



# Cosmic topology

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<https://cosmo.torun.pl/~boud/Roukema20240423KNSA.pdf>



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  - ◆ scalar (GR) averaging: statistically homogeneous spatial slices
- within this model, what is the shape of the Universe?

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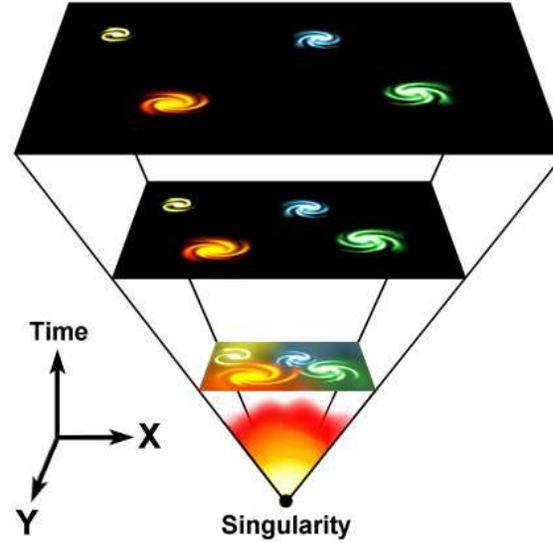
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  3. assume that  $(M, \mathbf{g})$  remains unchanged if we add density perturbations to an early time slice

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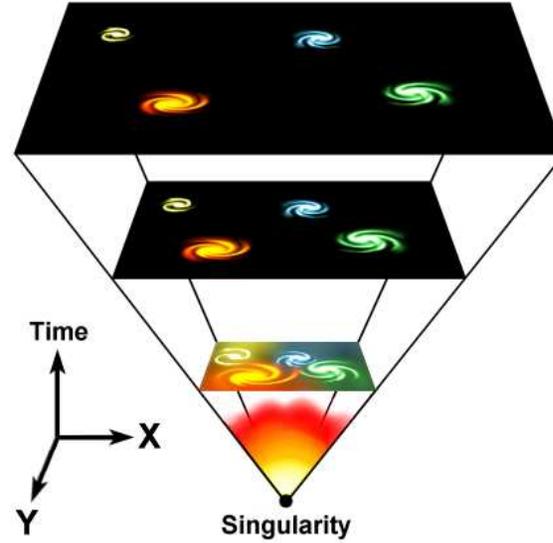
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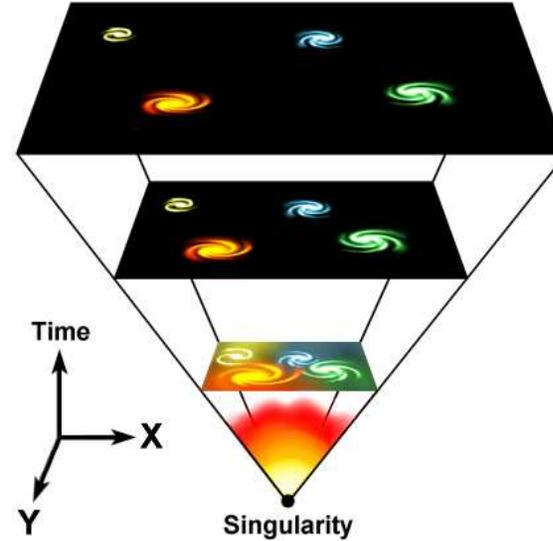
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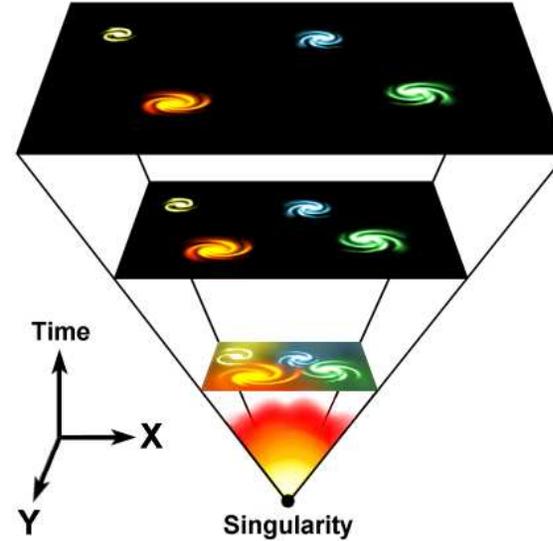


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- spherical coordinates for spatial slice

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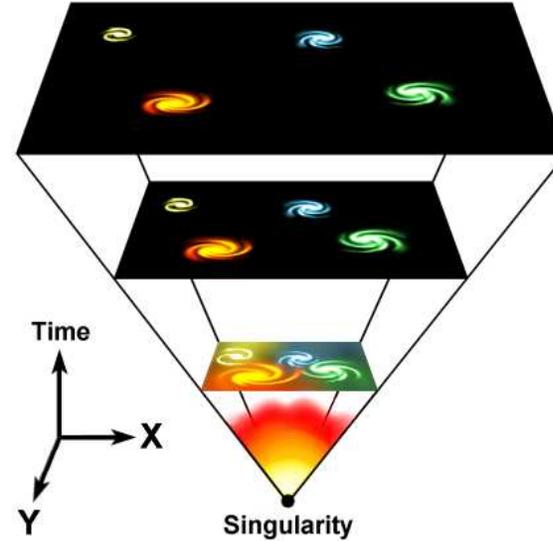


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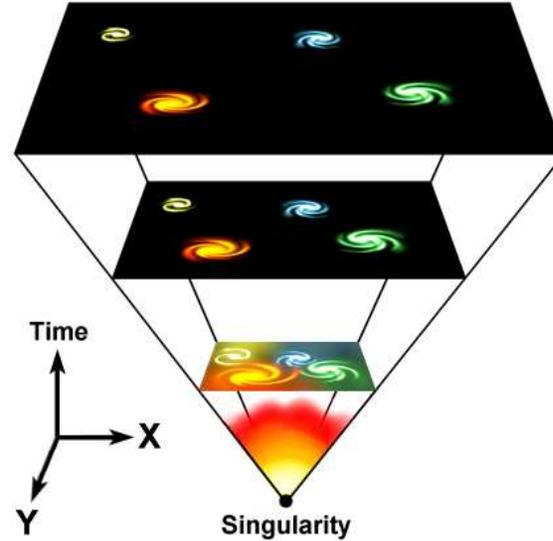


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- universe is static in comoving coordinates  $(r, \theta, \phi)$

# FLRW metric

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- w: w:Howard Percy Robertson  
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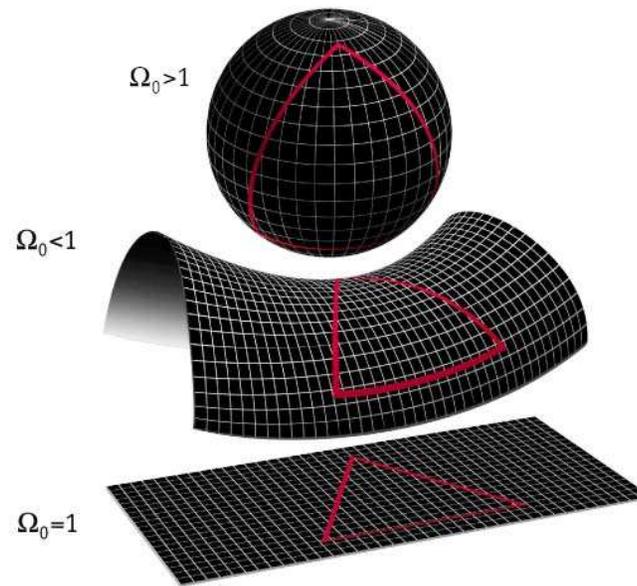
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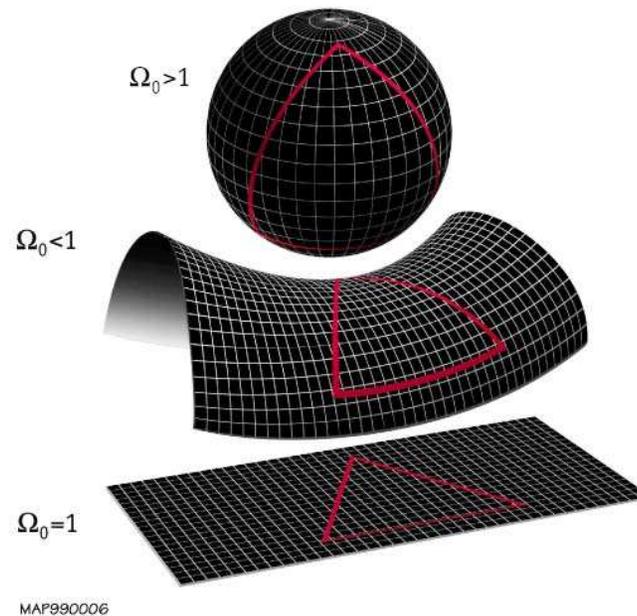


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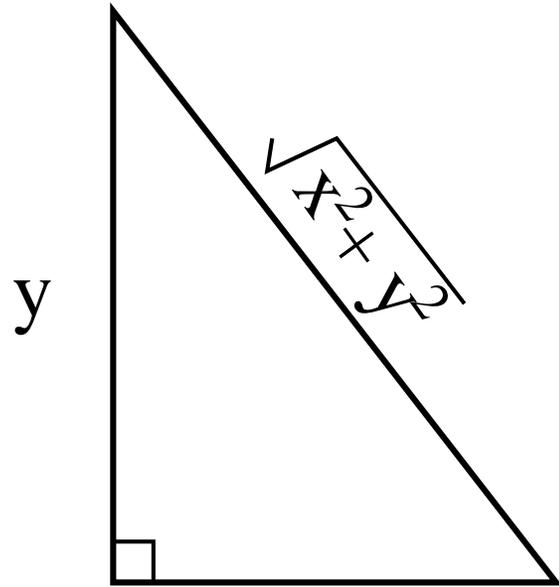


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for a comoving radius of curvature  $R_C$  and curvature of sign  $k$

# curvature

- on a spatial slice (fixed value of  $t$ ):

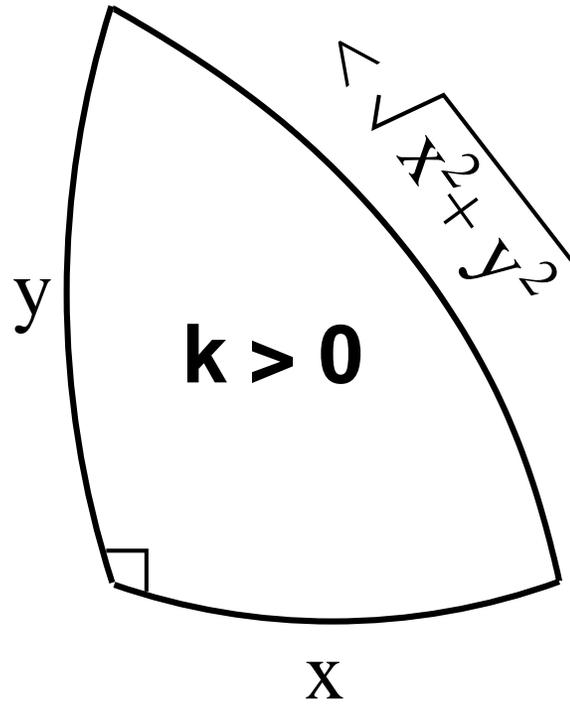


$x$

$$k = 0$$

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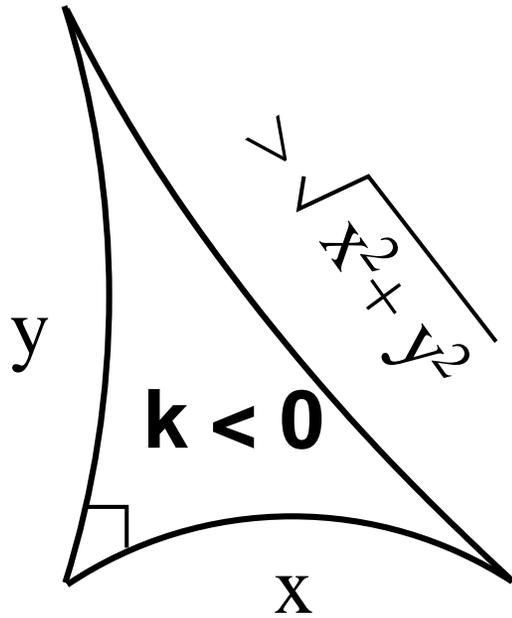
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# curvature

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$$k < 0$$

# 2D curvature intuition: $k > 0$

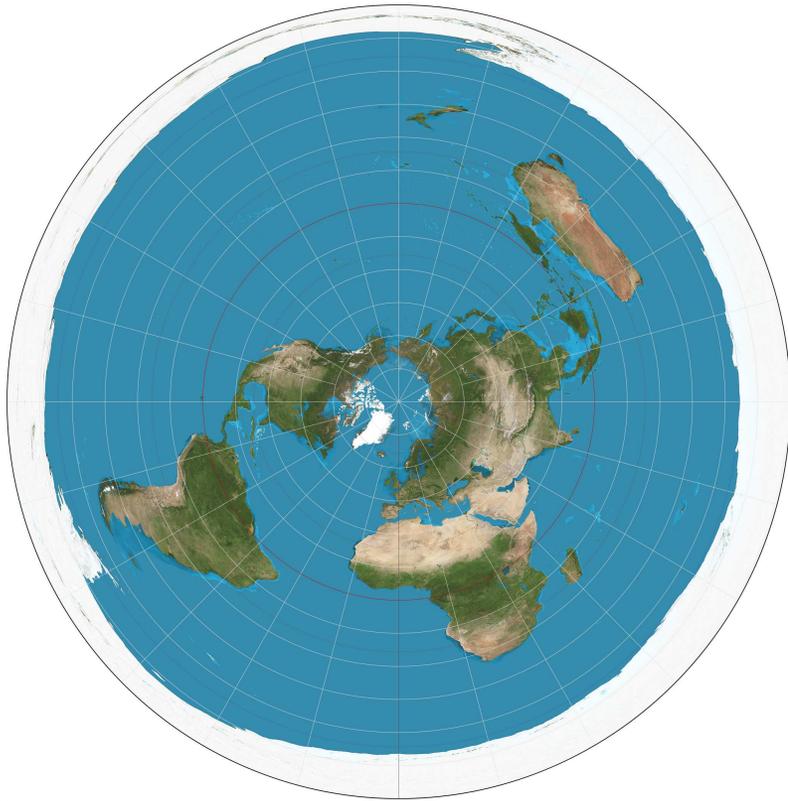
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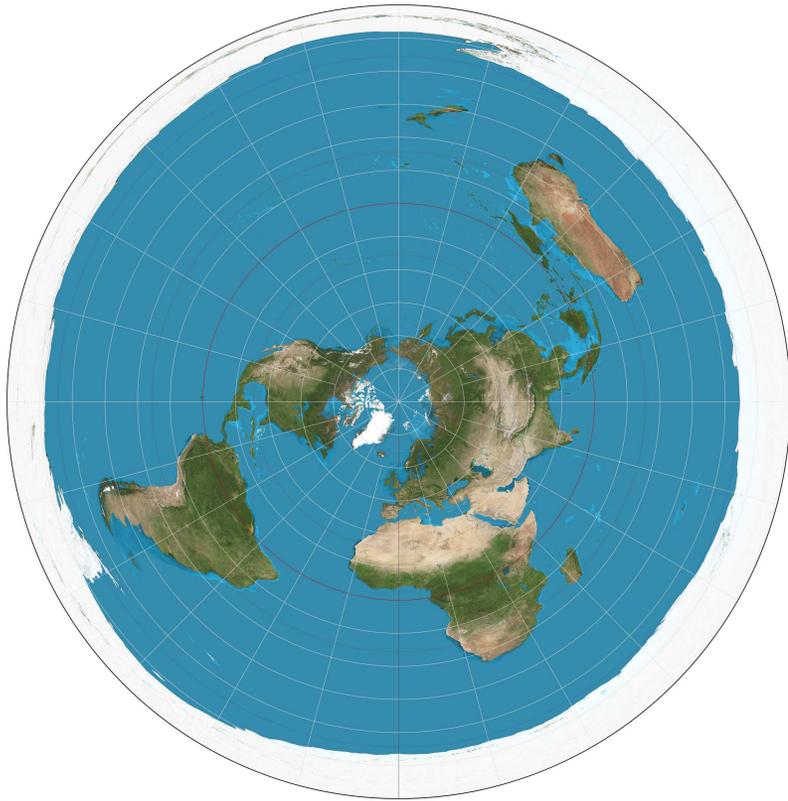


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- intuition switch:  $S^2$  easier vs  $S^3$  more physical

