion and glide reflection, or

AGNs—successive filters

Fujii & Yoshii (2013) arXiv:1103.1466

- method valid for compact flat spaces:
- pairs of Type II pairs = quadruples +
- require type I pairs , i.e. require:
  - translations (= vectorial filter), or
  - half-turn corkscrew motion and glide reflection, or
  - n-th–turn corkscrew motion, $n \in \{4, 3, 6\}$

AGNs—successive filters

Fujii & Yoshii (2013) arXiv:1103.1466

- method valid for compact flat spaces:
- pairs of Type II pairs = quadruples +
- require type I pairs (2nd filter) +
- δt filter—short QSO lifetimes +

AGNs—successive filters

Fujii & Yoshii (2013) arXiv:1103.1466

- method valid for compact flat spaces:
- pairs of Type II pairs = quadruples +
- require type I pairs (2nd filter) +
- δt filter—short QSO lifetimes +
- collect n-tuples:

AGNs—successive filters

Fujii & Yoshii (2013) arXiv:1103.1466

- method valid for compact flat spaces:
- pairs of Type II pairs = quadruples +
- require type I pairs (2nd filter) +
- δt filter—short QSO lifetimes +
- collect n-tuples:
- each i-th object $\in$ s_i quadruples

AGNs—successive filters

Fujii & Yoshii (2013) arXiv:1103.1466

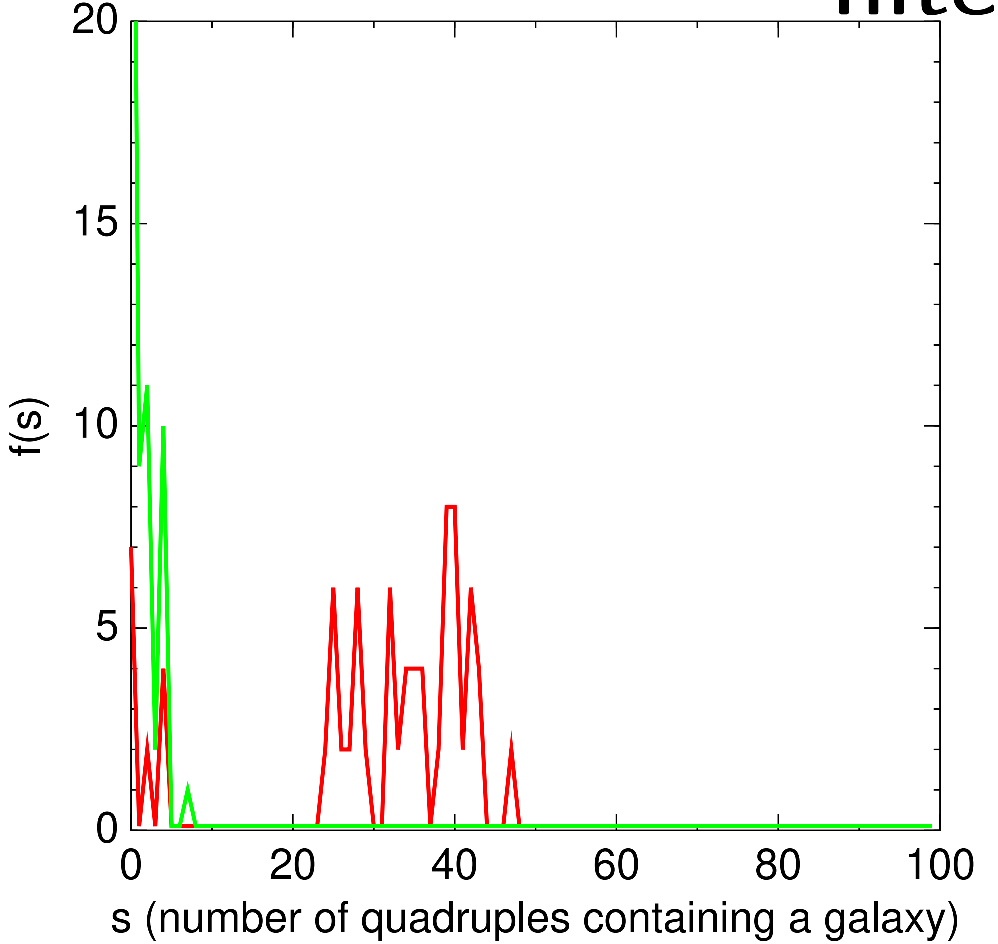
- method valid for compact flat spaces:
- pairs of Type II pairs = quadruples +
- require type I pairs (2nd filter) +
- δt filter—short QSO lifetimes +
- collect n-tuples:
- each i-th object $\in s_i$ quadruples
- plot histogram of frequency of s values

AGNs—successive filters

Fujii & Yoshii (2013) arXiv:1103.1466

- method valid for compact flat spaces:
- pairs of Type II pairs = quadruples +
- require type I pairs (2nd filter) +
- δt filter—short QSO lifetimes +
- collect n-tuples:
- each i-th object $\in s_i$ quadruples
- plot histogram of frequency of s values