# 2D topology intuition ( $k = 0$ )



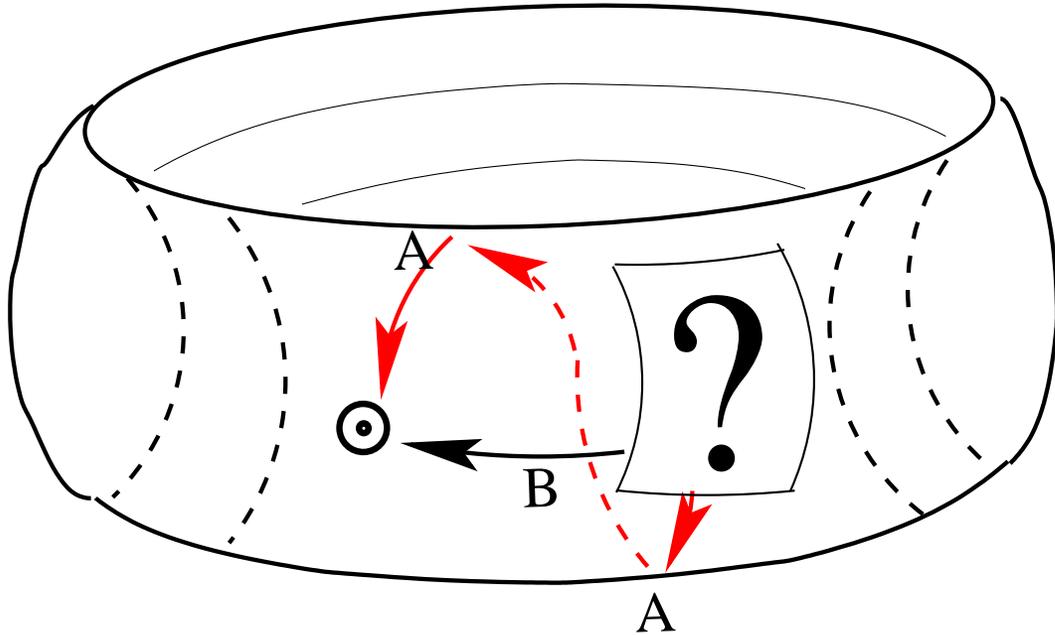
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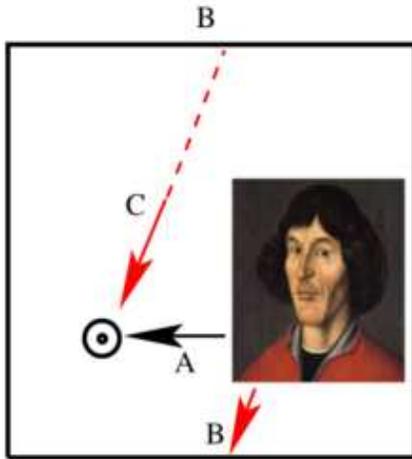


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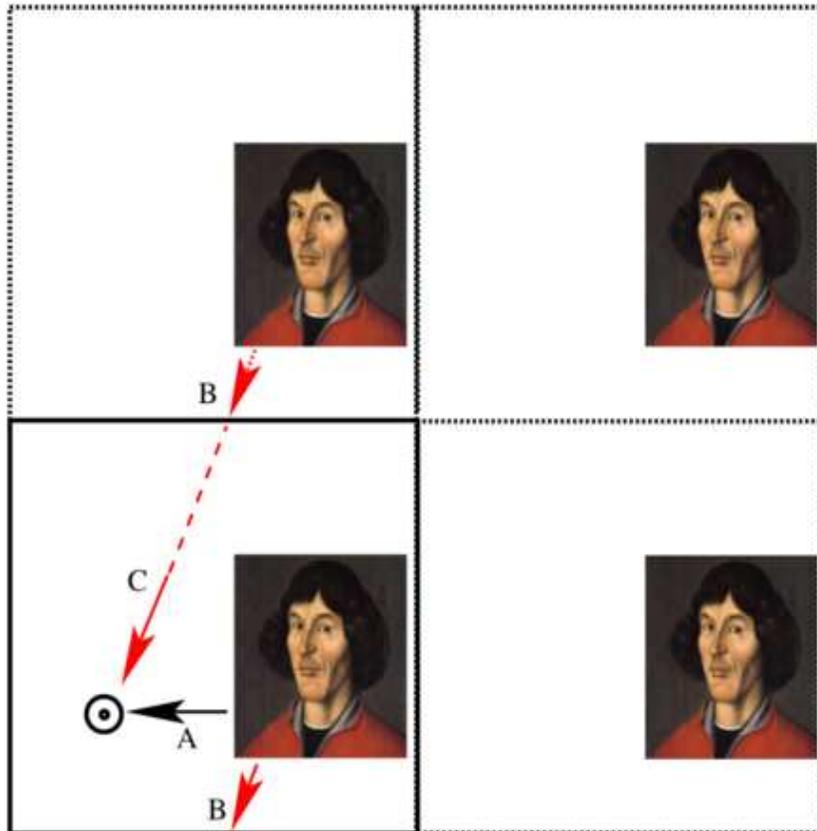


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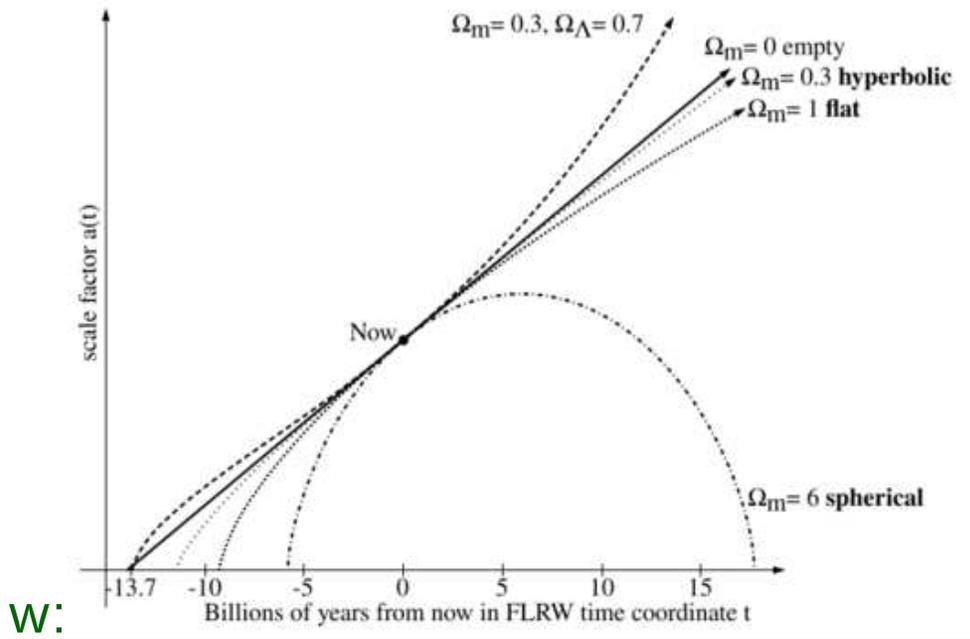
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- convention:  $t_b := 0$ , giving  $t \rightarrow 0^+ \Rightarrow a(t) \rightarrow 0^+$

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(Defn:  $a_0 := 1$ )

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$$1 + z = \frac{1}{a_{\text{em}}}$$

(Defn of redshift  $z$ )

# expansion

- matter density:  $\rho_m \propto a^{-3} = (1+z)^3$
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- radiation density:  $E = h\nu \Rightarrow \rho_r \propto a^{-4} = (1 + z)^4$

# Black body: COBE ( $\sim 1992$ )

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- $\Rightarrow$  black body + primordial nucleosynthesis

# CMB discovery: McKellar 1941

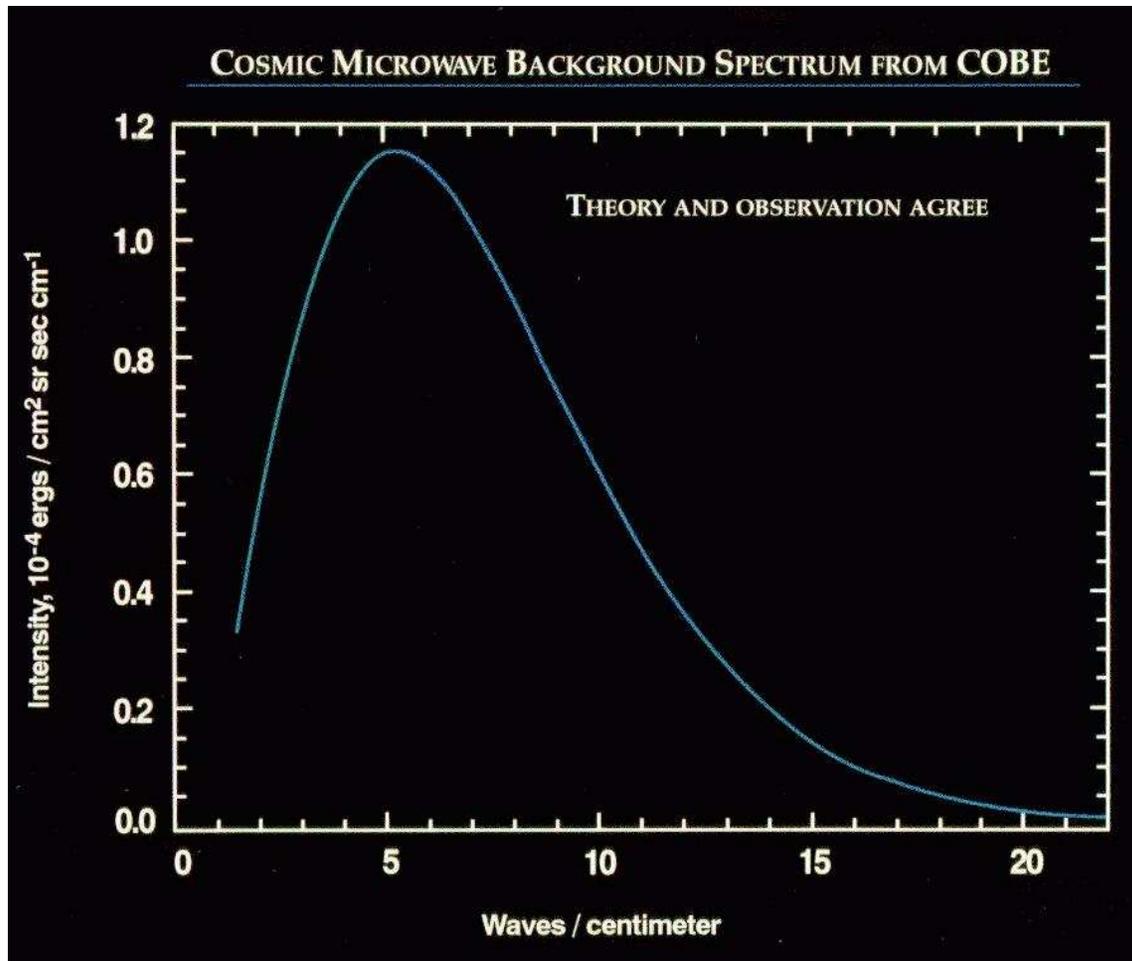
- $T \approx 2.3$  K — Andrew McKellar (1941; [ADS:1941PDAO....7..251M](#))  
from observations by Walter S. Adams (1941;  
[ADS:1941ApJ....93...11A](#))
- Penzias & Wilson rediscovery (1965 + Nobel prize)

# Black body: COBE ( $\sim 1992$ )

- COBE /FIRAS (Far Infrared Absolute Spectrophotometer)

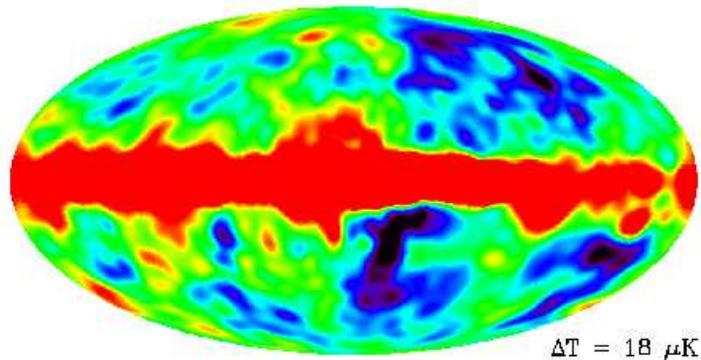
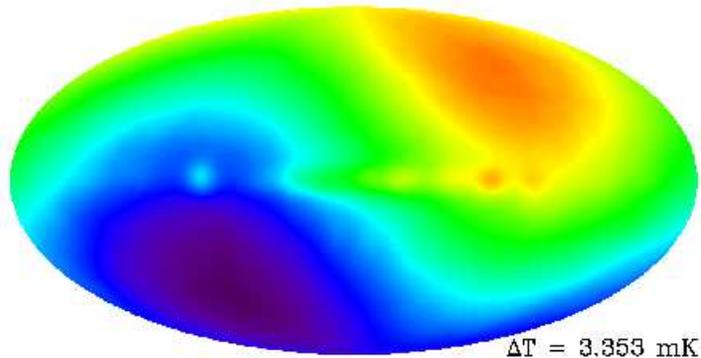
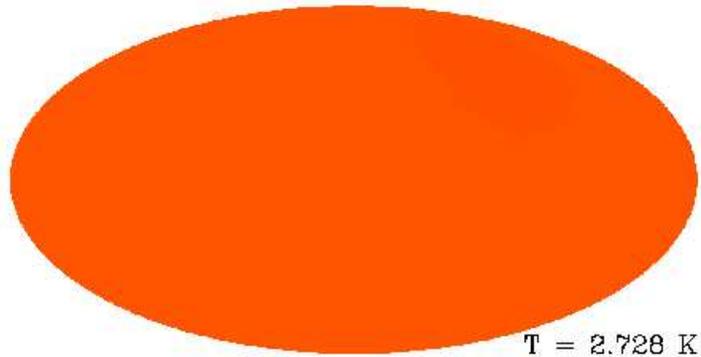
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- COBE /DMR (Differential Microwave Radiometer)



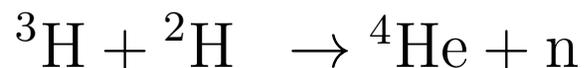
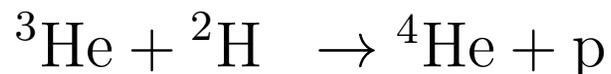
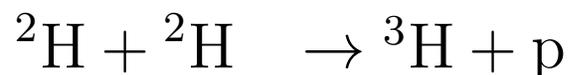
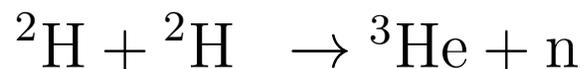
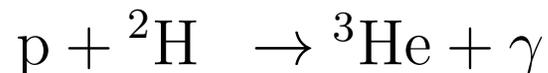
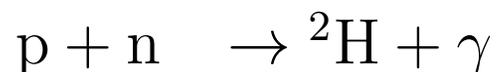
# BBN: Big bang nucleosynthesis

- Alpher, Bethe, & Gamow (1948; [ADS:1948PhRv...73..803A](#))

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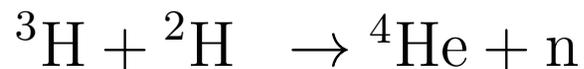
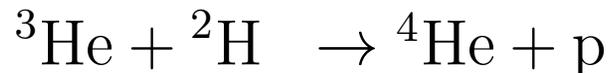
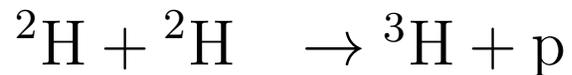
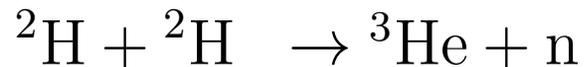
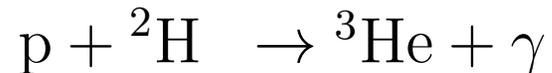
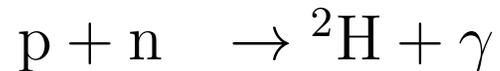
main reactions of varying probability:



# BBN: Big bang nucleosynthesis

- Alpher, Bethe, & Gamow (1948; [ADS:1948PhRv...73..803A](#))

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[w:Big Bang nucleosynthesis](#)

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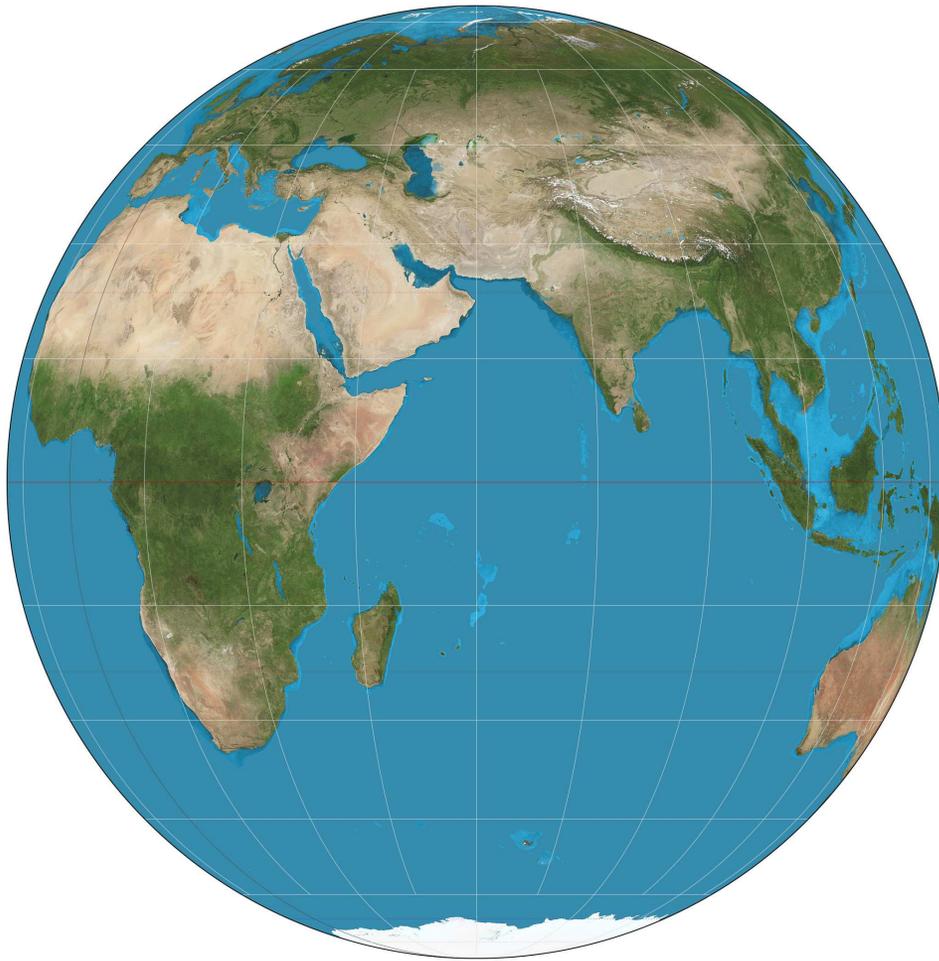
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Einstein–de Sitter model (EdS)

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- convenient conversion:  $1 \text{ km/s} \approx 1.04 \text{ kpc/Gyr} \approx 1 \text{ kpc/Gyr}$