AGN Catalogues—successive filters

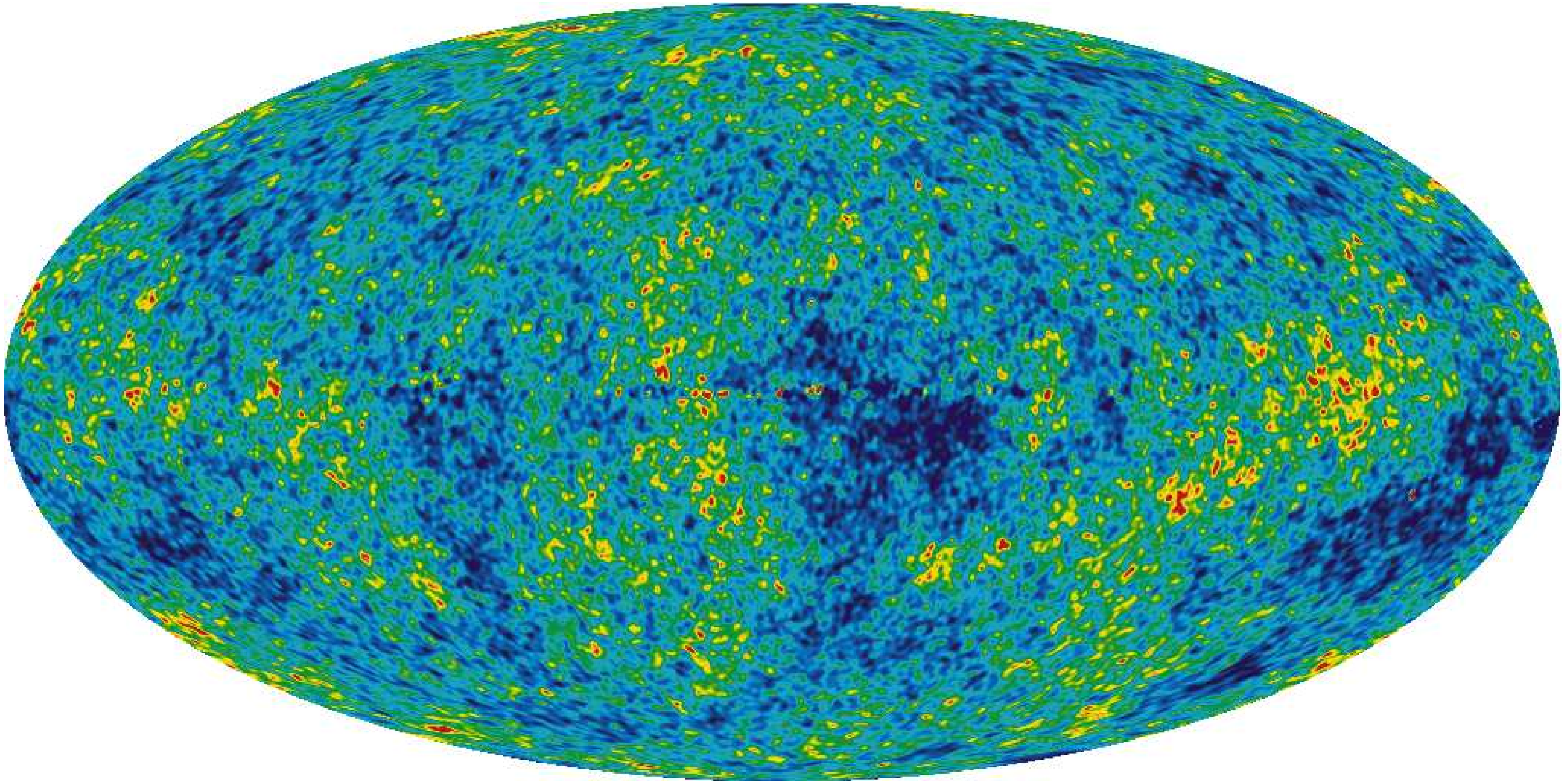


simulation of s histogram for Lyman break galaxies (LBGs) at $z \approx 6$

green: simply connected; red: T^3

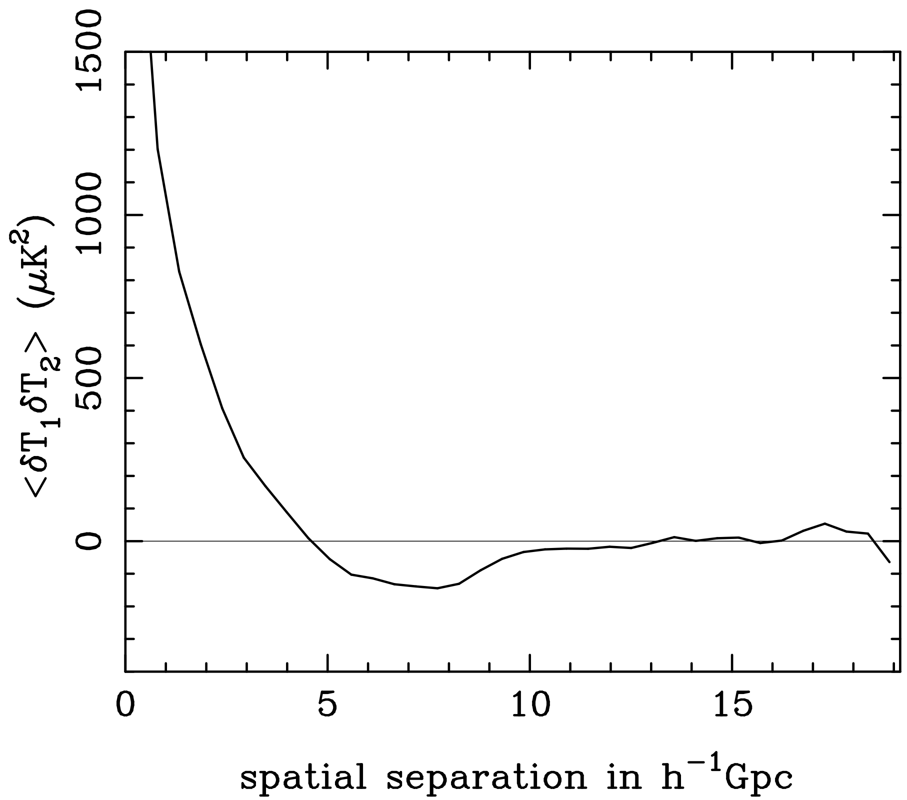
ADS:2014MNRAS.437.1096R

2D methods: structure cutoff



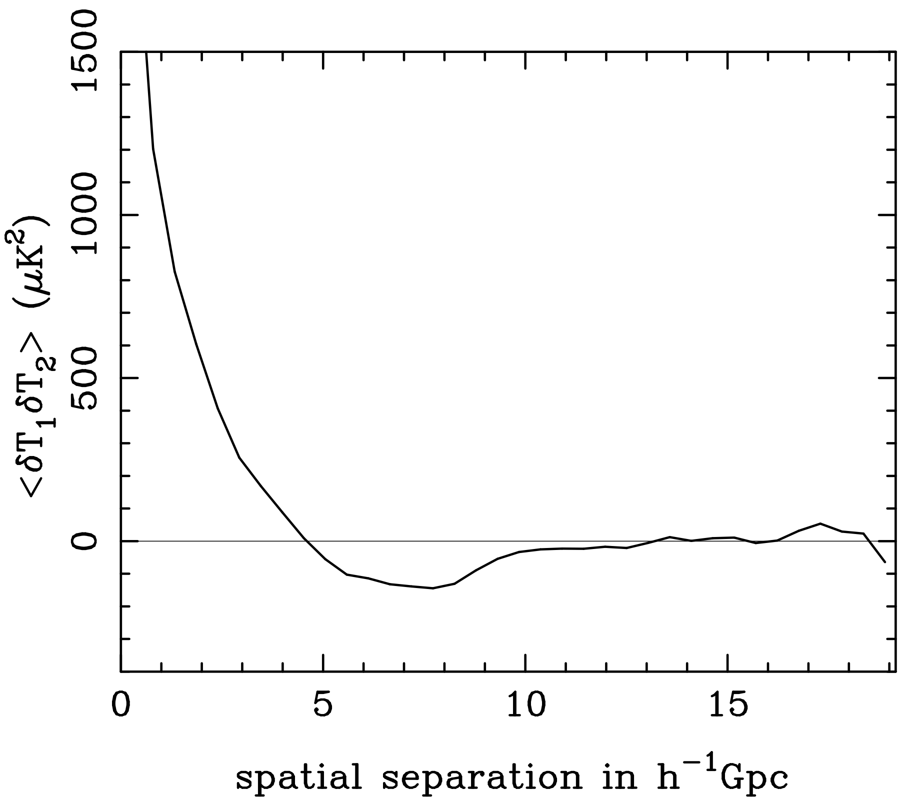
WMAP 5yr ILC (internal linear combination)

2D methods: structure cutoff



3D: structures bigger than FD cannot exist

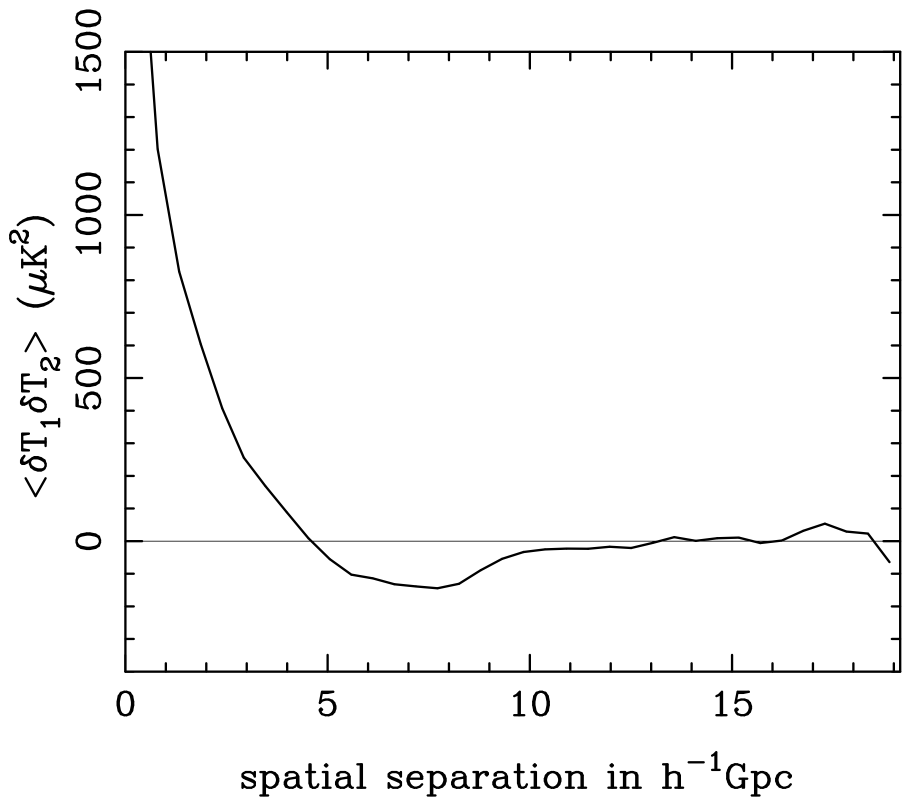
2D methods: structure cutoff



3D: structures bigger than FD cannot exist

roughly $\Rightarrow$ 2D structure cutoff

2D methods: structure cutoff



3D: structures bigger than FD cannot exist

roughly $\Rightarrow$ 2D structure cutoff

Starobinsky (1993); Stevens et al. (1993)

The Identified Circles Principle

- discovery of principle: Cornish, Spergel & Starkman (1996)

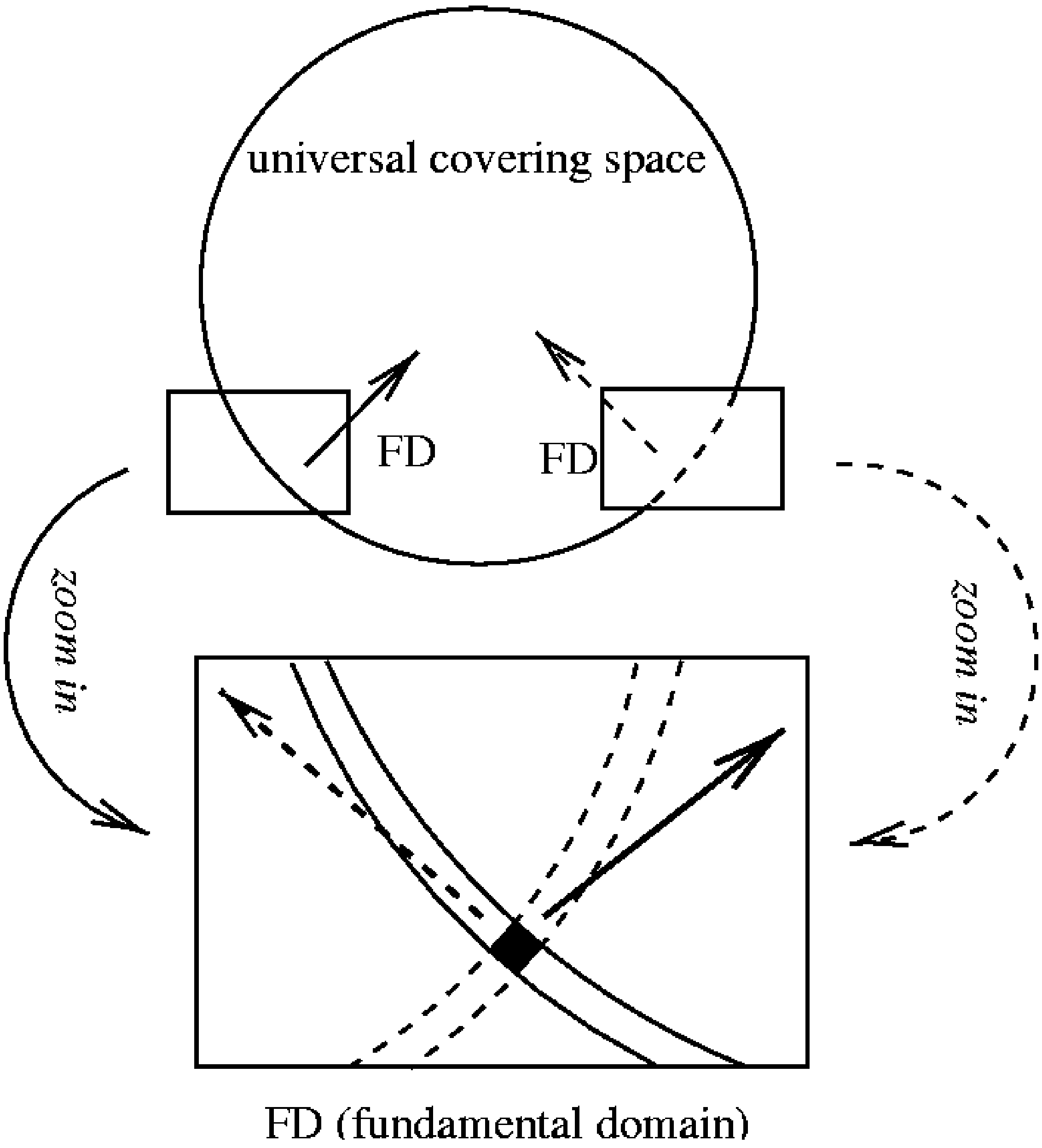
The Identified Circles Principle

- discovery of principle: Cornish, Spergel & Starkman (1996)
- original article only as preprint: arXiv:gr-qc/9602039

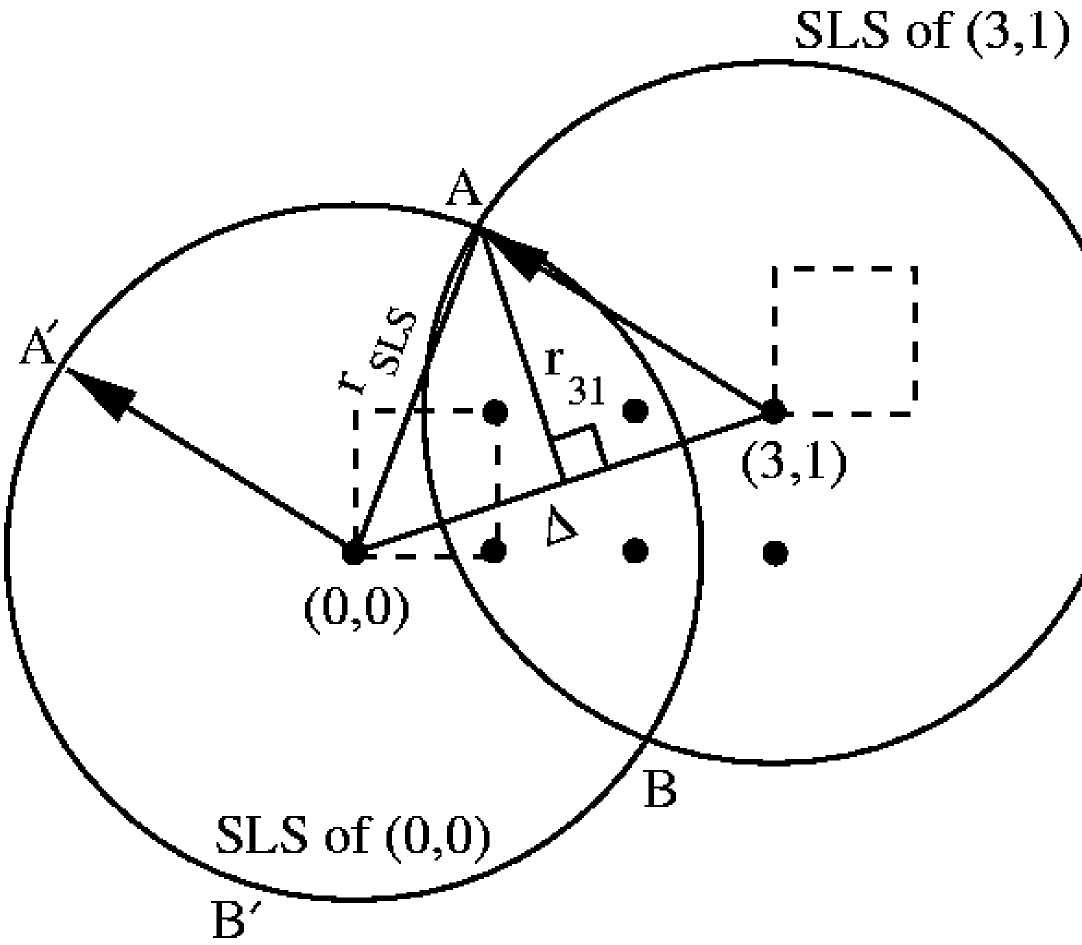
The Identified Circles Principle

- discovery of principle: Cornish, Spergel & Starkman (1996)
- original article only as preprint: arXiv:gr-qc/9602039
- closed access peer-reviewed article: CQG, 15, 2657 (1998)