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# FLRW: $\rho_{\text{crit}}$

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◆ azimuthal equidistant coords:  $R_C$

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- $\Omega_{\text{tot}0} < 1$  *hyperbolic*  $R_C$  imaginary (or use  $|R_C|$ )

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- *hint*: mixed index form of  $g$  is easy

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Friedmann Eqn ( $\Lambda \neq 0$ ):

$$\frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$

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[ADS:1917SPAW.....142E](#)

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- high-level frontends (e.g. python) should be easy to write

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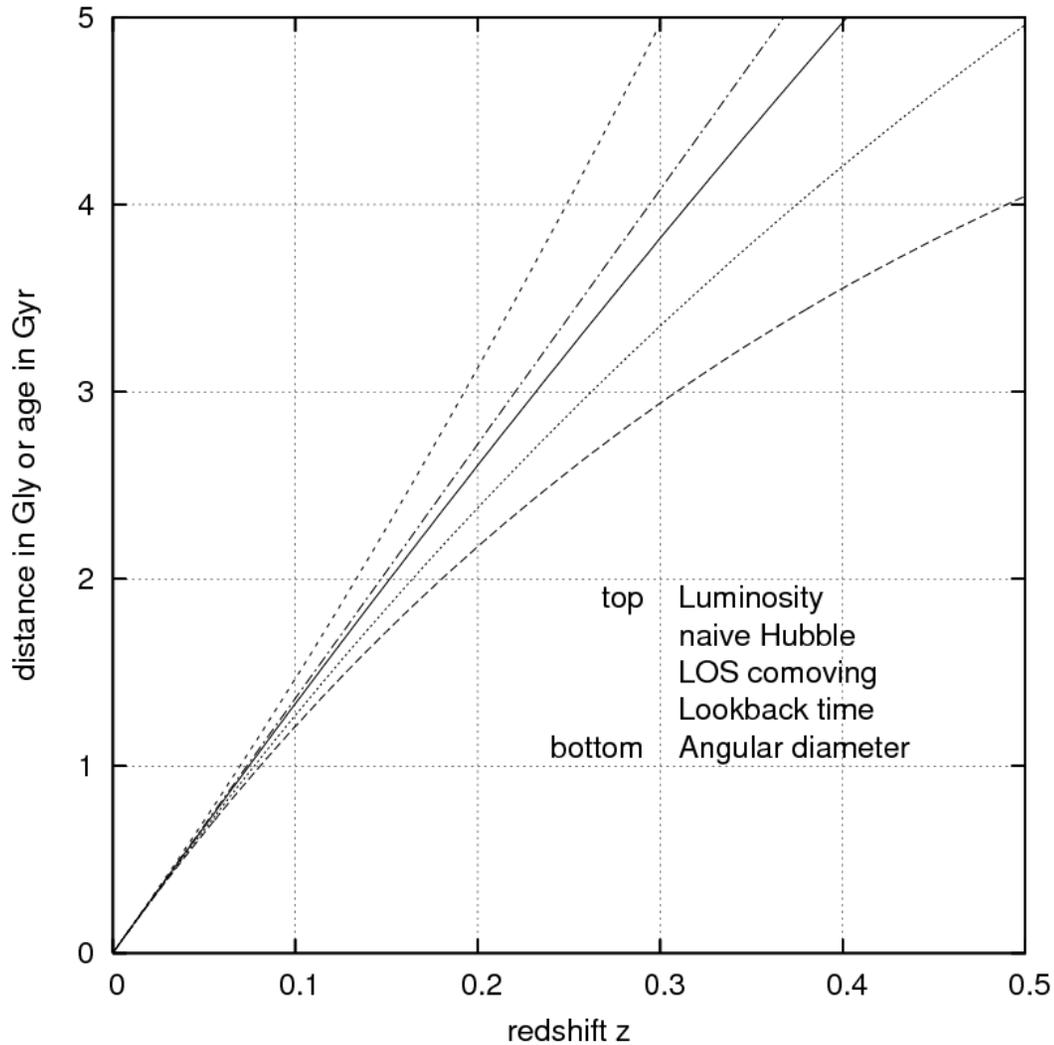
■ w:Distance measures (cosmology)



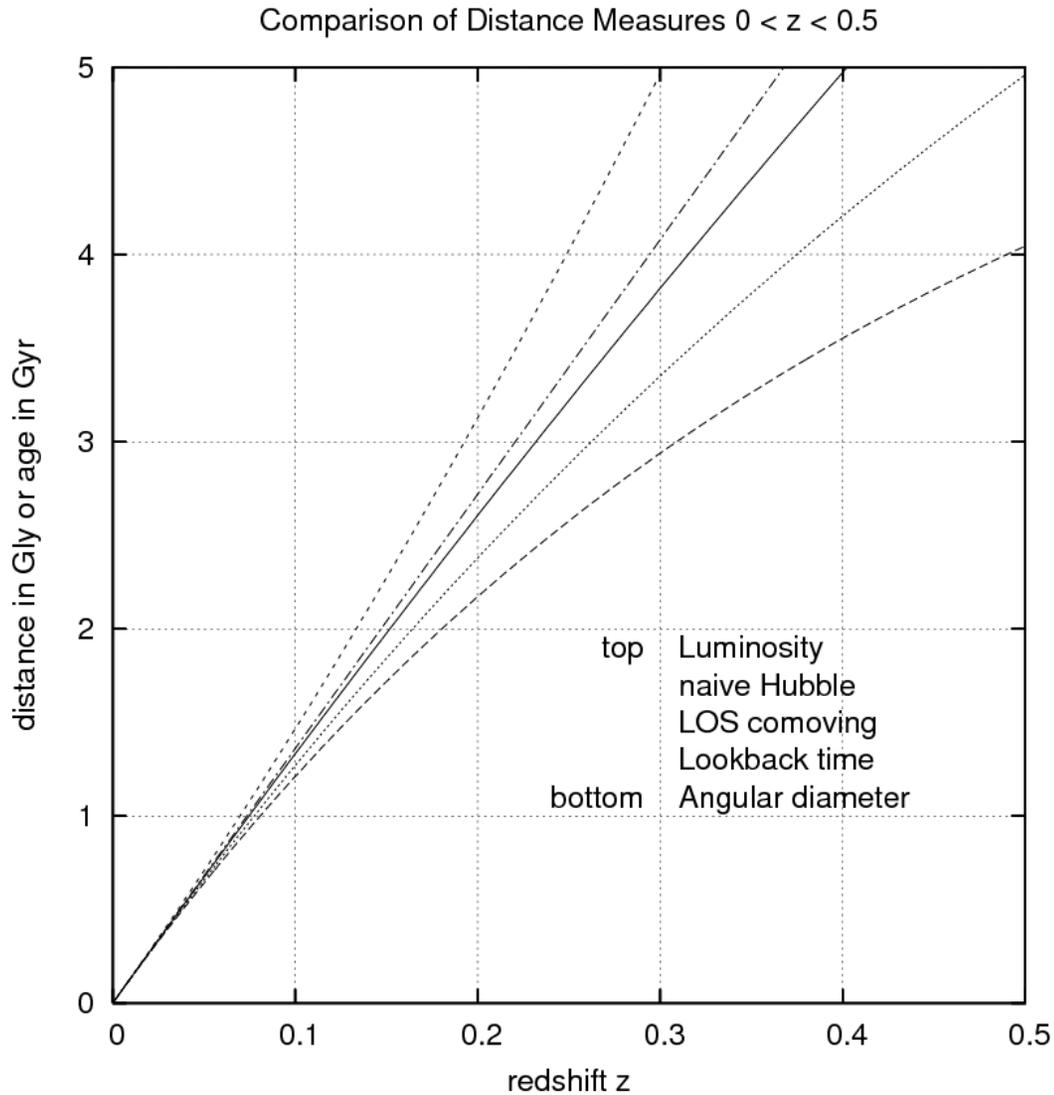
# FLRW distances e.g. $\Lambda$ CDM



Comparison of Distance Measures  $0 < z < 0.5$

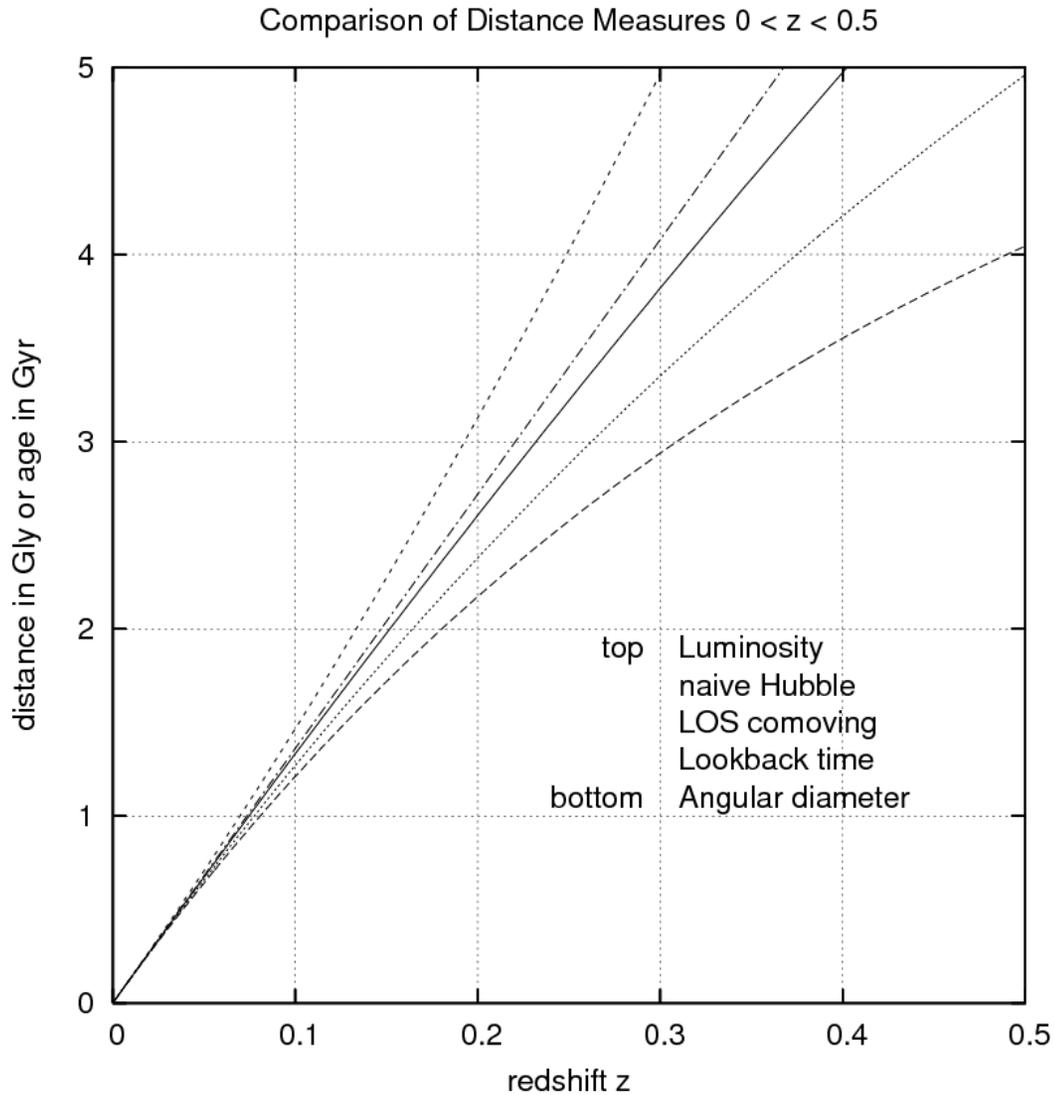


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Defn:  $h := H_0/100 \text{ km/s/Mpc}$  (without a "0" subscript on  $h$ )

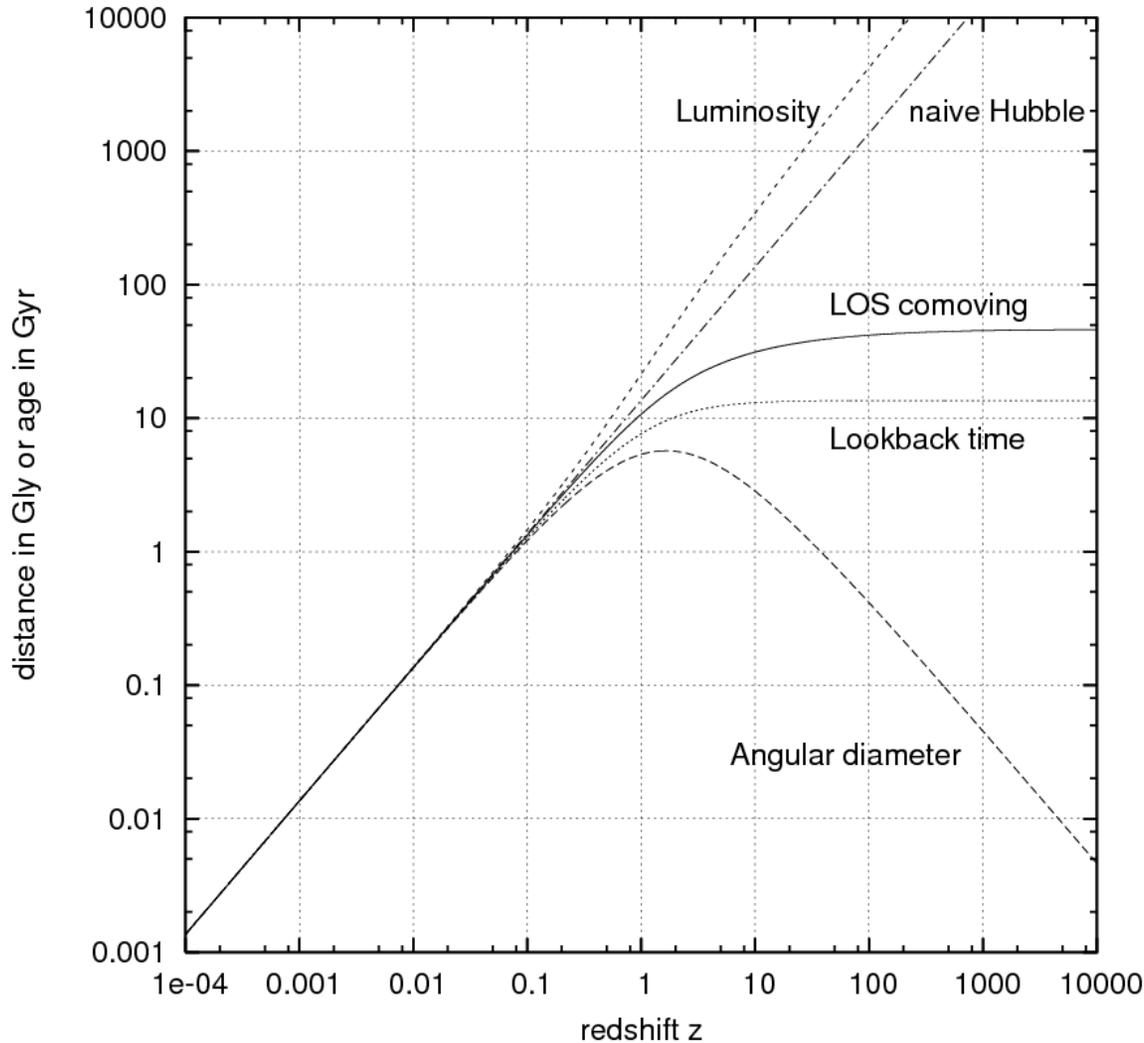
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$$\Omega_{m0} = 0.3, \Omega_{r0} = 10^{-4}, \Omega_{\Lambda 0} = 1.0 - (\Omega_{m0} + \Omega_{r0}), h = 0.7, \Omega_{k0} = 0$$

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Comparison of Distance Measures  $0 < z < 10,000$



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# Non-radial spatial geodesics

- What is the comoving distance between two objects at different celestial positions and different redshifts, for an arbitrary curvature (+, 0, -)?

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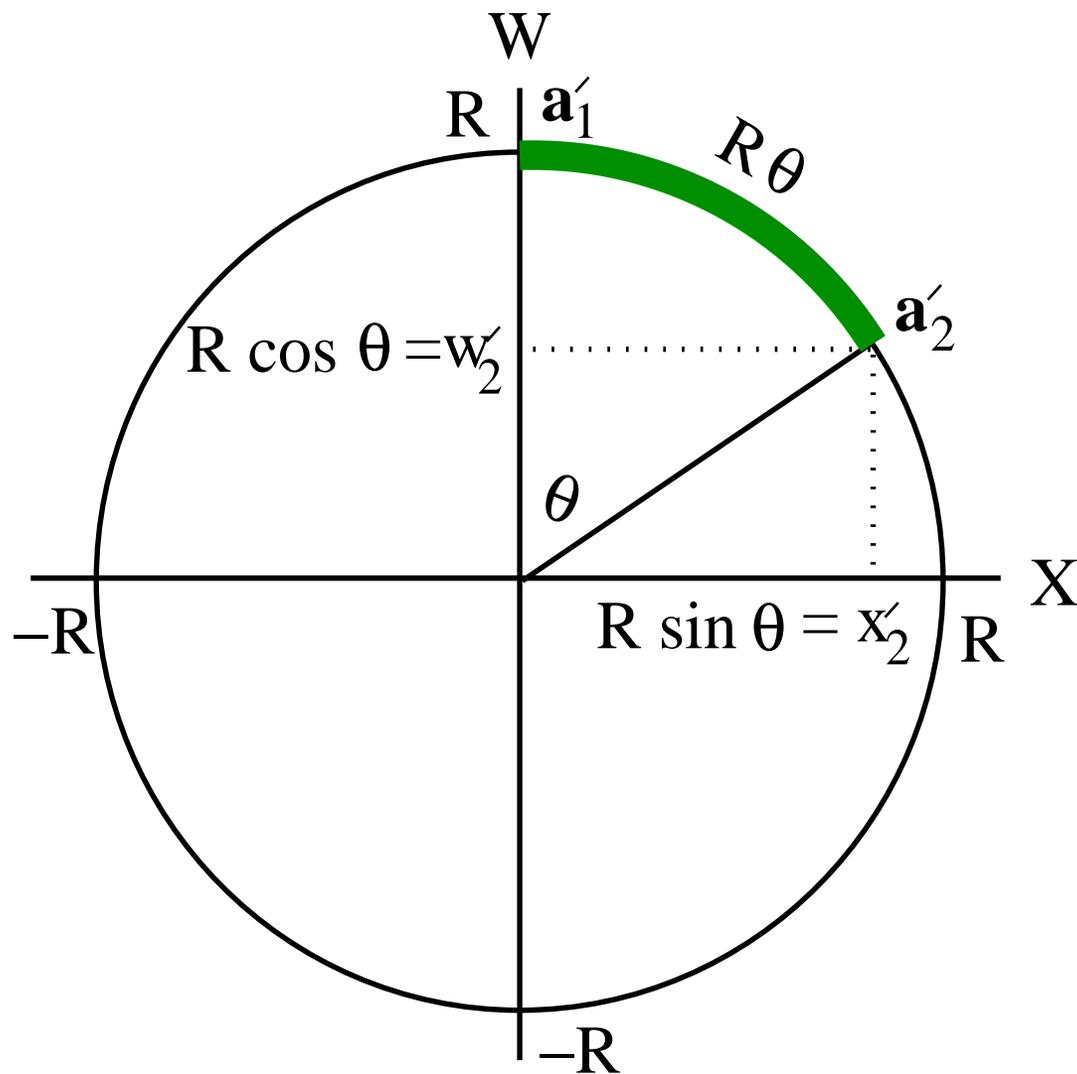
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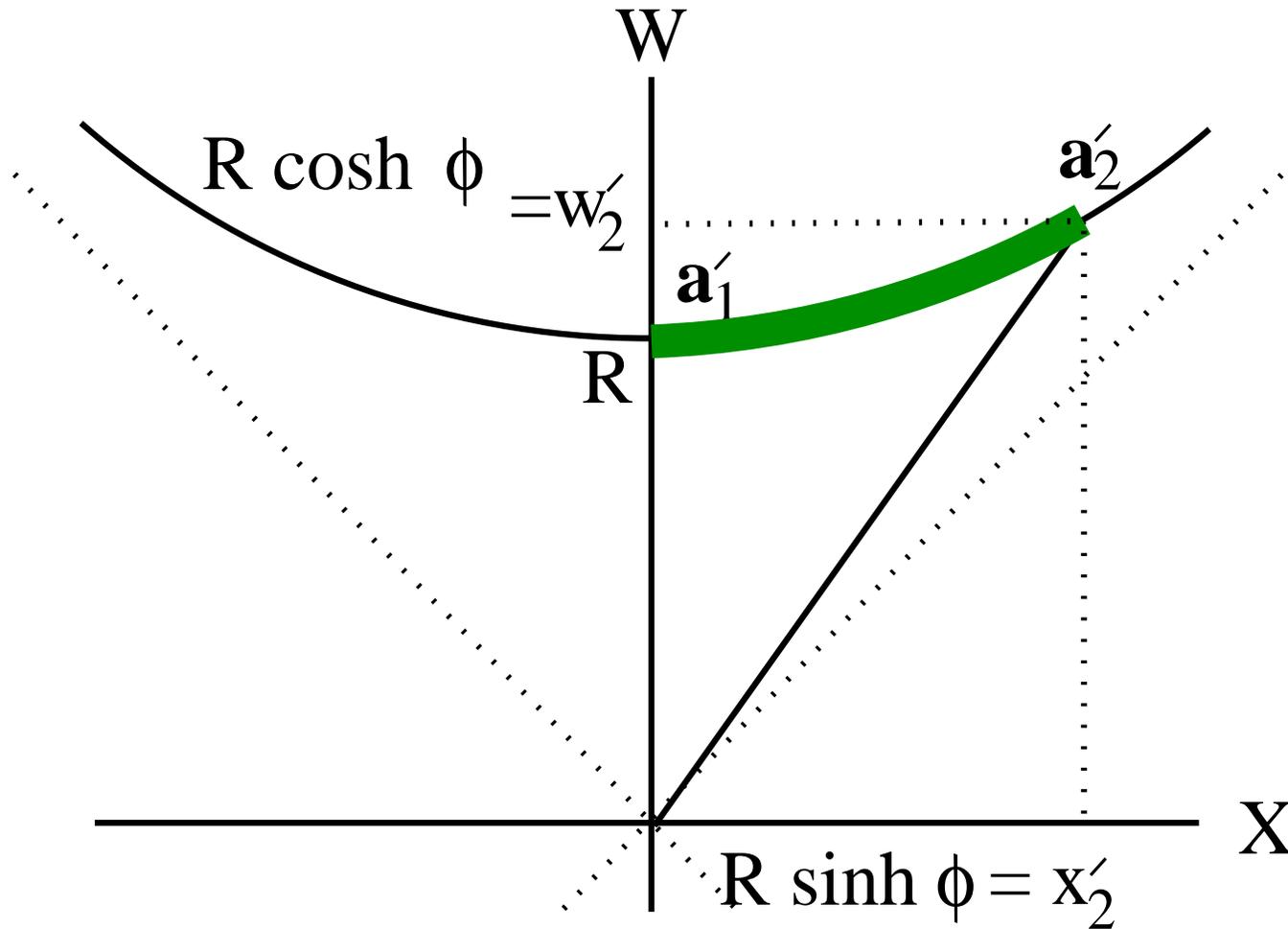


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metric on  $S^3$  (or  $\mathbb{R}^3$  or  $H^3$ ):

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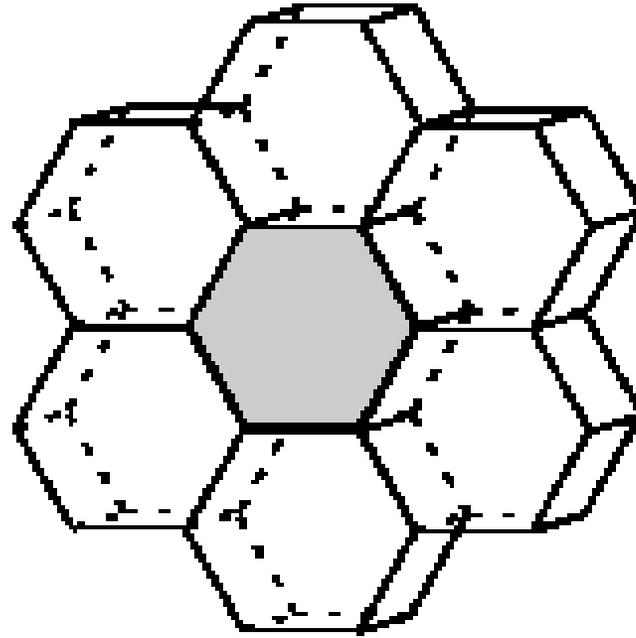
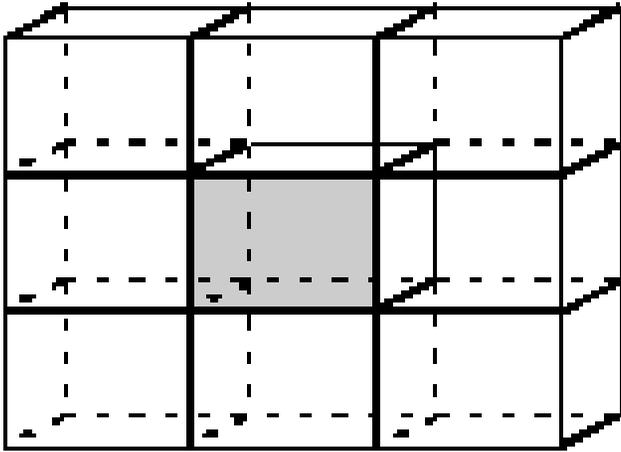
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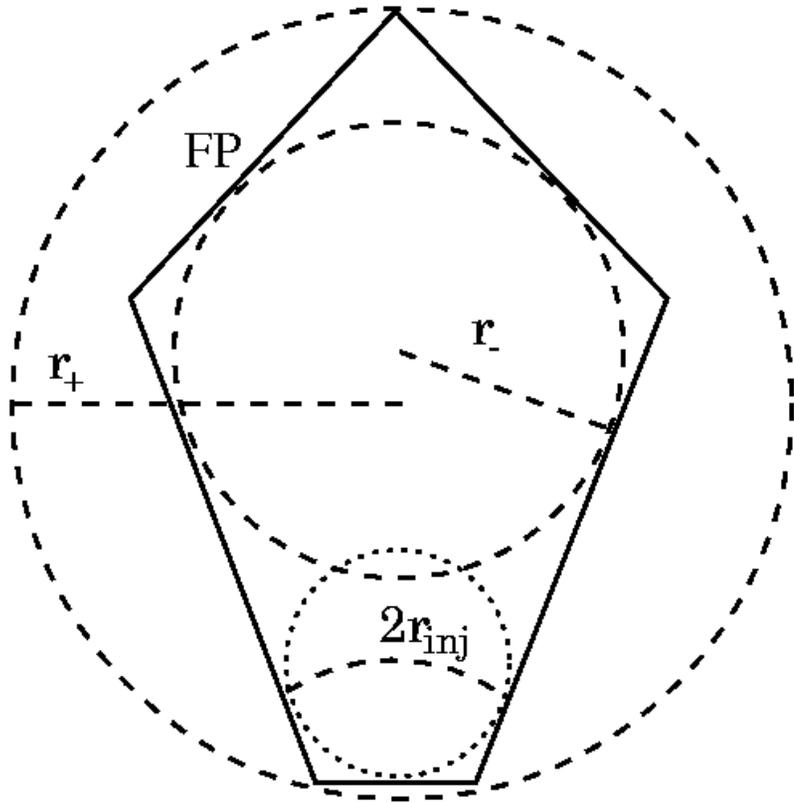
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# Cosmic topology: definitions



3D flat examples [arXiv:astro-ph/9901364](https://arxiv.org/abs/astro-ph/9901364)

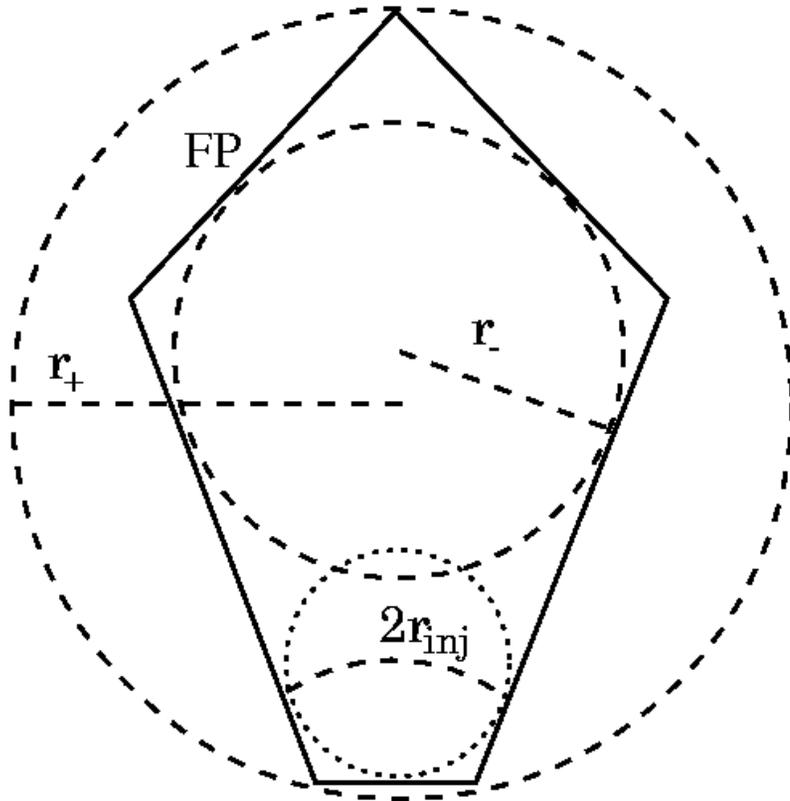
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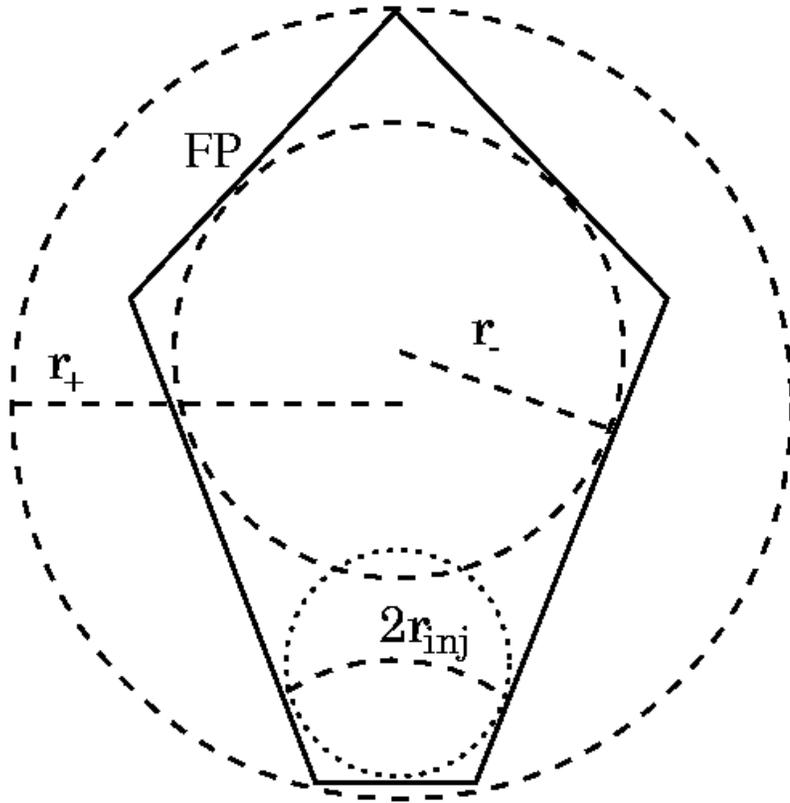
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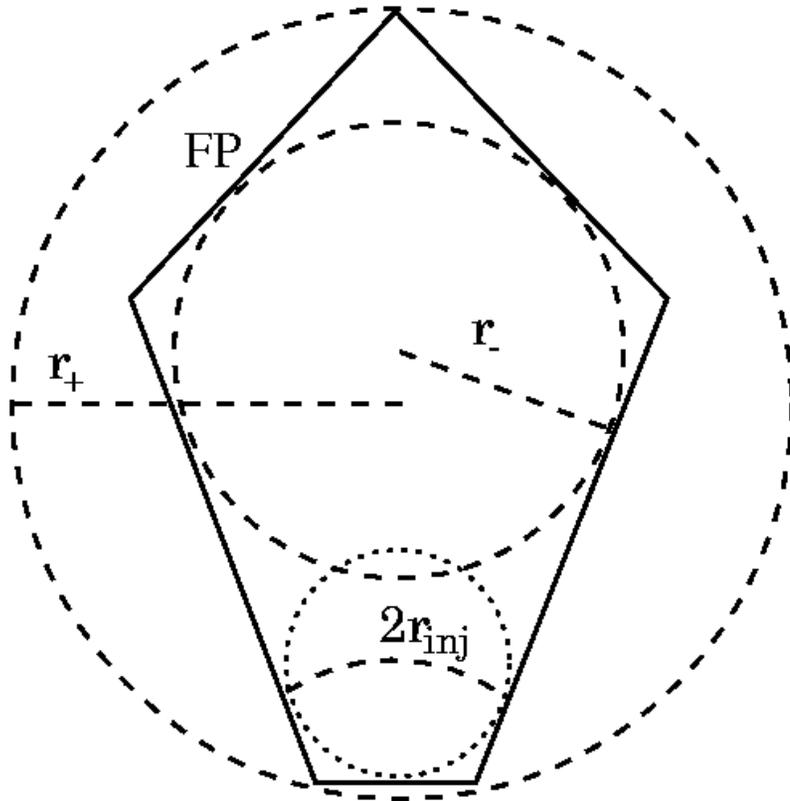
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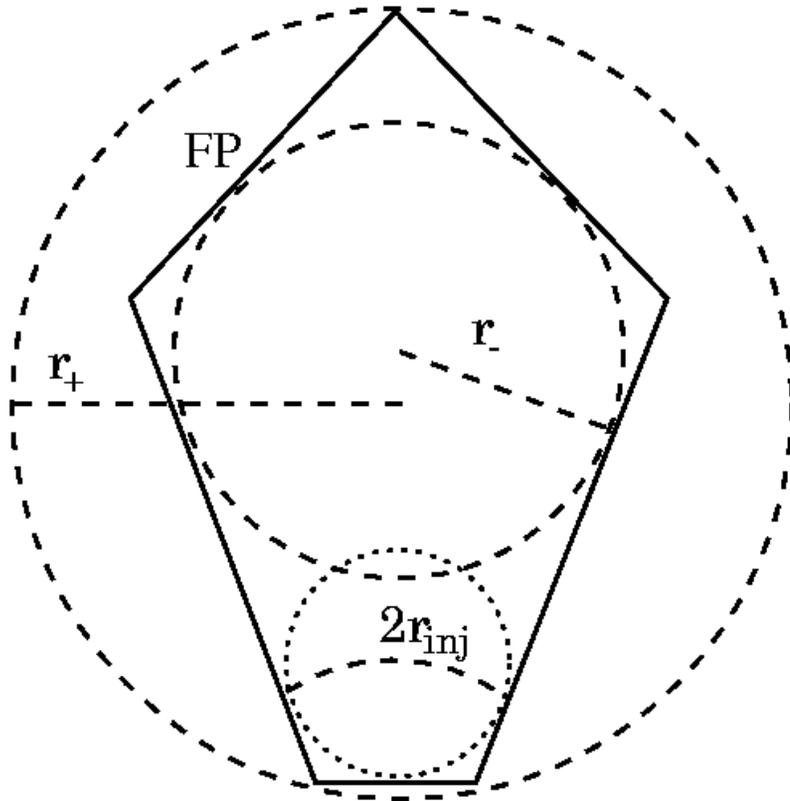
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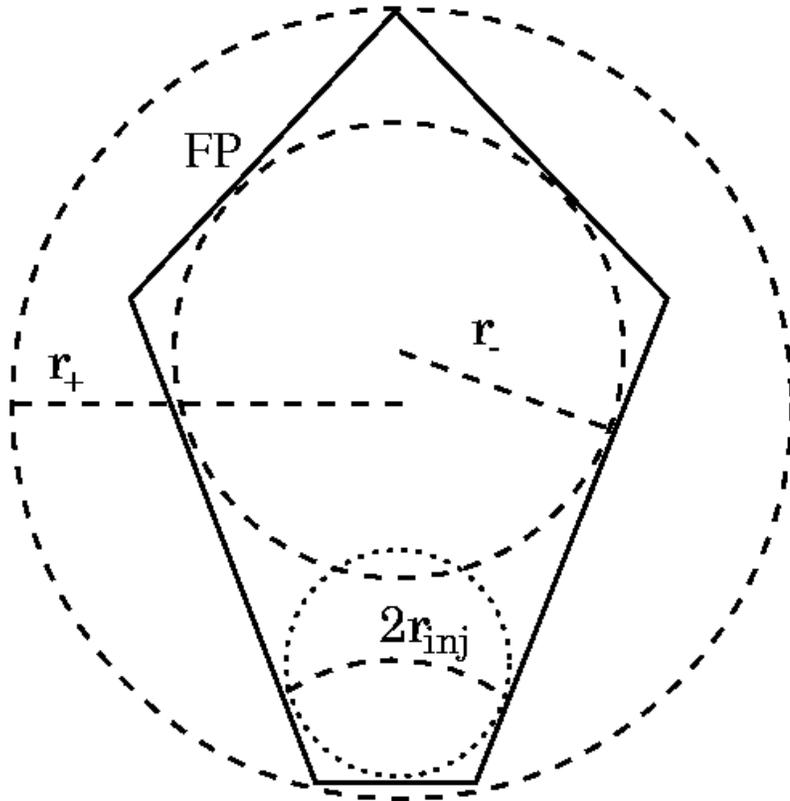
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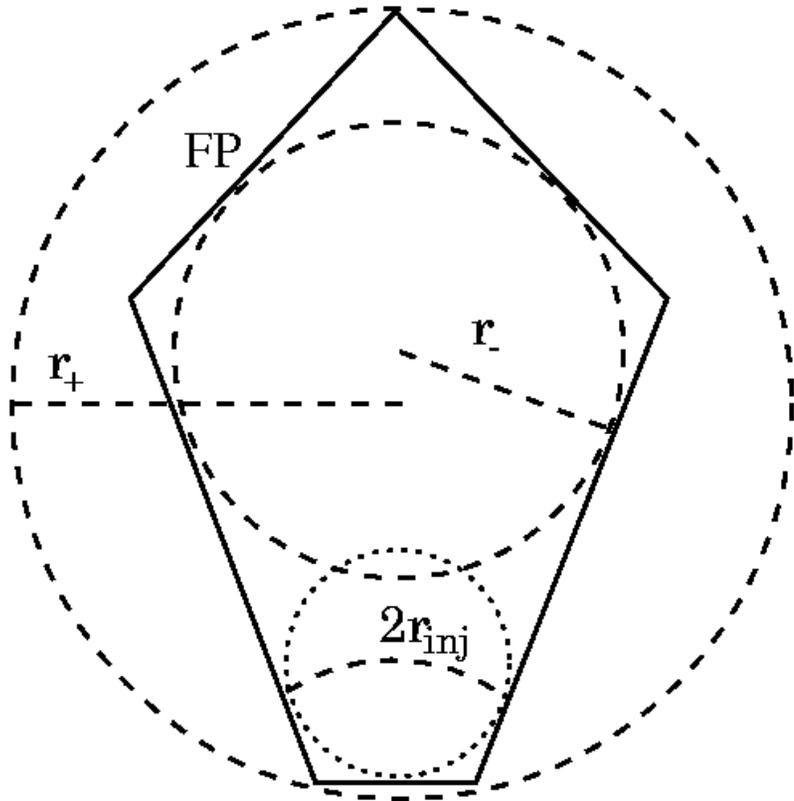
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# Families of const $k$ 3-spaces