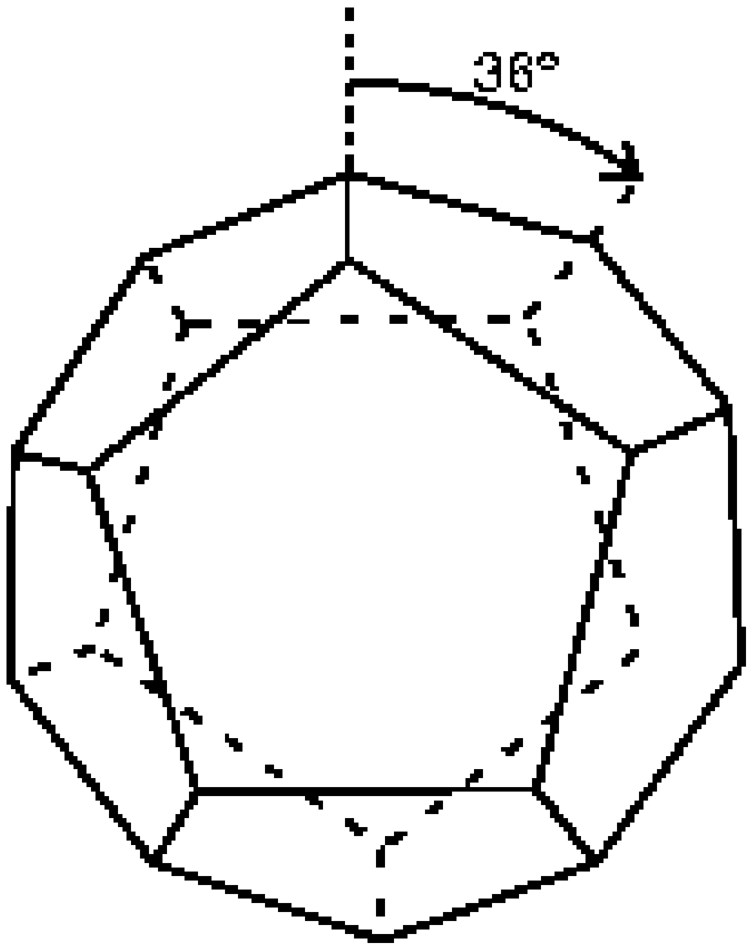
The Identified Circles Principle



The Identified Circles Principle

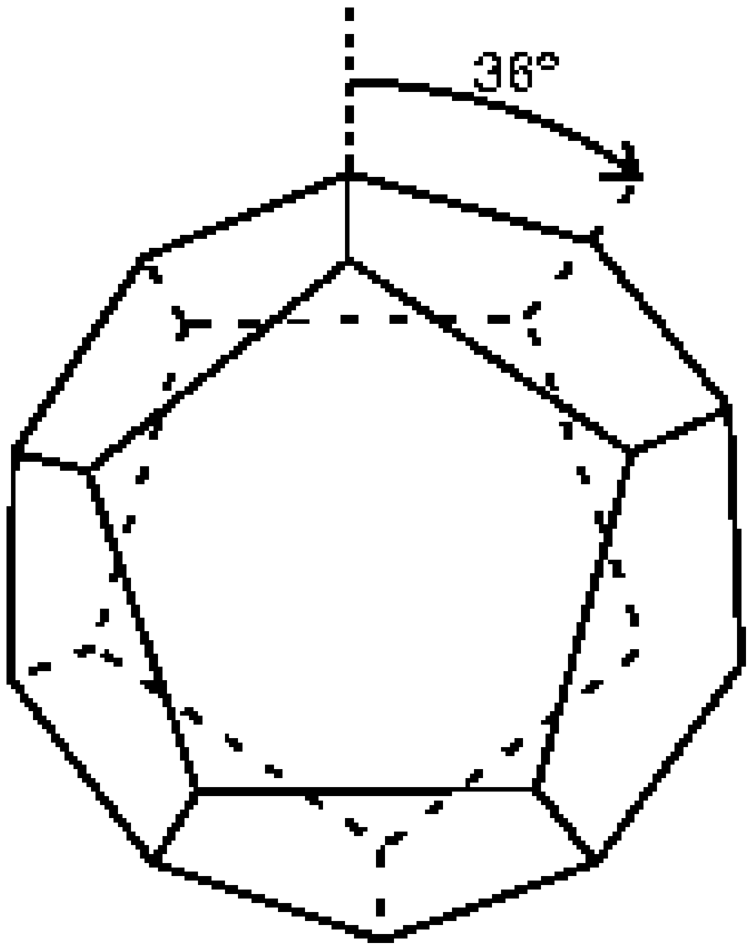


The Poincaré Dodecahedral 3-Manifold



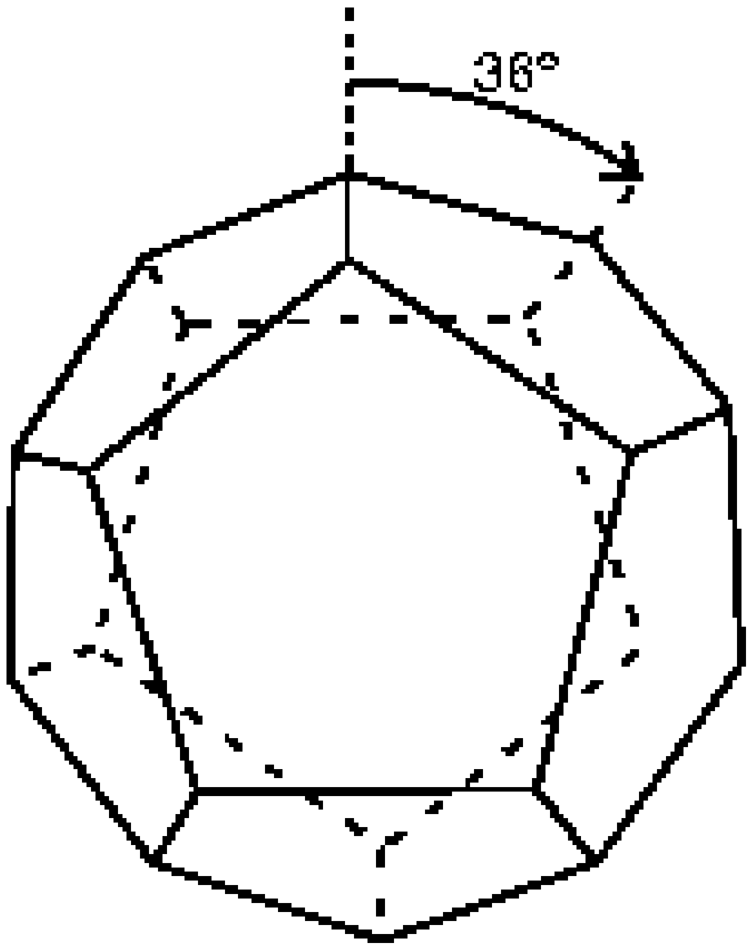
- FD = positively curved dodecahedron

The Poincaré Dodecahedral 3-Manifold



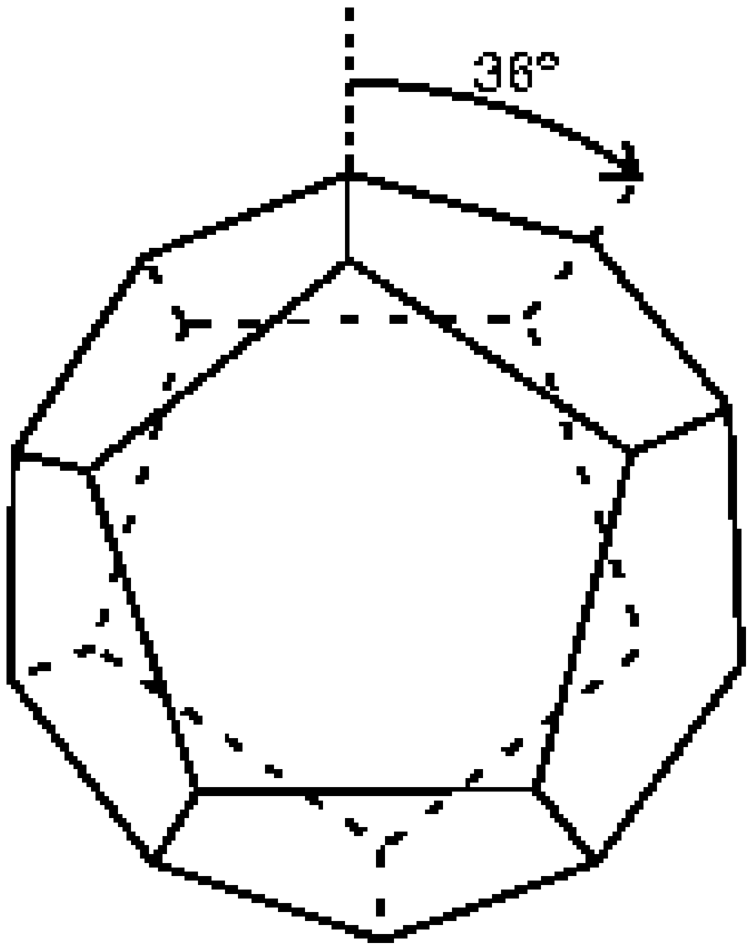
- FD = positively curved dodecahedron
- covering space is S^3

The Poincaré Dodecahedral 3-Manifold



- FD = positively curved dodecahedron
- covering space is S^3
- 120 copies of FD tile (exactly fill) S^3

The Poincaré Dodecahedral 3-Manifold



- FD = positively curved dodecahedron
- covering space is S^3
- 120 copies of FD tile (exactly fill) S^3
- Luminet et al. (2003): S^3/I^* favoured by WMAP statistics