■ flat 3-spaces: 18



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w:Grigori Perelman, [arXiv:math/0211159](https://arxiv.org/abs/math/0211159) + [arXiv:math/0303109](https://arxiv.org/abs/math/0303109) + [arXiv:math/0307245](https://arxiv.org/abs/math/0307245)

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  - ◆ active research area, e.g. [arXiv:0705.4325](https://arxiv.org/abs/0705.4325) min. vol.
- 5 other Thurston classes – [w:Geometrization conjecture](#)

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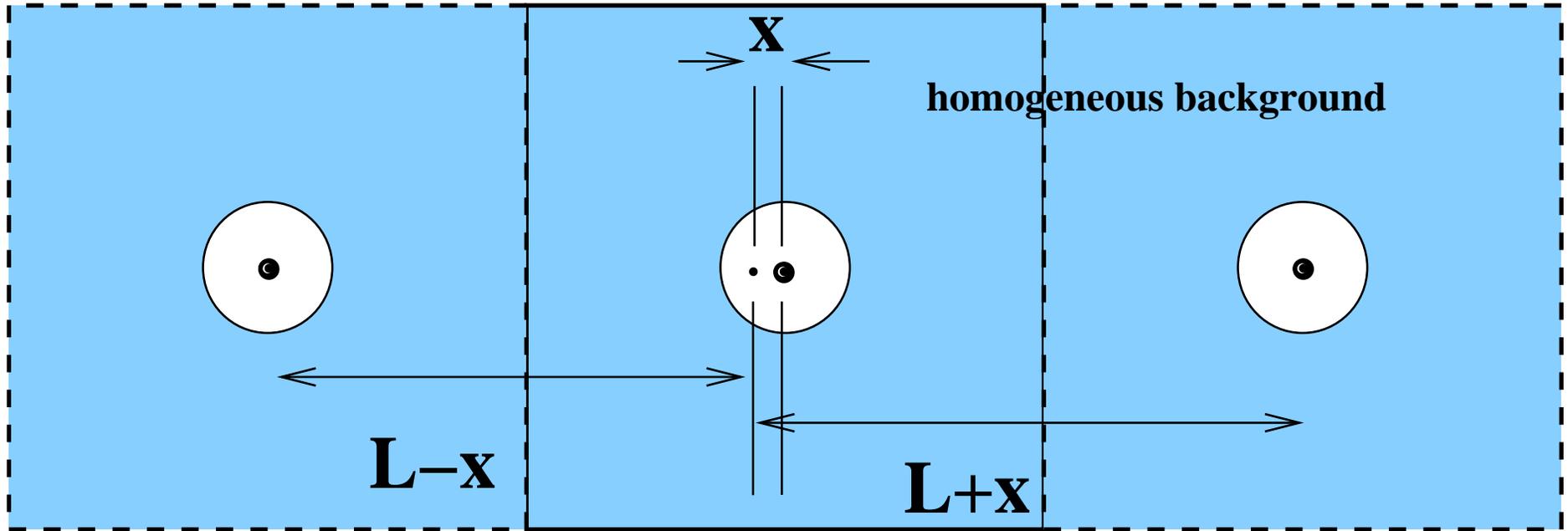
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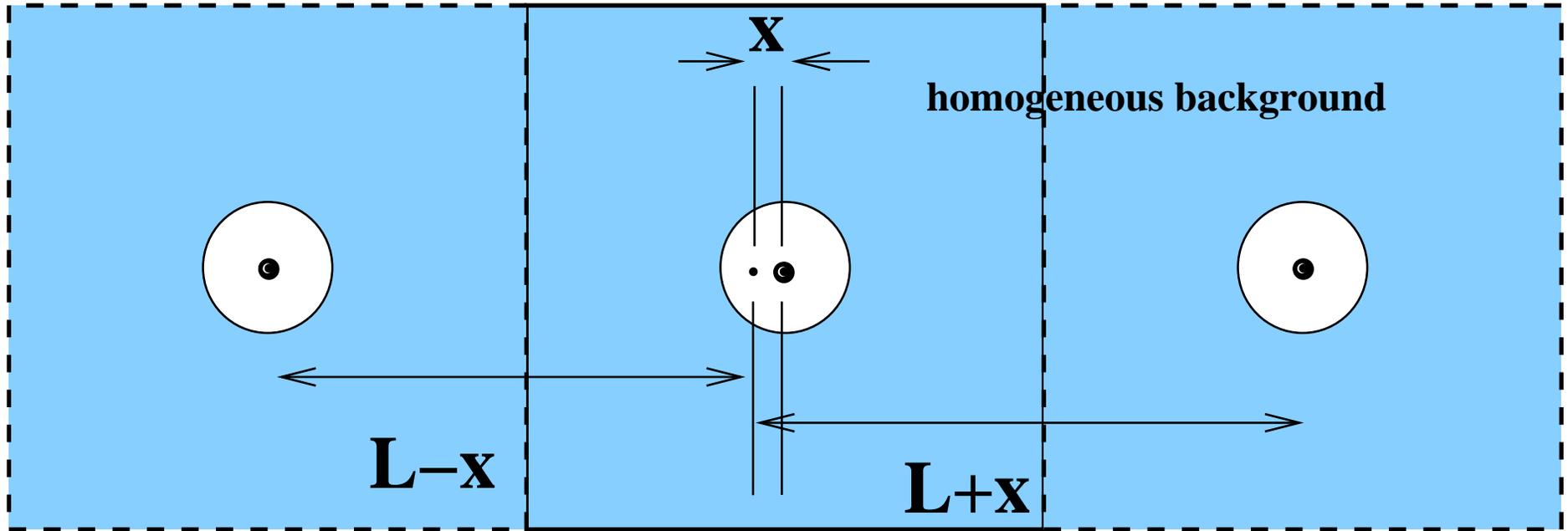
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- scalar averaging and dynamical topology change (e.g. black holes): Brunswic & Buchert (CQG, 2020) [arXiv:2002.08336](https://arxiv.org/abs/2002.08336)

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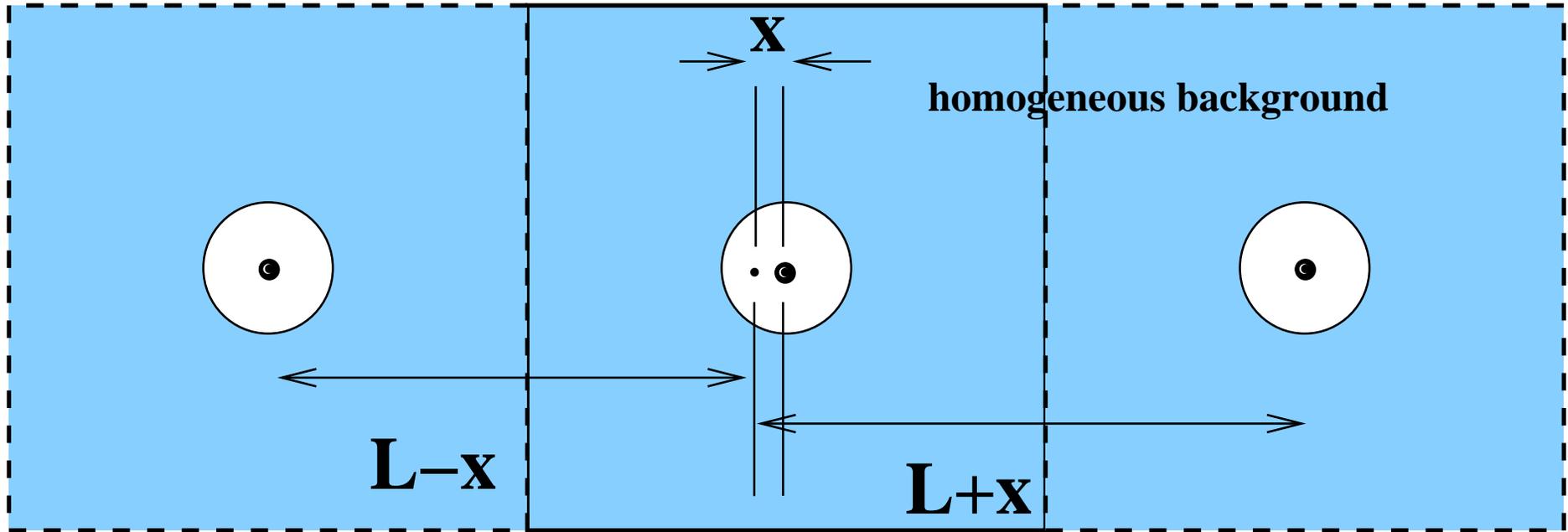


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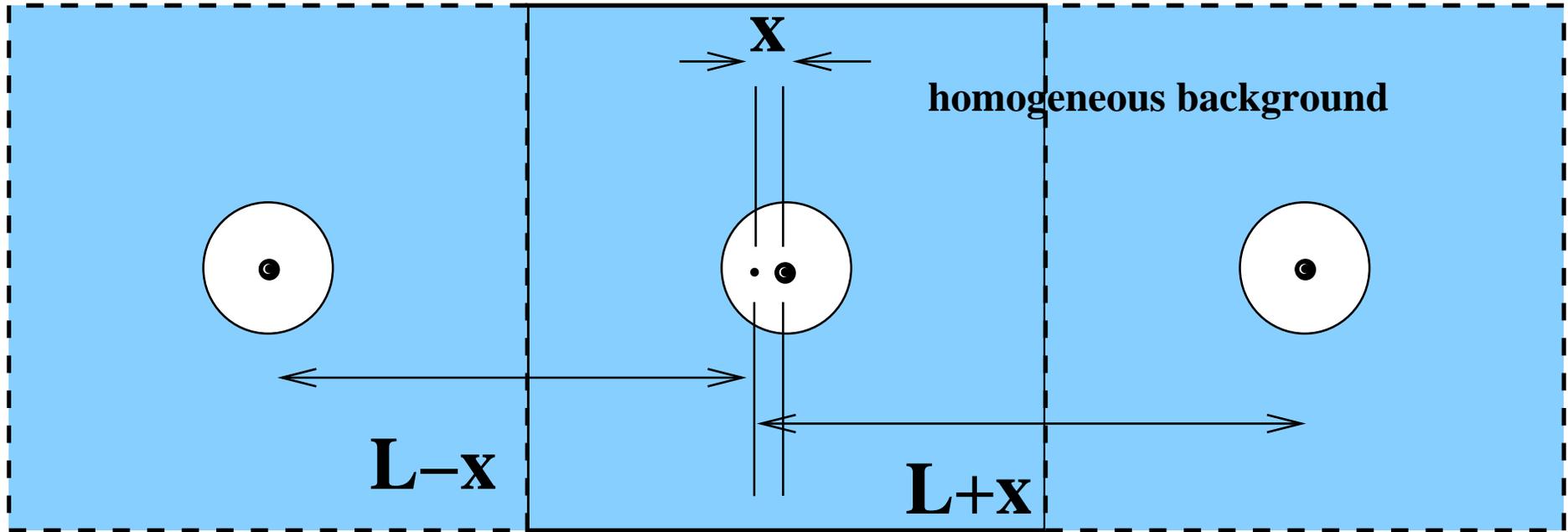
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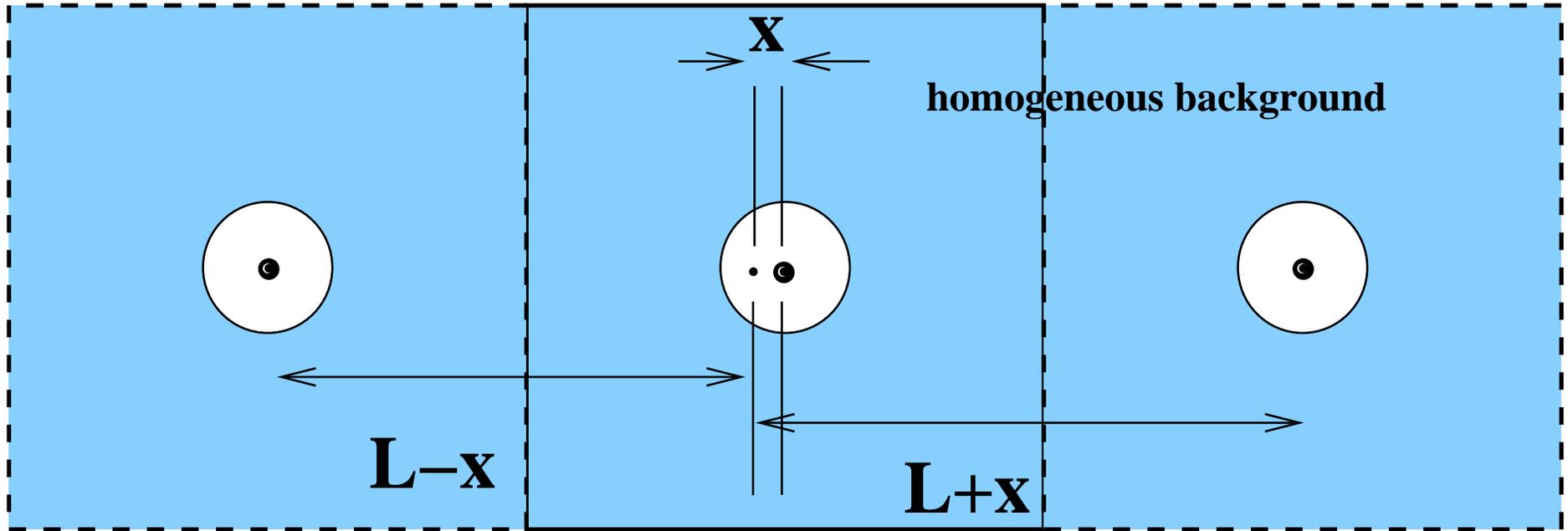
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- consider only first layer of topological images (e.g. particle horizon)
  - $\mathbb{T}^3 = \mathbb{E}^3 / \mathbb{Z}^3 \Rightarrow \ddot{x}_{\text{resid}} \propto (x/L)^3 + \dots$
  - $\mathbb{S}^3 / T^* \equiv M_6$  (octahedral space)  $\Rightarrow \ddot{x}_{\text{resid}} \propto (x/R_C)^3 + \dots$
  - $\mathbb{S}^3 / O^*$  (truncated cube space)  $\Rightarrow \ddot{x}_{\text{resid}} \propto (x/R_C)^3 + \dots$
  - $\mathbb{S}^3 / I^* \equiv M_8$  (Poincaré dodecahedral space)  
 $\Rightarrow \ddot{x}_{\text{resid}} \propto (x/R_C)^5 + \dots$
- *topological acceleration is manifold-dependent*  
Roukema & Rózański [arXiv:0902.3402](https://arxiv.org/abs/astro-ph/09023402), A&A, 502, 27