Optimal cross-correlation method

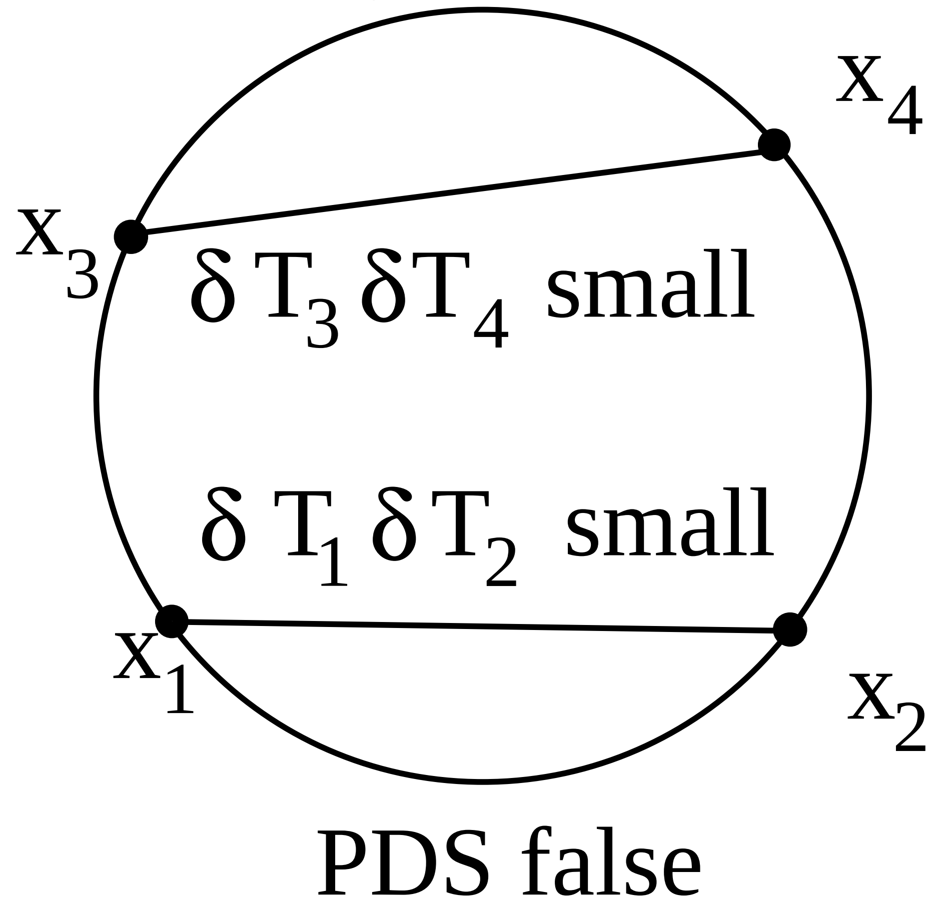
- extension to identified circles principle:

Optimal cross-correlation method

- for a given manifold, e.g. S^3/I^*:

Optimal cross-correlation method

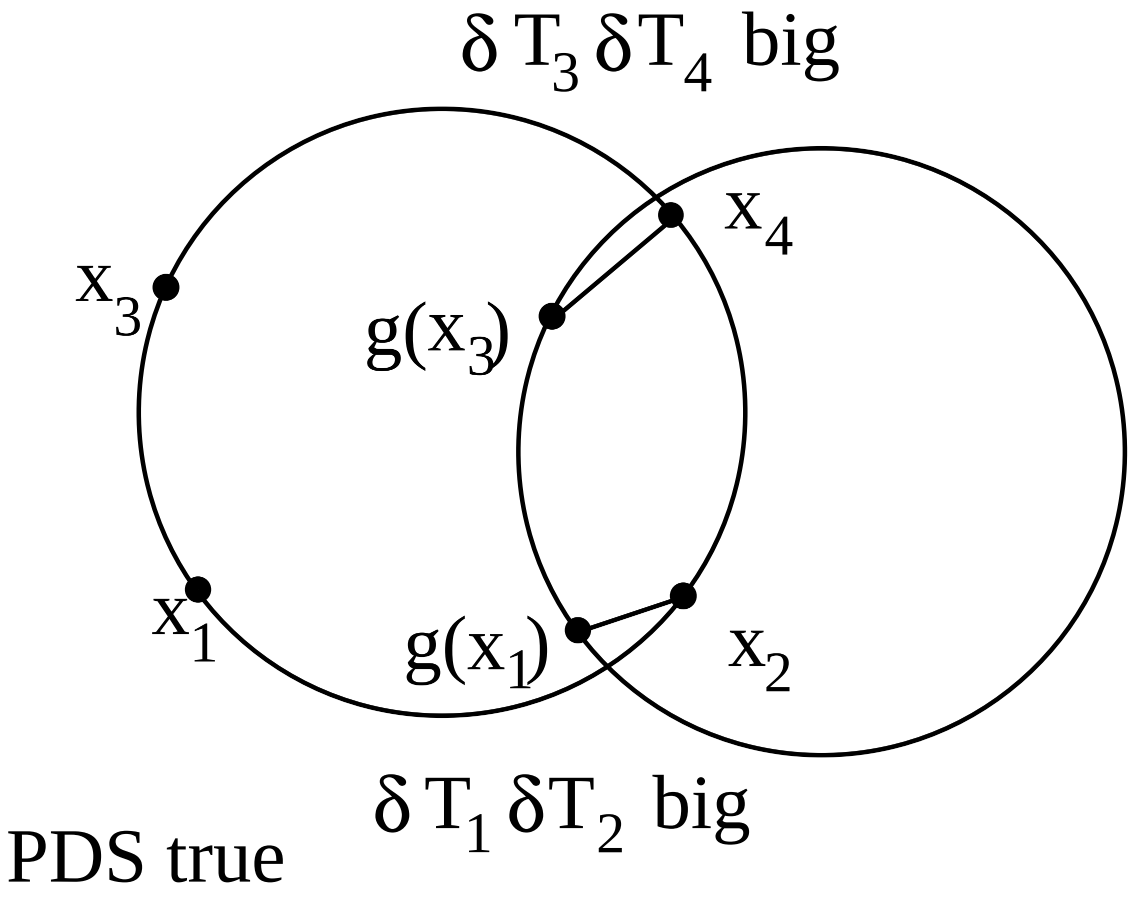
- for a given manifold, e.g.



S^3/I^*:

Optimal cross-correlation method

- for a given manifold, e.g. S^3/I^*:



Optimal cross-correlation method

- for a given manifold, e.g. S^3/I^*:
- prediction: $\exists$ solution $(l, b, \theta, \alpha, \phi = \pm 36°)$ where

Optimal cross-correlation method

- for a given manifold, e.g. S^3/I^*:
- prediction: $\exists$ solution $(l, b, \theta, \alpha, \phi = \pm 36^\circ)$ where
- $\xi_{\mathrm{C}}(r \lesssim 4\mathsf{h}^{-1}$ Gpc) $\sim \xi_{\mathrm{A}}(r \lesssim 4\mathsf{h}^{-1}$ Gpc) (cf ξ_{A}(WMAP))

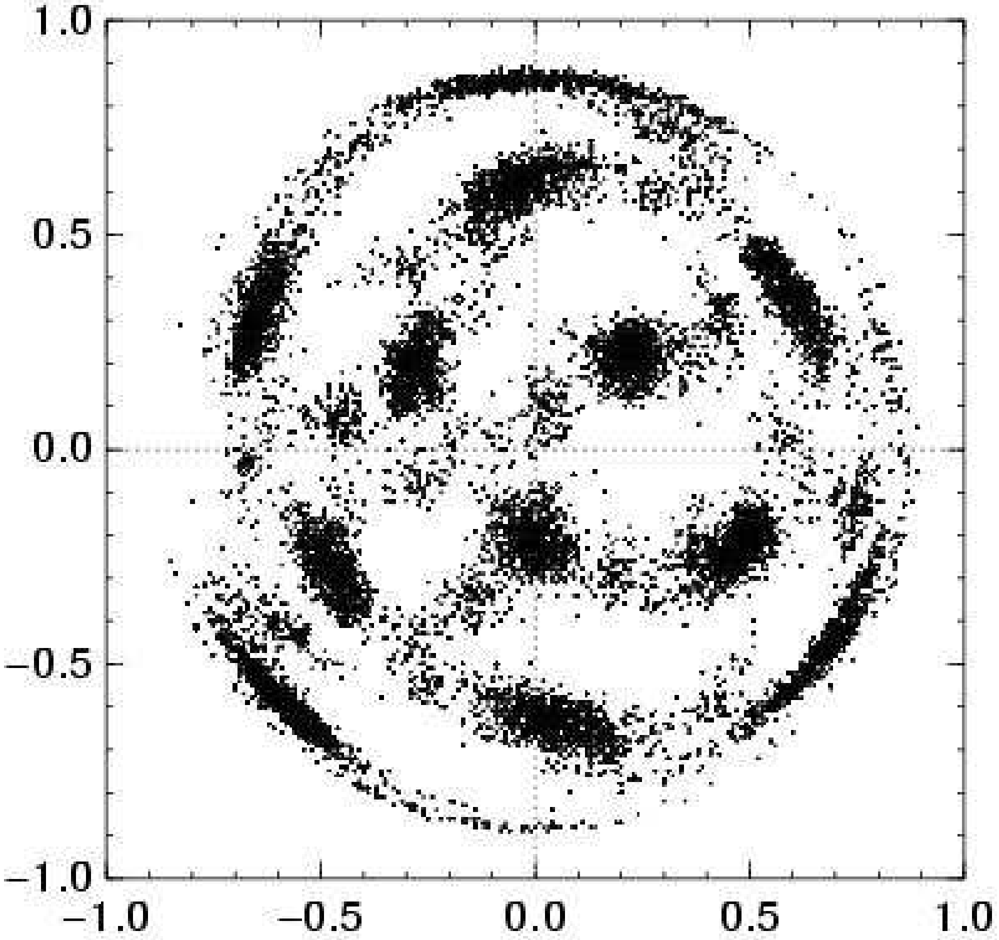
Optimal cross-correlation method

- for a given manifold, e.g. S^3/I^*:
- prediction: $\exists$ solution $(l, b, \theta, \alpha, \phi = \pm 36^\circ)$ where
- $\xi_{\rm C}(r \lesssim 4\mathsf{h}^{-1}$ Gpc$) \sim \xi_{\rm A}(r \lesssim 4\mathsf{h}^{-1}$ Gpc$)$ (cf $\underline{\xi_{\rm A}(\text{WMAP})}$)
- search for best solution with Monte Carlo Markov Chain

Optimal cross-correlation method

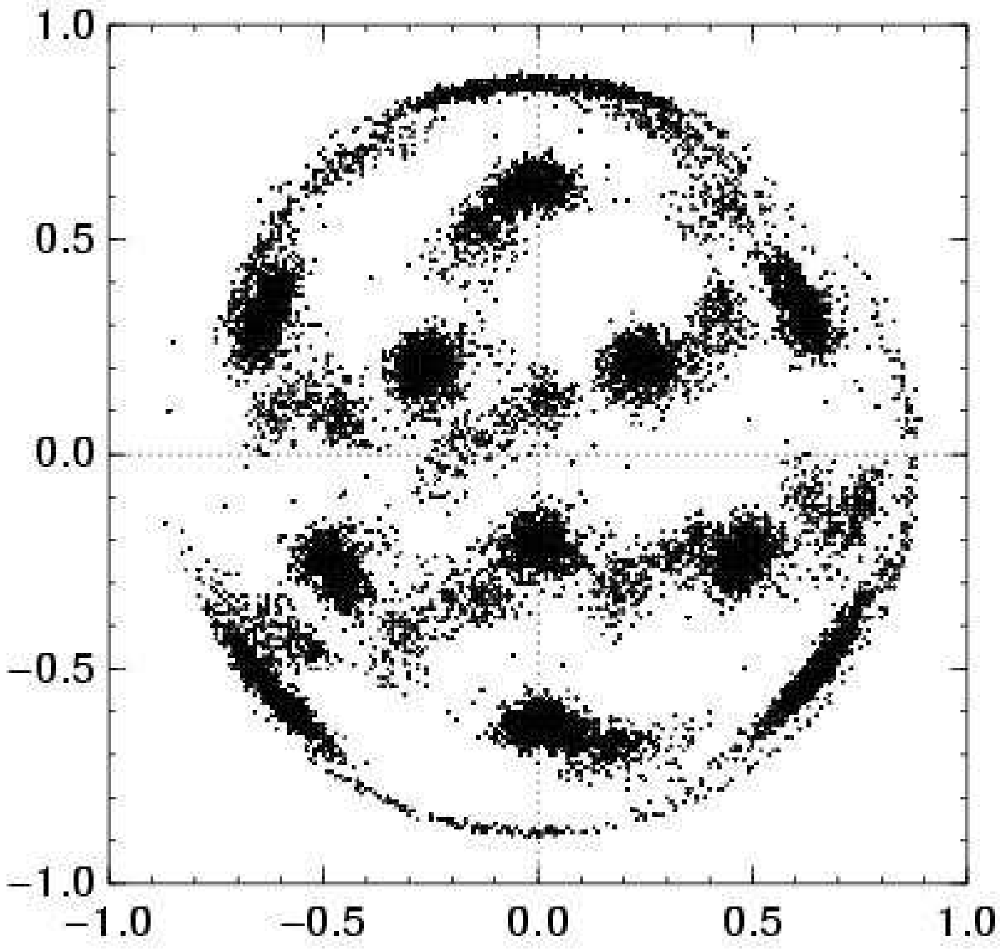
- for a given manifold, e.g. S^3/I^*:
- prediction: $\exists$ solution $(l, b, \theta, \alpha, \phi = \pm 36°)$ where
- $\xi_{\rm C}(r \lesssim 4\mathsf{h}^{-1}$ Gpc) $\sim \xi_{\rm A}(r \lesssim 4\mathsf{h}^{-1}$ Gpc) (cf $\xi_{\rm A}$(WMAP))
- search for best solution with Monte Carlo Markov Chain
- allow arbitrary ϕ so that accidental correlations are likely to give an invalid value $\phi \neq \pm 36°$