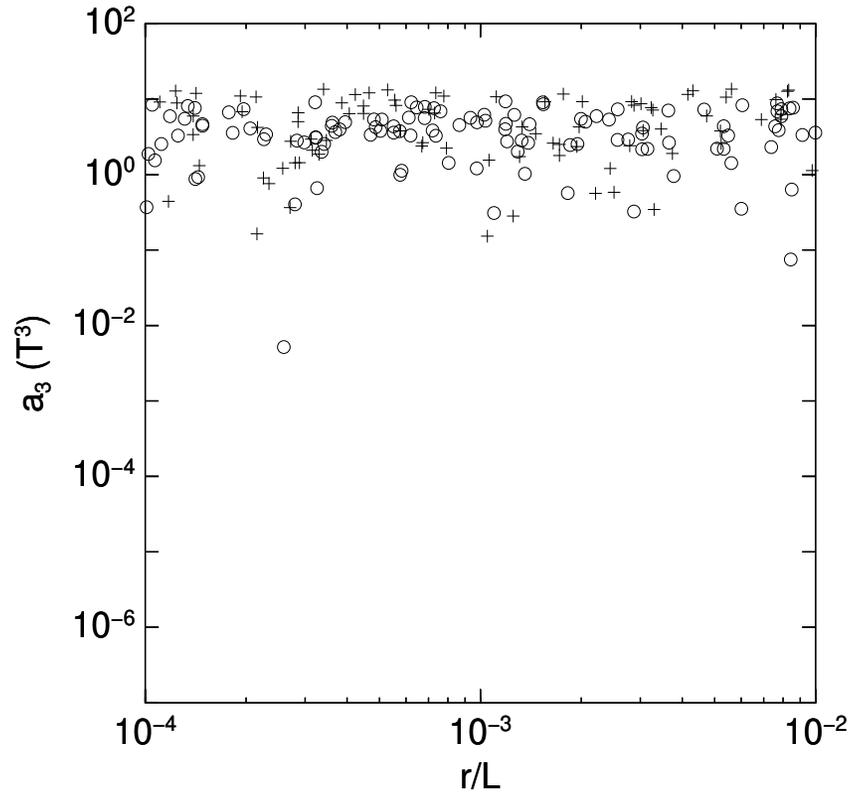
# Newt. non-Euclid. top.accel.

NEN:  $\Phi_S(\xi) \propto -\cot \xi (1 - \xi/\pi) + A$

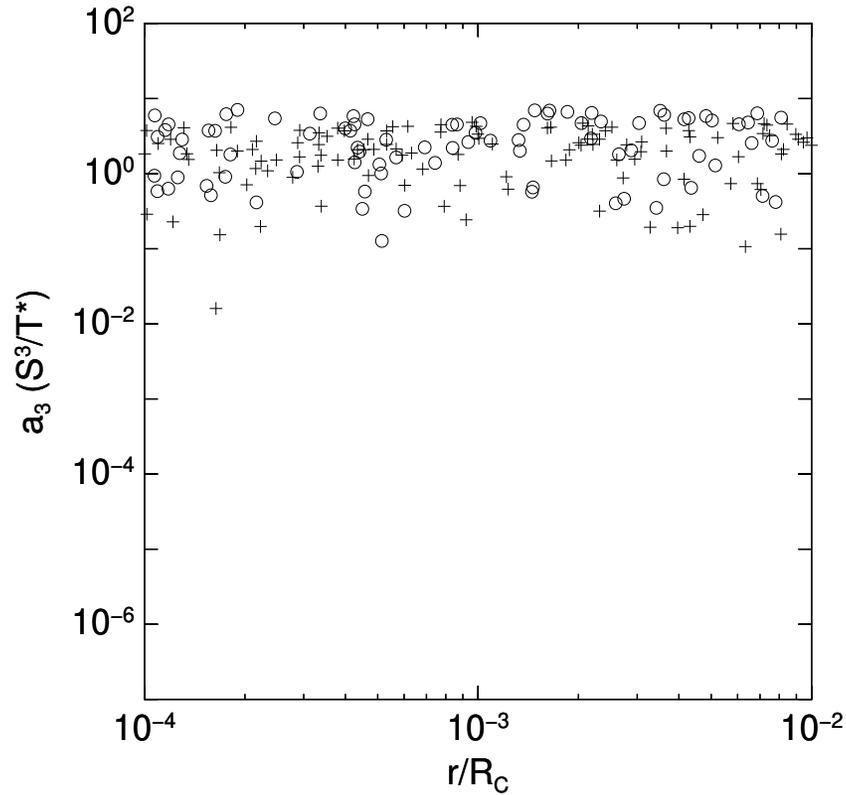
Topology	$N_\Sigma$	$\Phi_{-1}$	$\Phi_1$	$\Phi_2$	$\Phi_3$	$\Phi_4$	$\Phi_5$
Euclidean (infinite or Thurston-type)							
$\mathbb{E}^3$		-1	0	0	0	0	0
$\mathbb{T}^3$		-1	0	$-\frac{2\pi}{3} \frac{1}{V_\Sigma}$	0	-	0
Spherical							
$S^3$	1	-1	$\frac{1}{3} \frac{\mathcal{R}}{6}$	$-\frac{2\pi}{3} \frac{1}{V_\Sigma}$	$\frac{1}{45} \left(\frac{\mathcal{R}}{6}\right)^2$	$-\frac{2\pi}{45} \frac{\mathcal{R}/6}{V_\Sigma}$	$\frac{2}{945} \left(\frac{\mathcal{R}}{6}\right)^3$
$M_3$	8	-1	$\frac{1}{3} \frac{\mathcal{R}}{6}$	$-\frac{2\pi}{3} \frac{1}{V_\Sigma}$	$\frac{1}{45} \left(\frac{\mathcal{R}}{6}\right)^2$	-	$\frac{2}{945} \left(\frac{\mathcal{R}}{6}\right)^3$
$M_6$	24	-1	$\frac{1}{3} \frac{\mathcal{R}}{6}$	$-\frac{2\pi}{3} \frac{1}{V_\Sigma}$	$\frac{1}{45} \left(\frac{\mathcal{R}}{6}\right)^2$	-	$\frac{2}{945} \left(\frac{\mathcal{R}}{6}\right)^3$
$M_7$	120	-1	$\frac{1}{3} \frac{\mathcal{R}}{6}$	$-\frac{2\pi}{3} \frac{1}{V_\Sigma}$	$\frac{1}{45} \left(\frac{\mathcal{R}}{6}\right)^2$	$-\frac{2\pi}{45} \frac{\mathcal{R}/6}{V_\Sigma}$	$\frac{2}{945} \left(\frac{\mathcal{R}}{6}\right)^3$
Hyperbolic (infinite)							
$\mathbb{H}^3$		-1	$\frac{1}{3} \frac{\mathcal{R}}{6}$	0	$\frac{1}{45} \left(\frac{\mathcal{R}}{6}\right)^2$	0	$\frac{2}{945} \left(\frac{\mathcal{R}}{6}\right)^3$

even terms  $\Rightarrow$  closed; odd terms  $\Rightarrow$  curved

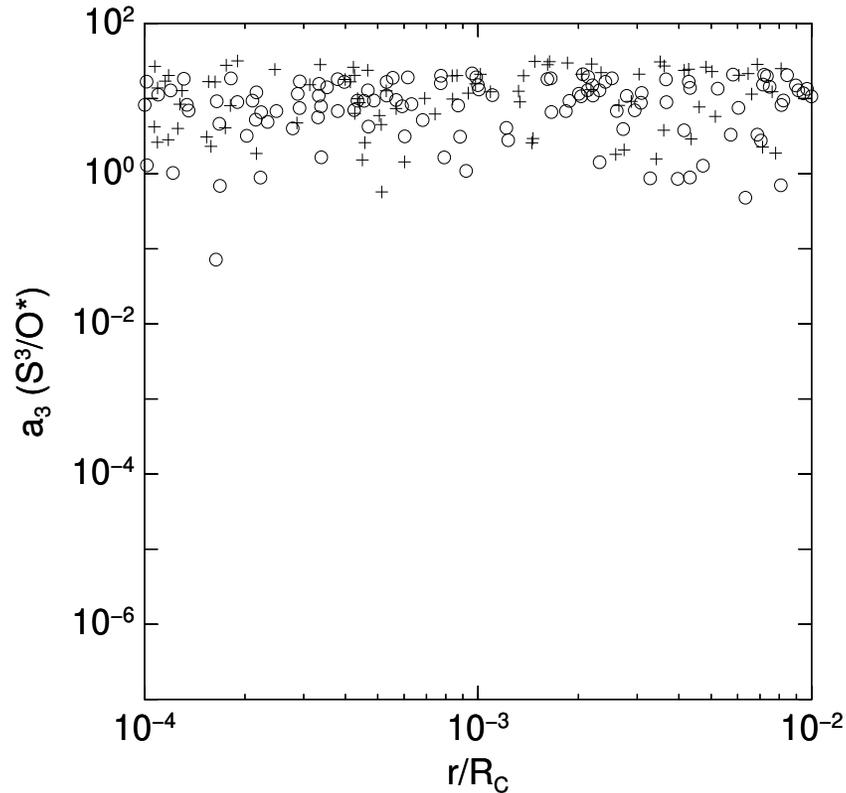
Vigneron & Roukema (2022) [arXiv:2201.09102](https://arxiv.org/abs/2201.09102)

$T^3, S^3/\Gamma$ 

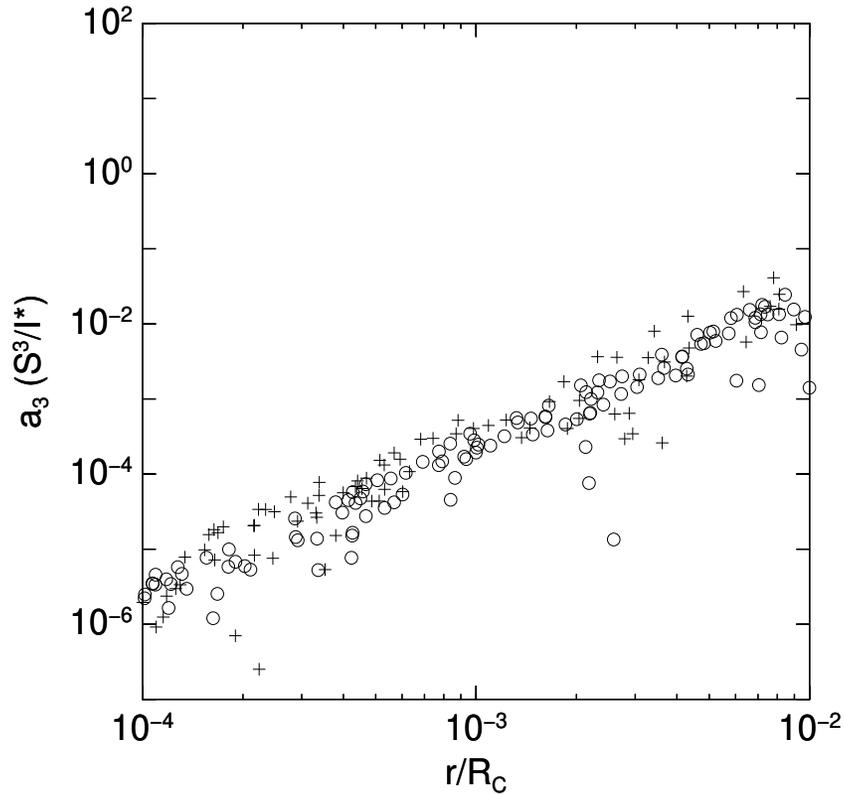
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# $T^3, S^3/\Gamma$

- Some spaces are more equal than others.
- Roukema & Róžański [arXiv:0902.3402](#), A&A, 502, 27
- Newton–Cartan approach for preparing for full GR approach:  
Vigneron (2020, PRD) [arXiv:2010.10247](#); Vigneron (2021, PRD) [arXiv:2012.10213](#); Vigneron (2022a, PRD) [arXiv:2109.10336](#);  
Vigneron (2022b, CQG) [arXiv:2201.02112](#); Vigneron & Roukema (2022) [arXiv:2201.09102](#)

# Is topolog. acceleration relativistic?

- Korotkin & Nikolai (1994) [arXiv:gr-qc/9403029](#) solution: Schwarzschild-like BH in  $S^1 \times E^2$  (slab space =  $T^1$ )
- outside event horizon, inside topology scale:

$$\ddot{x} = 4\zeta(3)G \frac{M}{L^3} x \propto x$$

Ostrowski, Roukema & Buliński (2012) [arXiv:1109.1596](#)

⇒ Yes.

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empirical strategies: [arXiv:astro-ph/0010189](https://arxiv.org/abs/astro-ph/0010189)

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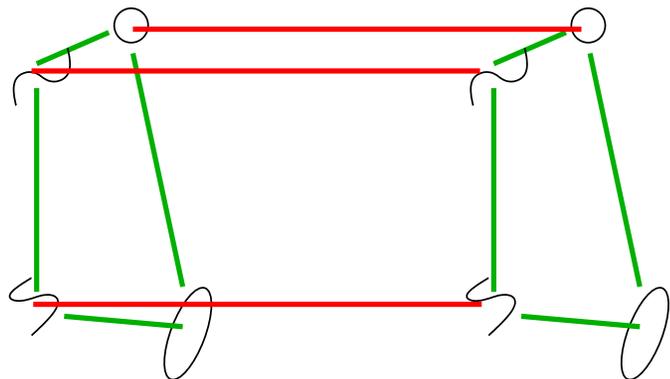
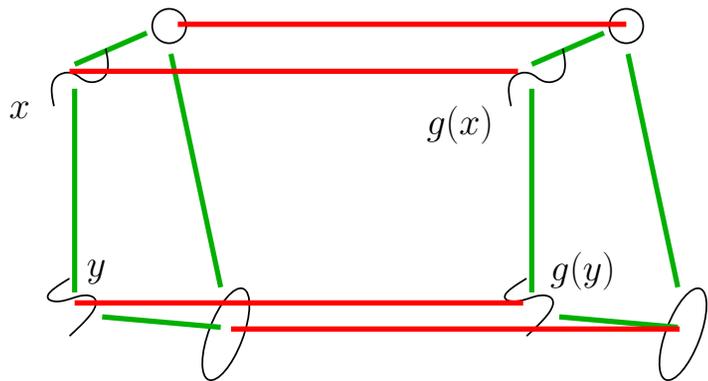
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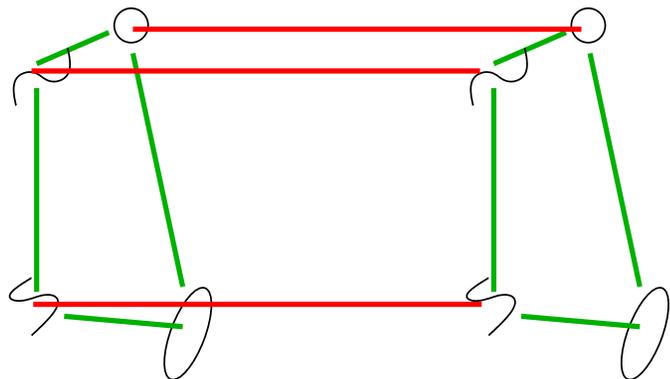
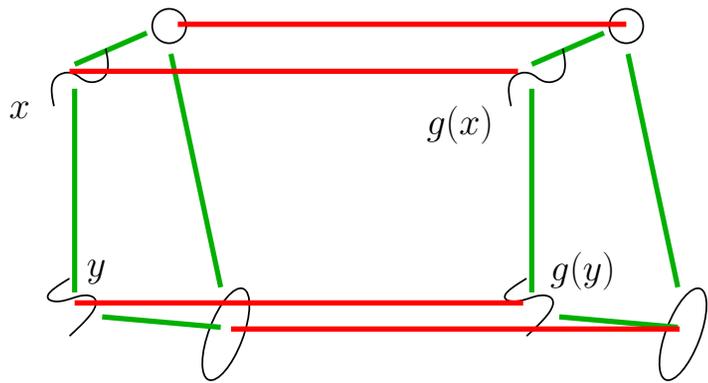
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# 3D strategies—pair types

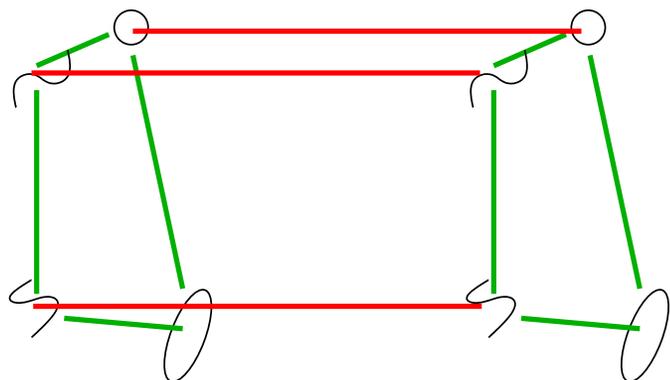
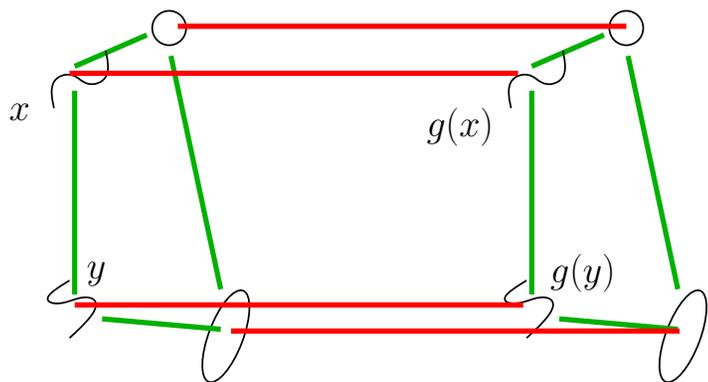