WMAP + Poincaré S^3/I^*



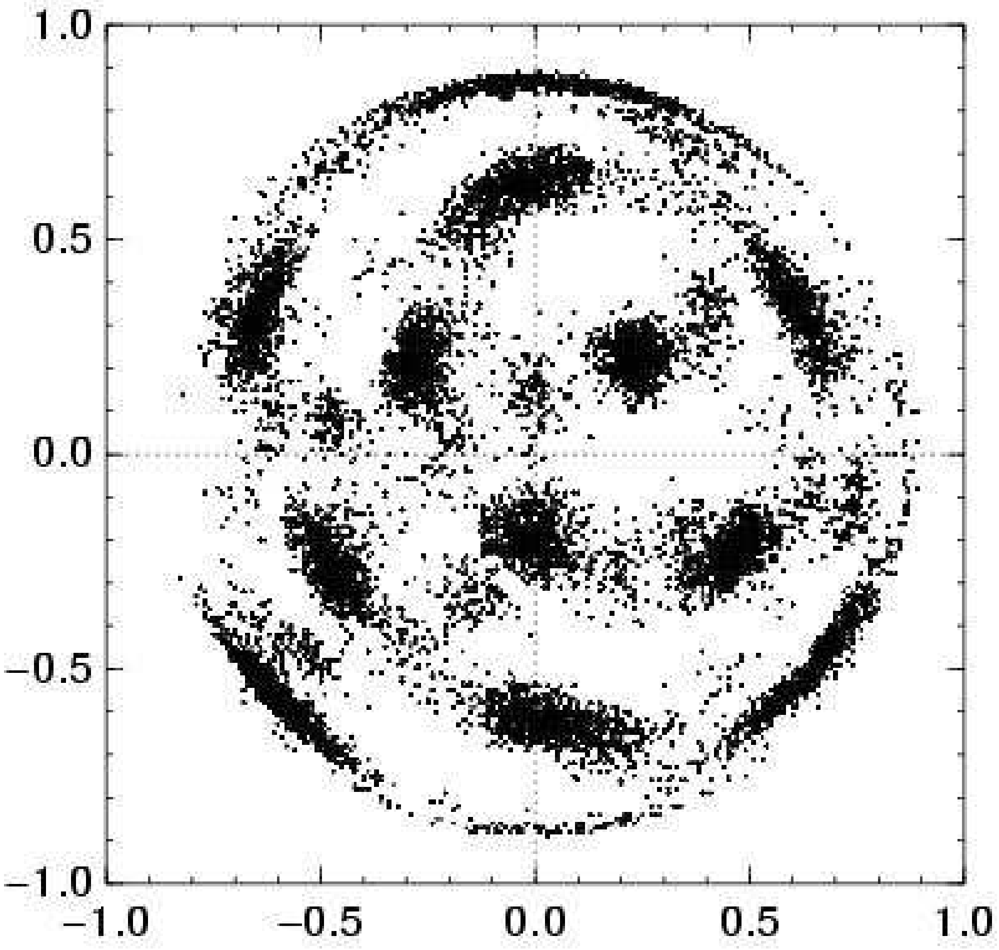
ILC + kp2

WMAP + Poincaré S^3/I^*



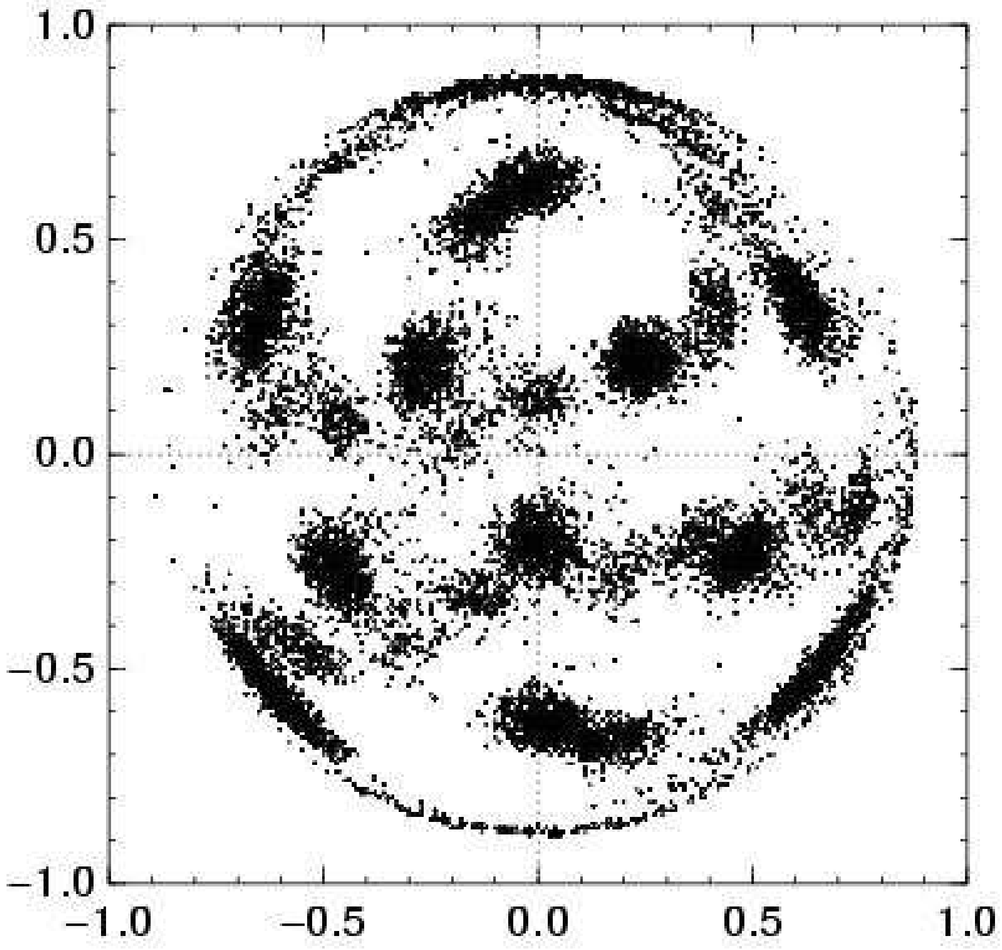
ILC + nomask

WMAP + Poincaré S^3/I^*



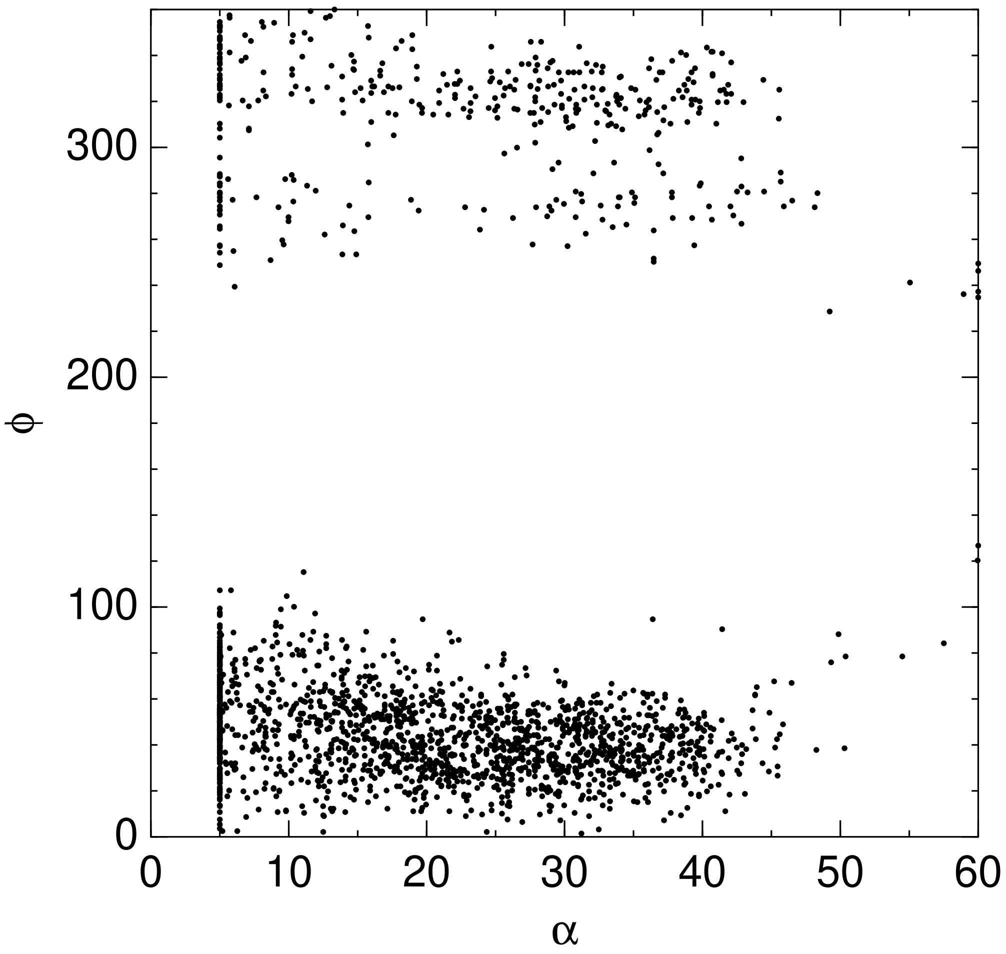
TOH + kp2

WMAP + Poincaré S^3/I^*



TOH + nomask

WMAP + Poincaré S^3/I^*

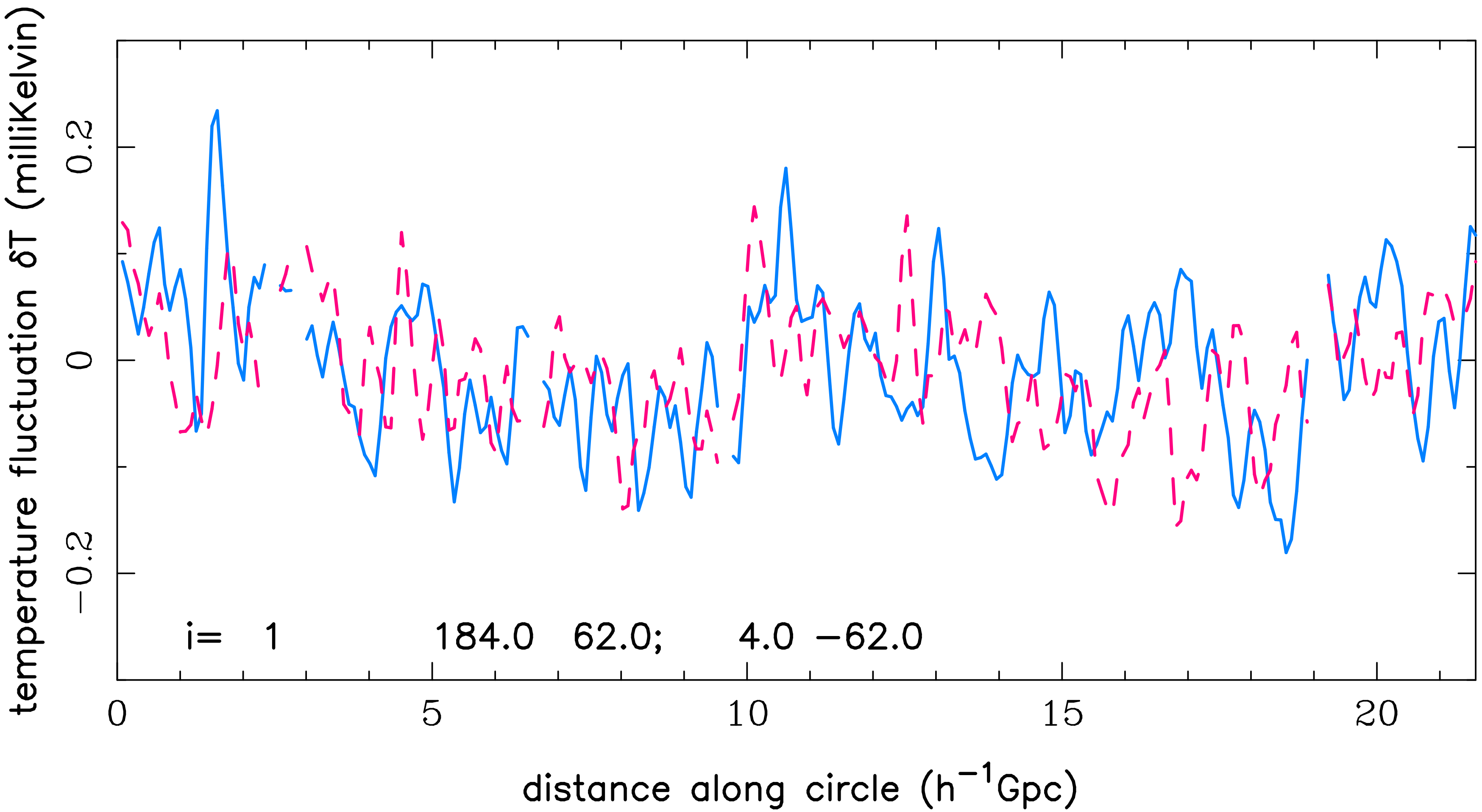


ϕ vs matched circle size α

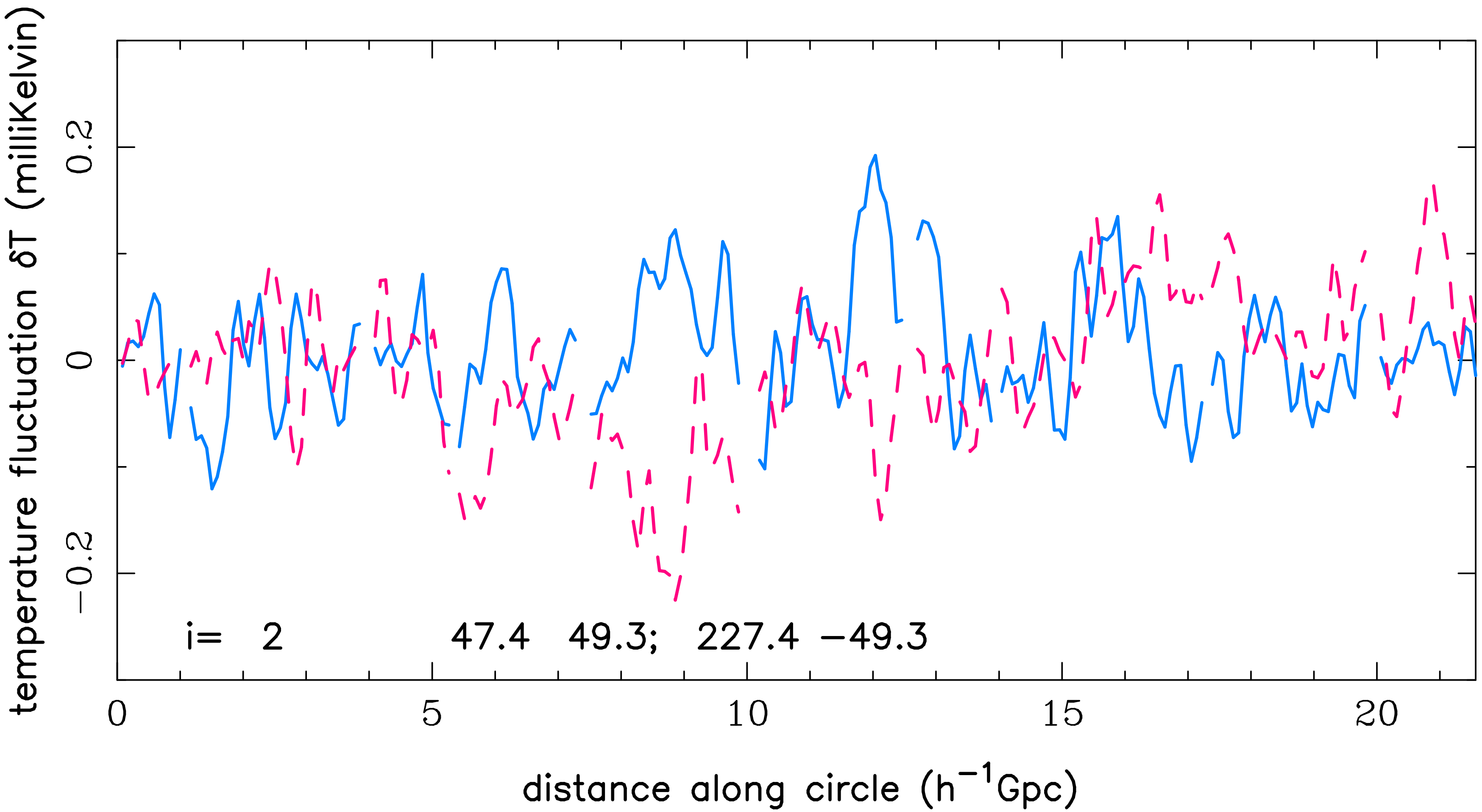
WMAP + Poincaré S^3/I^*

$P_{\min}$	n	α°	$\sigma^\circ_{\langle\alpha\rangle}$	ϕ°	$\sigma^\circ_{\langle\phi\rangle}$
0.4	12589.0	20.6	0.6	**39.0**	**2.4**
0.5	6537.5	20.8	0.7	**38.7**	**2.2**
0.6	2961.0	22.1	0.5	**37.4**	**2.1**