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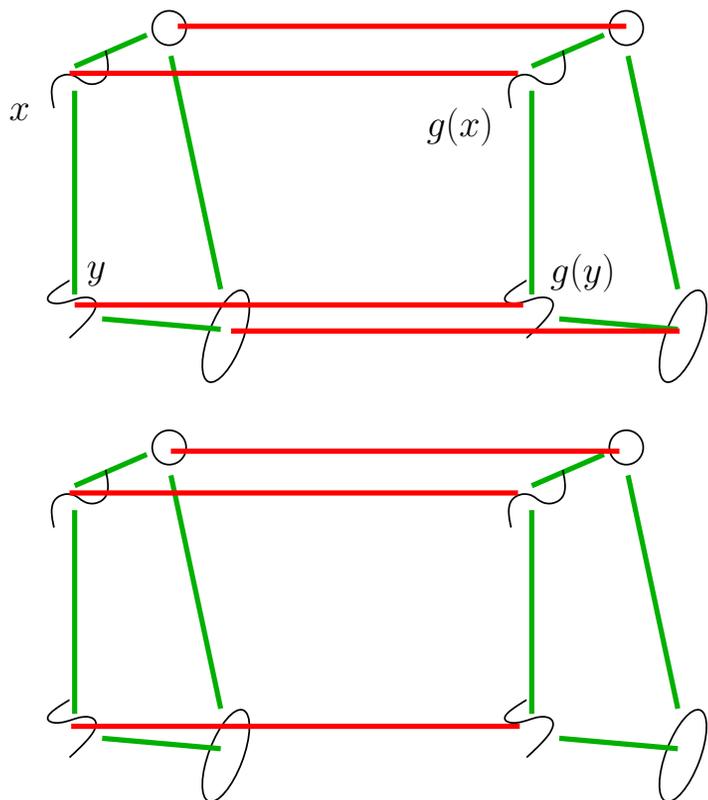
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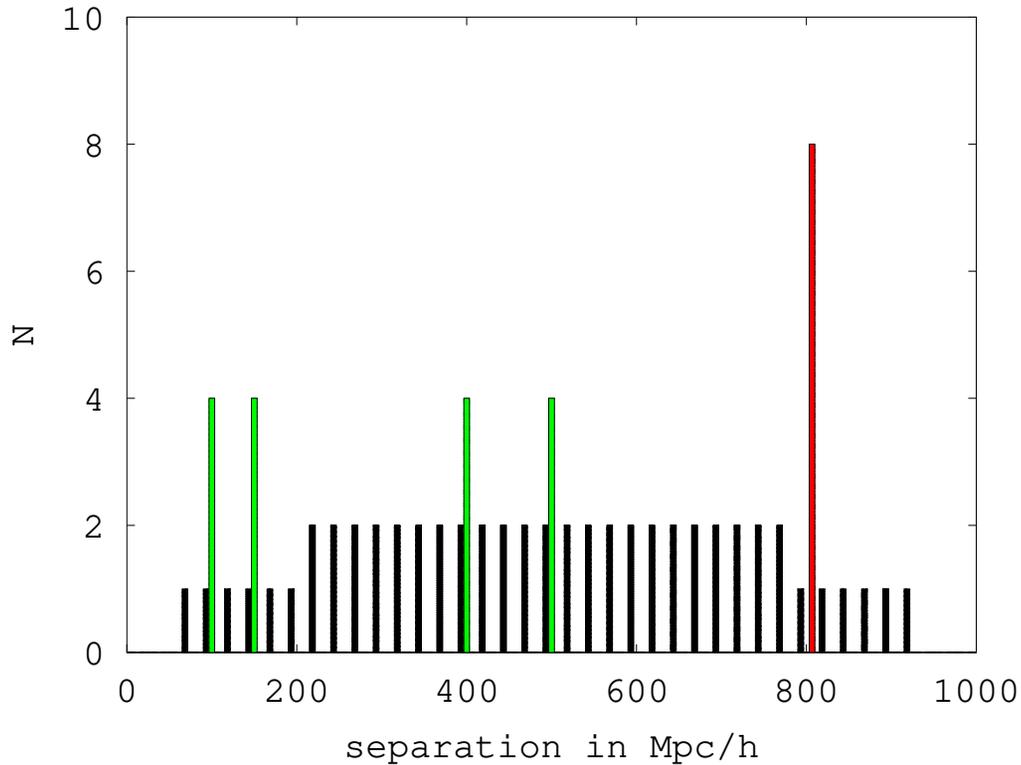


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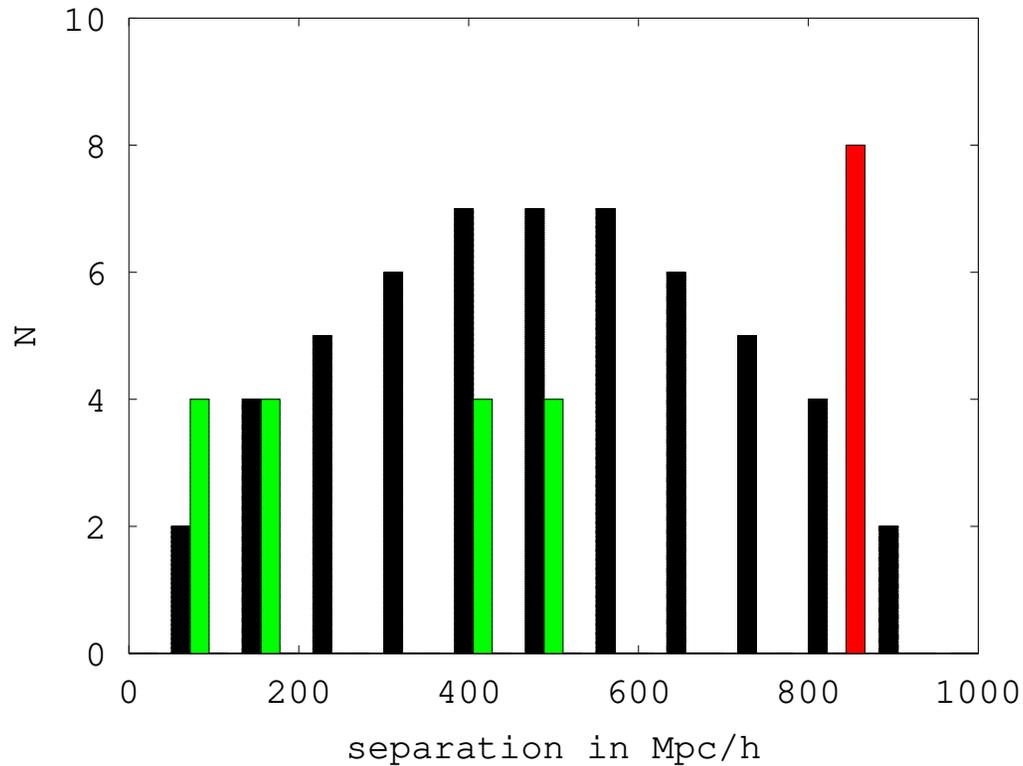


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- quadruples + successive filters + collect membership  $s$  of quadruples  
[Fujii & Yoshii \(2013\)](#)

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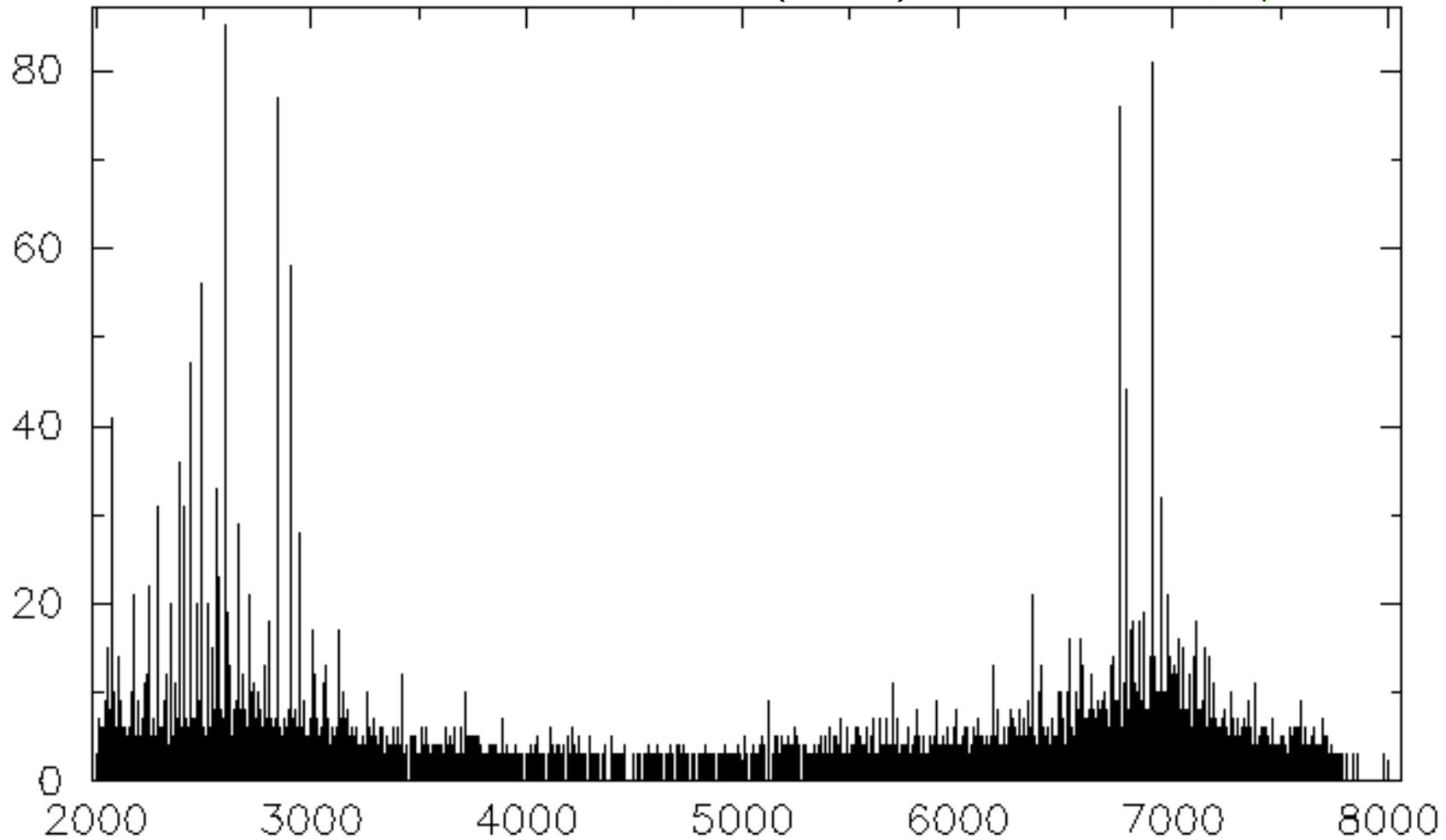
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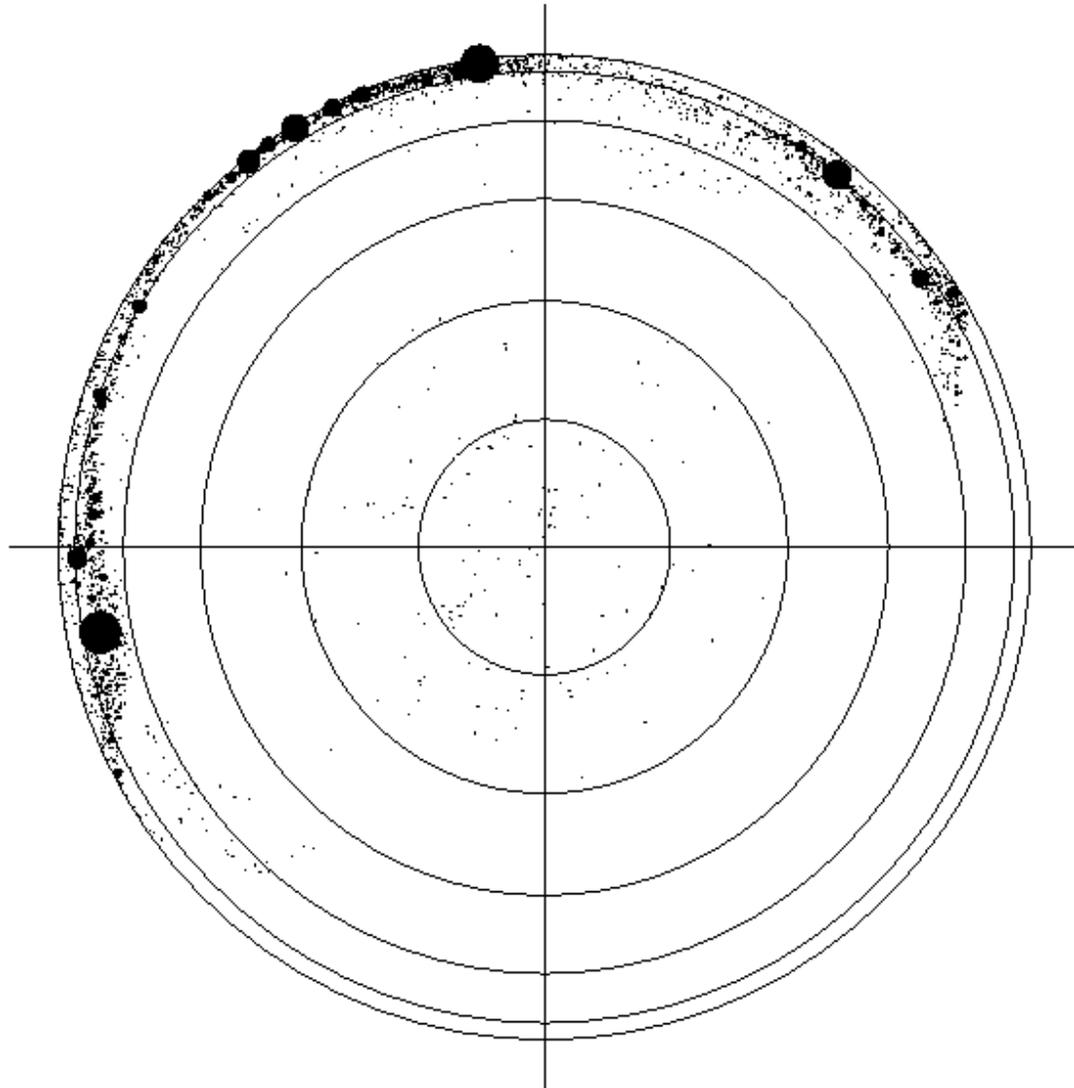
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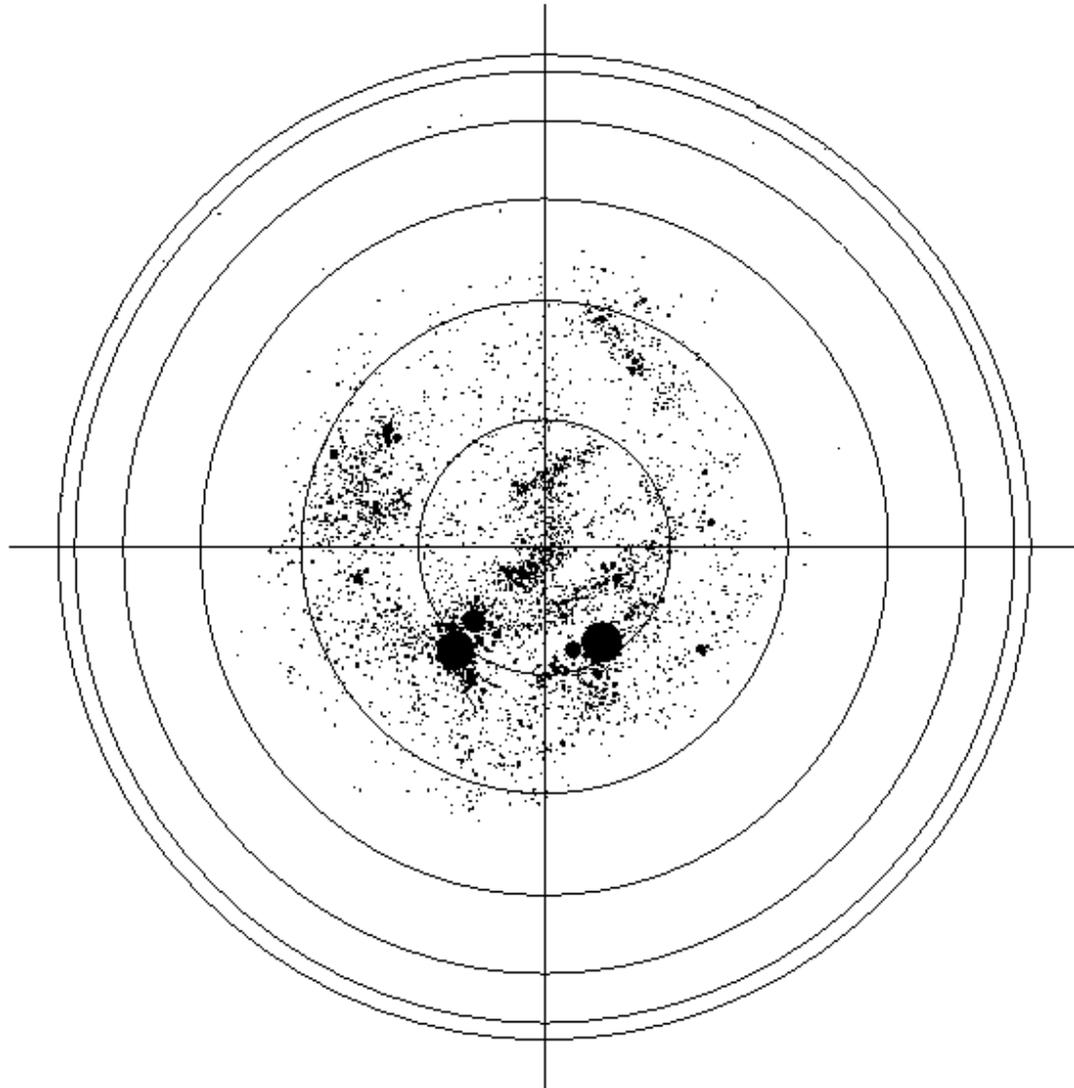
# AGN Catalogues

gmod range: 2000 – 5000  
ztol=0.50%, gtol=1.00%, angtol=0.0050 rad  
gtolfact=100., angtolfact= 30., gmodmin= 50.  
input file: analysepairs.qso\_results, Omega\_m=0.30