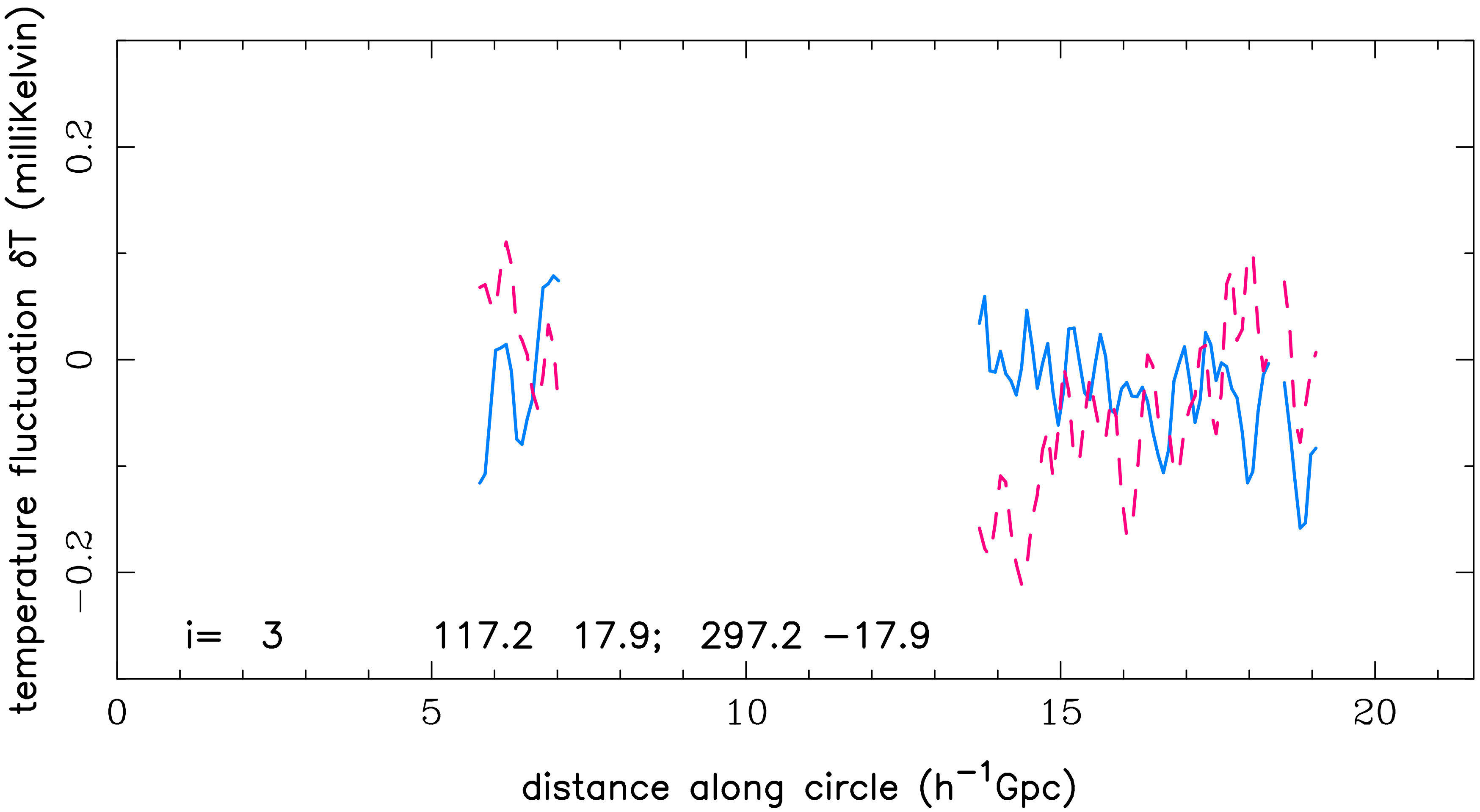
RBSG08 matched circles



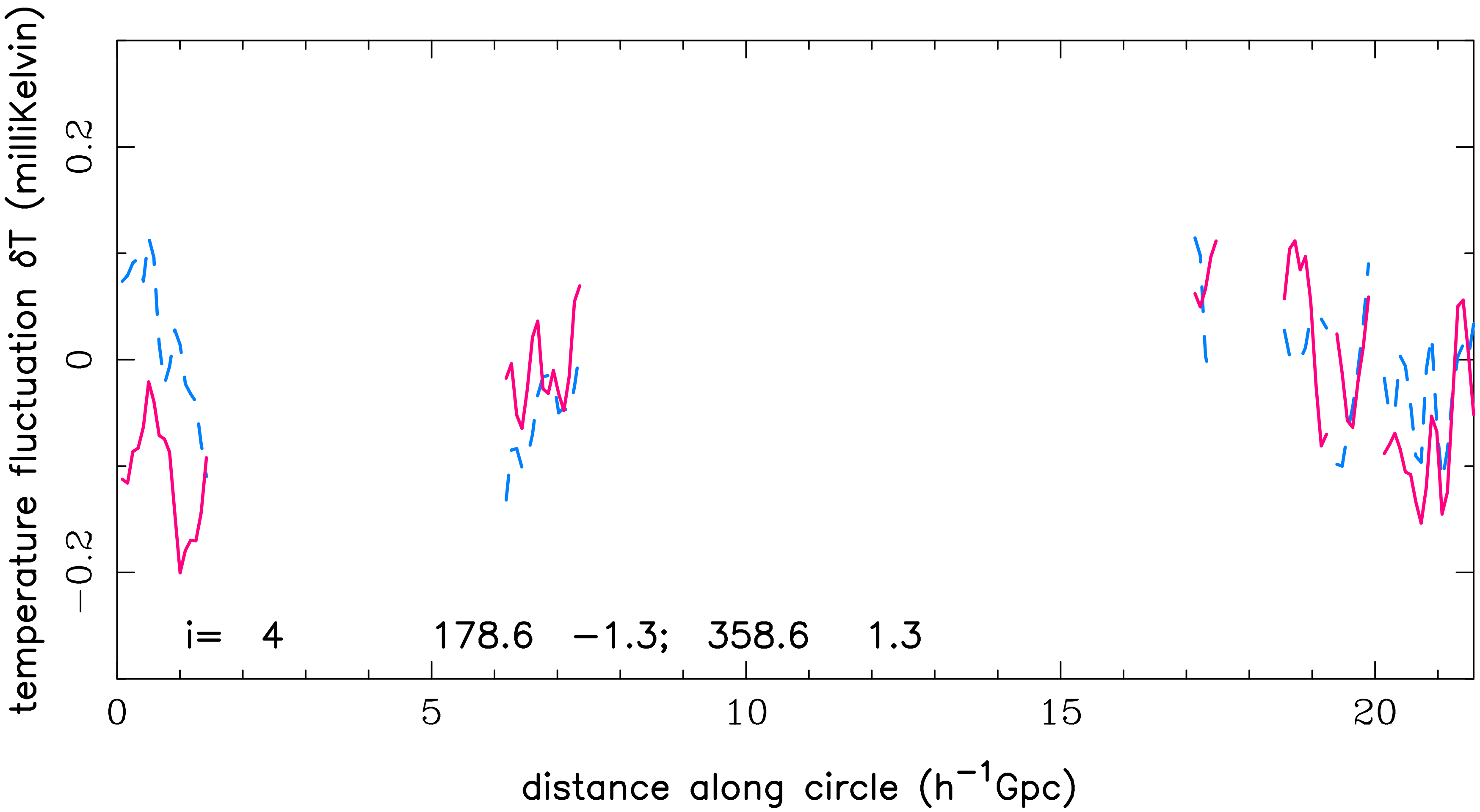
RBSG08 matched circles



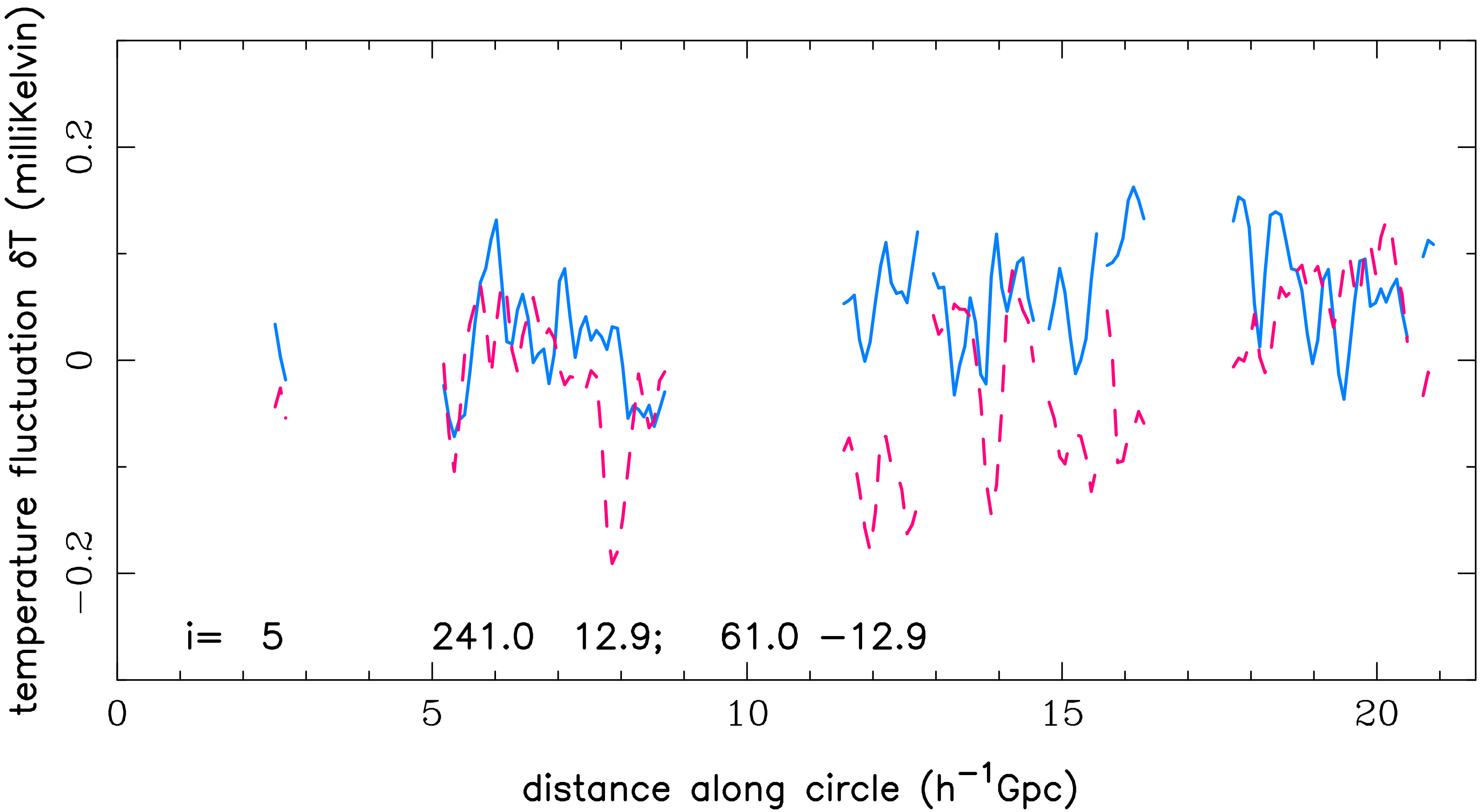
RBSG08 matched circles



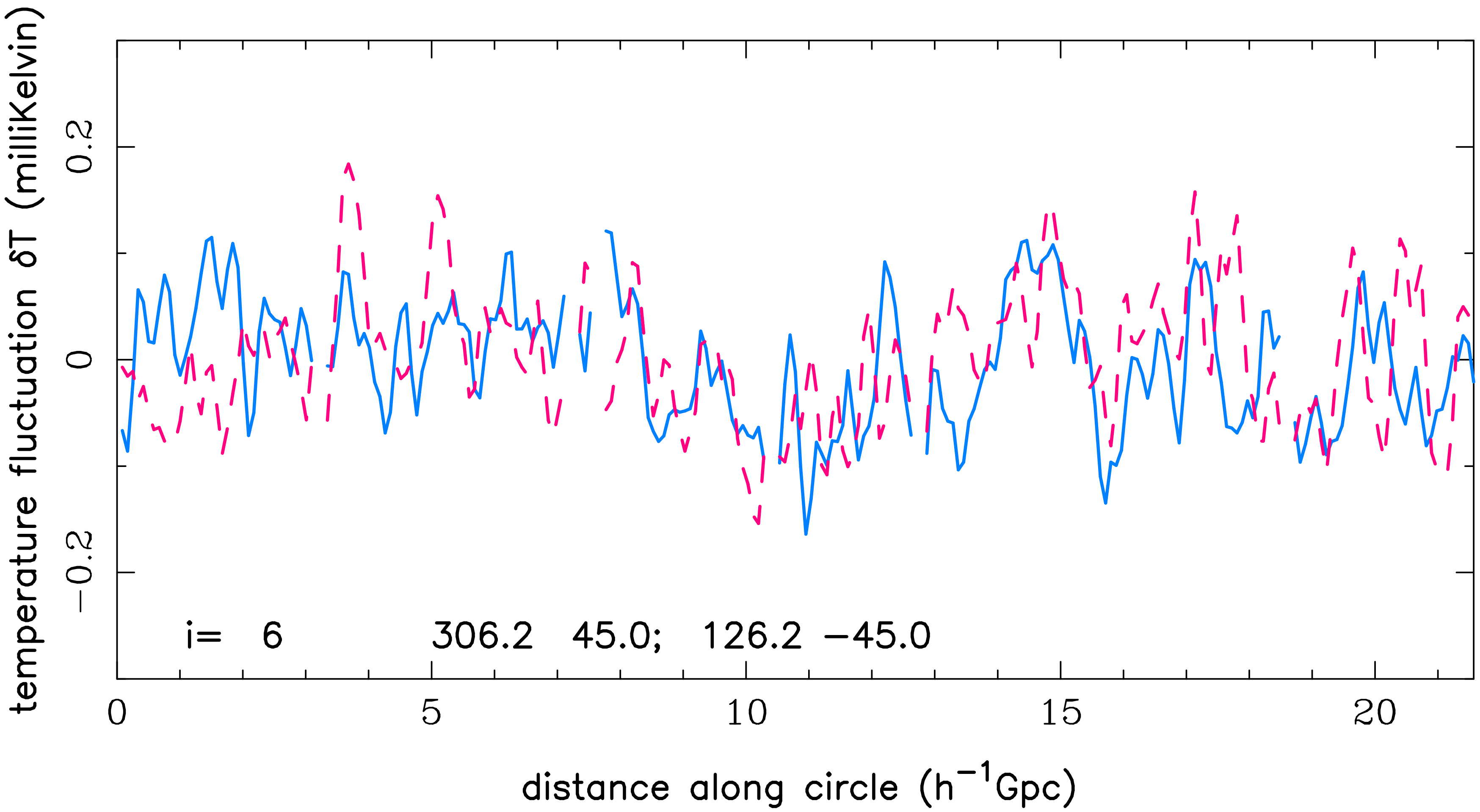
RBSG08 matched circles



RBSG08 matched circles



RBSG08 matched circles

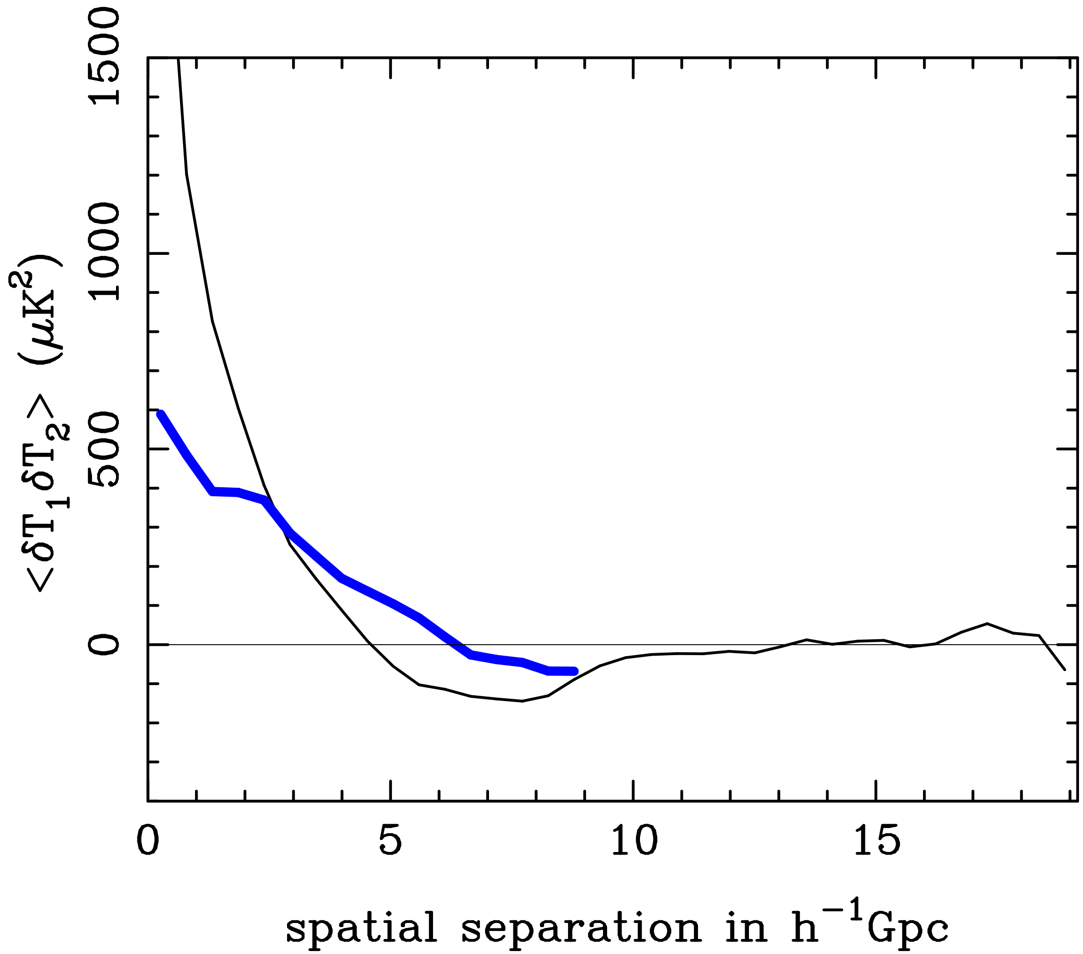


WMAP + Poincaré S^3/I^*

- $\rightarrow$ favoured Poincaré dodecahedral space orientation/size, RBSG08 arXiv:0801.0006

WMAP + Poincaré S^3/I^*

- $\rightarrow$ favoured Poincaré dodecahedral space orientation/size, RBSG08 arXiv:0801.0006



-

WMAP + Poincaré S^3/I^*

- $\rightarrow$ favoured Poincaré dodecahedral space orientation/size, RBSG08 arXiv:0801.0006
- $\{(l,b)\}_{i=1,6} \approx \{(184°, 62°), (305°, 44°), (46°, 49°), (117°, 20°), (176°, -4°), (240°, 13°)\}$ ($\pm \approx 2°$)

WMAP + Poincaré S^3/I^*

- $\rightarrow$ favoured Poincaré dodecahedral space orientation/size, RBSG08 arXiv:0801.0006
- $\{(l,b)\}_{i=1,6} \approx \{(184^\circ, 62^\circ), (305^\circ, 44^\circ), (46^\circ, 49^\circ), (117^\circ, 20^\circ), (176^\circ, -4^\circ), (240^\circ, 13^\circ)\}$ ($\pm \approx 2^\circ$)
- matched discs method, RK11 arXiv:1106.0727: $2r_{\rm inj} = 18.2 \pm 0.5\mathrm{h}^{-1}$ Gpc

WMAP + Poincaré S^3/I^*

- $\rightarrow$ favoured Poincaré dodecahedral space orientation/size, RBSG08 arXiv:0801.0006
- $\{(l,b)\}_{i=1,6} \approx \{(184°,62°),(305°,44°),(46°,49°),(117°,20°),(176°,-4°),(240°,13°)\}$ ($\pm \approx 2°$)
- matched discs method, RK11 arXiv:1106.0727: $2r_{\mathrm{inj}} = 18.2 \pm 0.5\mathrm{h}^{-1}$ Gpc
- Planck (2013): (i) perturbation statististics assumption method; + (ii) identified circles: small correlation signal from S^3/I^* and other well-proportioned spaces, but consistent with noise

Cosmic topology 2022+ ?

- topological acceleration method at $\sim 10\mathrm{h}^{-1}$ Mpc likely to be very difficult to separate from artefacts:

Cosmic topology 2022+ ?

- topological acceleration method at $\sim 10\mathrm{h}^{-1}$ Mpc likely to be very difficult to separate from artefacts:
  - needs detailed GR modelling

Cosmic topology 2022+ ?

- topological acceleration method at $\sim 10\mathrm{h}^{-1}$ Mpc likely to be very difficult to separate from artefacts:
  - needs detailed GR modelling
  - needs excellent quality surveys

Cosmic topology 2022+ ?

- topological acceleration method at $\sim 10\mathrm{h}^{-1}$ Mpc likely to be very difficult to separate from artefacts:
  - needs detailed GR modelling
  - needs excellent quality surveys
- numerical simulations — Buliński 2015 PhD thesis
- 3D and 2D statistical methods at 1–10h^{-1} Gpc:

Cosmic topology 2022+ ?

- topological acceleration method at $\sim 10\mathrm{h}^{-1}$ Mpc likely to be very difficult to separate from artefacts:
  - needs detailed GR modelling
  - needs excellent quality surveys
- numerical simulations — Buliński 2015 PhD thesis
- 3D and 2D statistical methods at 1–10h^{-1} Gpc:
  - searches for many 3-manifolds not yet done

Cosmic topology 2022+ ?

- topological acceleration method at $\sim 10\mathrm{h}^{-1}$ Mpc likely to be very difficult to separate from artefacts:
  - needs detailed GR modelling
  - needs excellent quality surveys
- numerical simulations — Buliński 2015 PhD thesis
- 3D and 2D statistical methods at 1–10h^{-1} Gpc:
  - searches for many 3-manifolds not yet done
  - 2D (CMB) data: WMAP, Planck: no detections

Cosmic topology 2022+ ?

- topological acceleration method at $\sim 10\mathrm{h}^{-1}$ Mpc likely to be very difficult to separate from artefacts:
  - needs detailed GR modelling
  - needs excellent quality surveys
- numerical simulations — Buliński 2015 PhD thesis
- 3D and 2D statistical methods at 1–10h^{-1} Gpc:
  - searches for many 3-manifolds not yet done
  - 2D (CMB) data: WMAP, Planck: no detections
  - much h^{-1} Gpc scale 3D data: 4MOST, LSST, EUCLID, VLT/MUSE, . . . , 2022–2030

Cosmic topology 2022+ ?

- topological acceleration method at $\sim 10\mathrm{h}^{-1}$ Mpc likely to be very difficult to separate from artefacts:
  - needs detailed GR modelling
  - needs excellent quality surveys
- numerical simulations — Buliński 2015 PhD thesis
- 3D and 2D statistical methods at 1–10h^{-1} Gpc:
  - searches for many 3-manifolds not yet done
  - 2D (CMB) data: WMAP, Planck: no detections
  - much h^{-1} Gpc scale 3D data: 4MOST, LSST, EUCLID, VLT/MUSE, . . . , 2022–2030
- cosmic topology with inhomogeneities: very much unexplored . . .

Topological acceleration

- heuristic argument — Roukema+2007 A&A arXiv:astro-ph/0602159

Topological acceleration

- heuristic argument — Roukema+2007 A&A arXiv:astro-ph/0602159
- KN94 exact GR example – Ostrowski+2012 CQG arXiv:1109.1596
- effect depends on choice of topological manifold — Roukema & Różański 2009 A&A arXiv:0902.3402
- Newton–Cartan approach:
  even Taylor terms ⇒ closed; odd terms ⇒ curved — Vigneron & Roukema (2022) arXiv:2201.09102

Topological acceleration

- heuristic argument — Roukema+2007 A&A arXiv:astro-ph/0602159
- KN94 exact GR example – Ostrowski+2012 CQG arXiv:1109.1596
- effect depends on choice of topological manifold — Roukema & Różański 2009 A&A arXiv:0902.3402
- Newton–Cartan approach:
  even Taylor terms ⇒ closed; odd terms ⇒ curved — Vigneron & Roukema (2022) arXiv:2201.09102
- patterns of time-integrated effects of topological acceleration should exist at ~ 10–$1000h^{-1}$ Mpc
  - very difficult to separate from artefacts
  - need detailed GR modelling
  - need excellent quality surveys
- numerical simulations — Buliński 2015 PhD thesis NCU