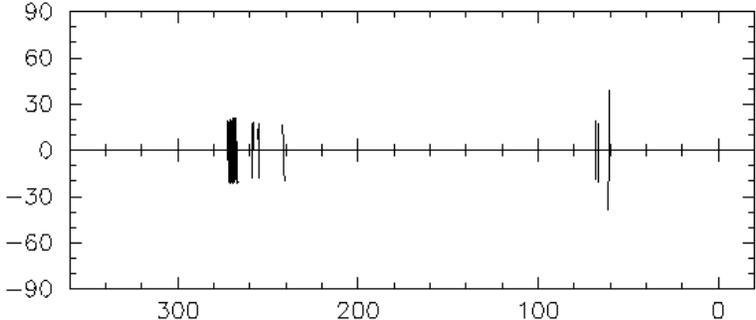
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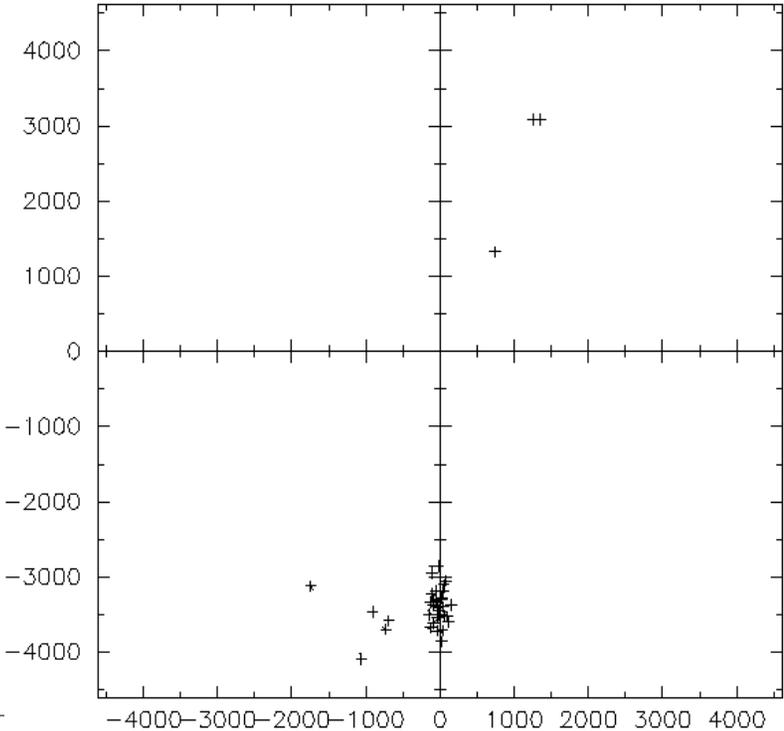


# AGN Catalogues

Positions of objects on matched discs (weighted cleaned data)



RA=17 47 00.0 Dec=10 31 54 l= 35.253 b=19 02 03  
group # 2285, number of pairs= 36, gmod=2387.1314 Mpc



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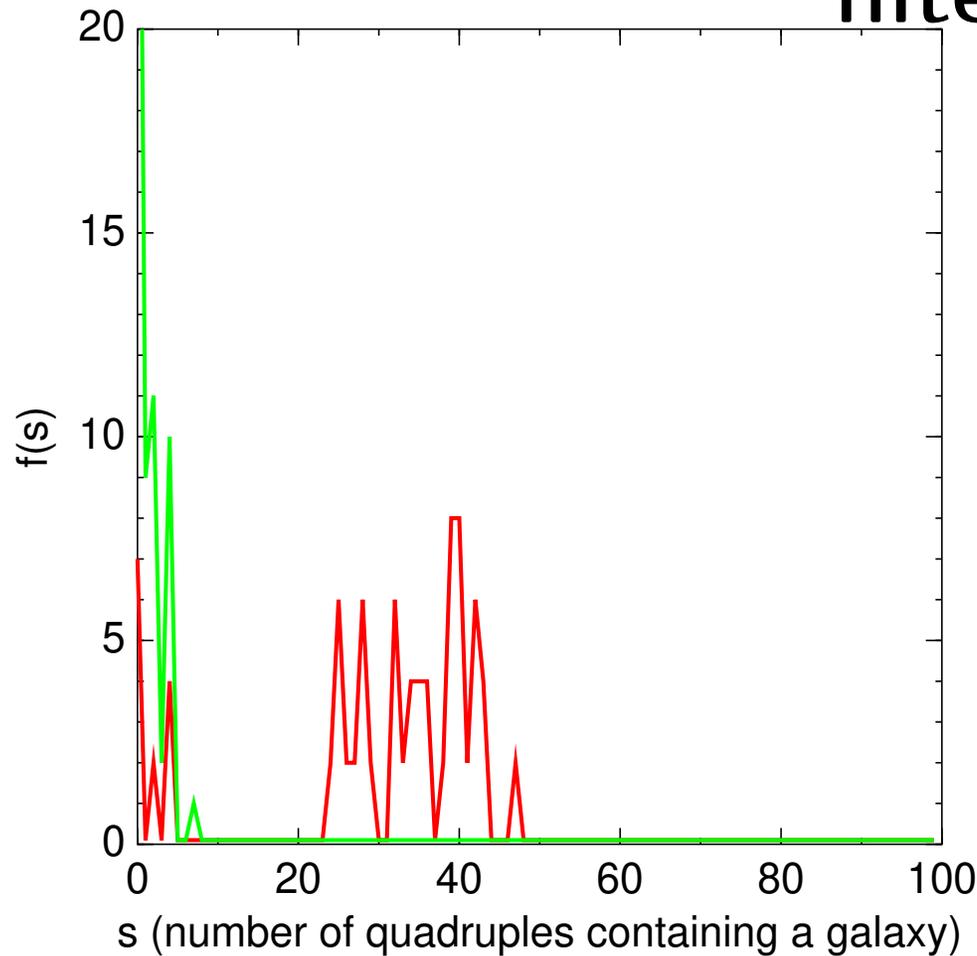
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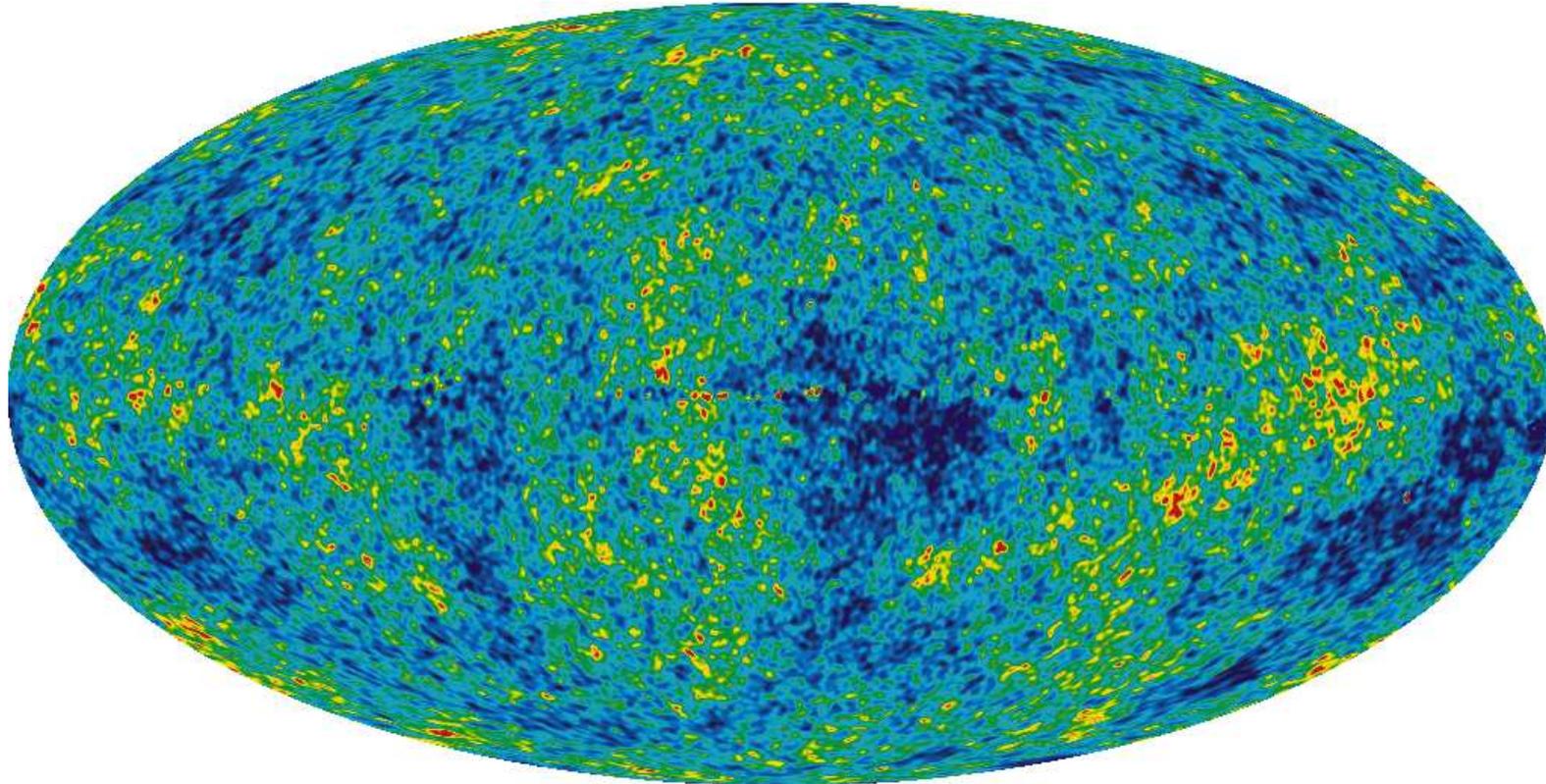


simulation of  $s$  histogram for Lyman break galaxies (LBGs) at  $z \approx 6$

green: simply connected; red:  $T^3$

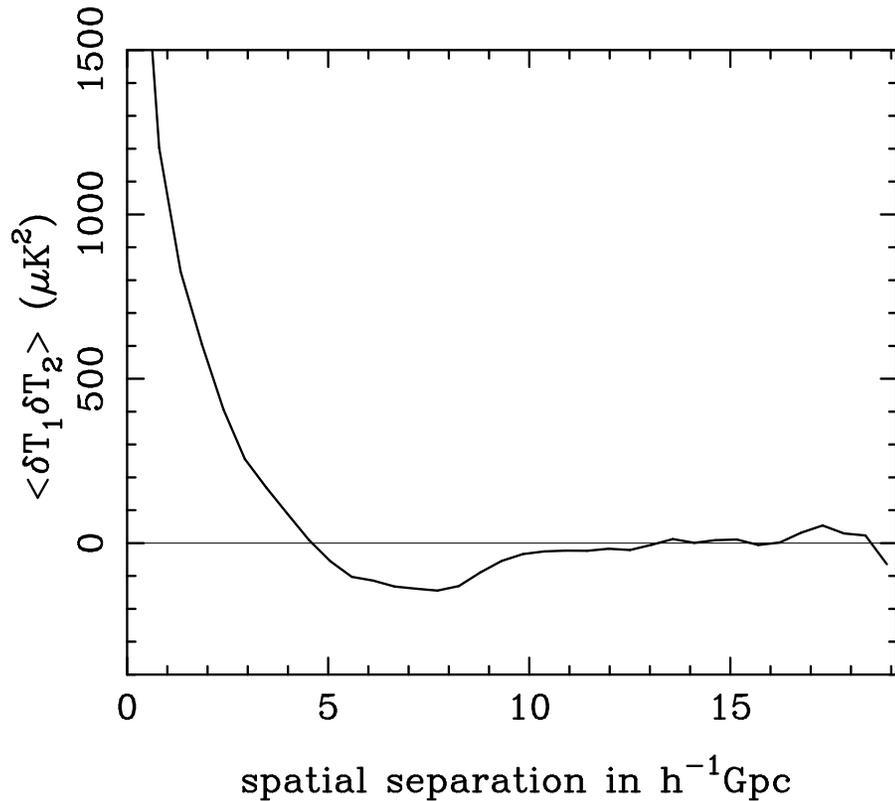
ADS:2014MNRAS.437.1096R

# 2D methods: structure cutoff



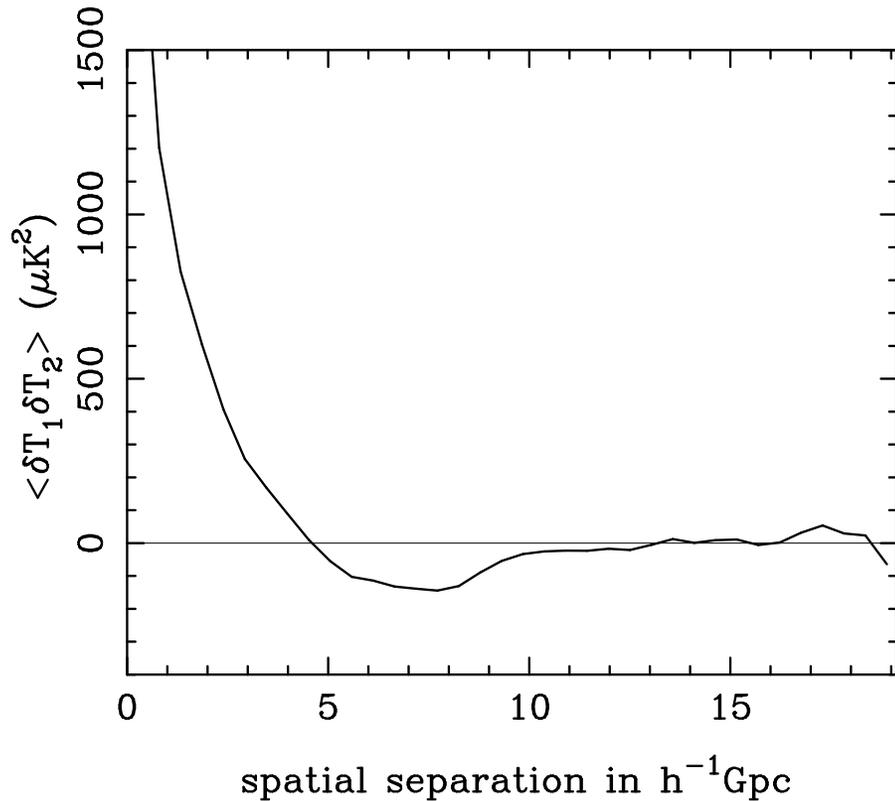
WMAP 5yr ILC (internal linear combination)

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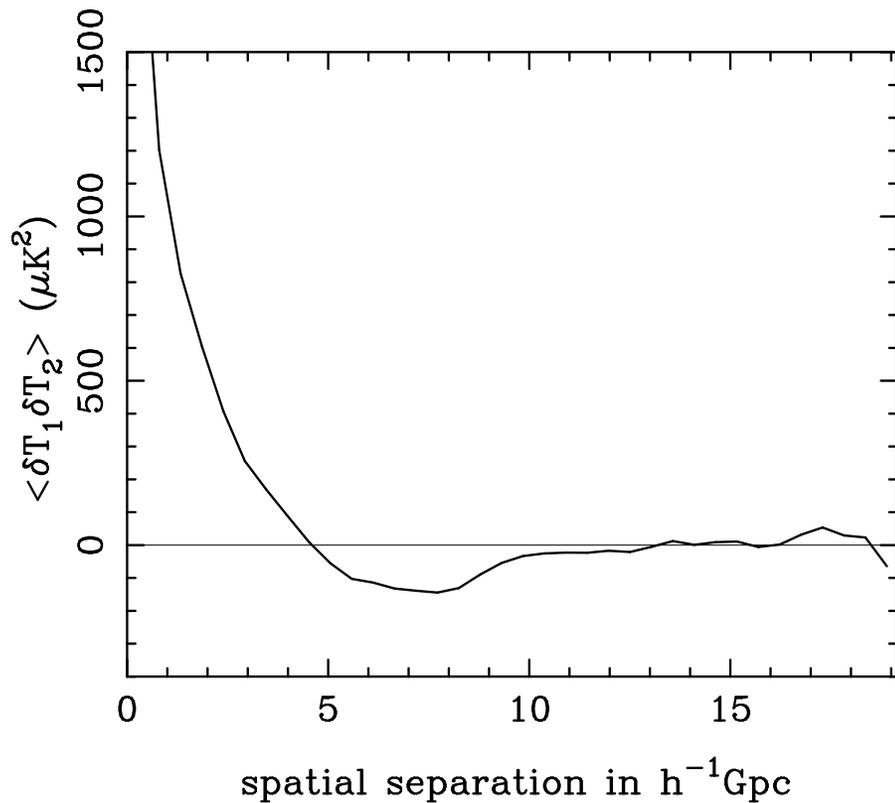
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Starobinsky (1993); Stevens et al. (1993)

# The Identified Circles Principle

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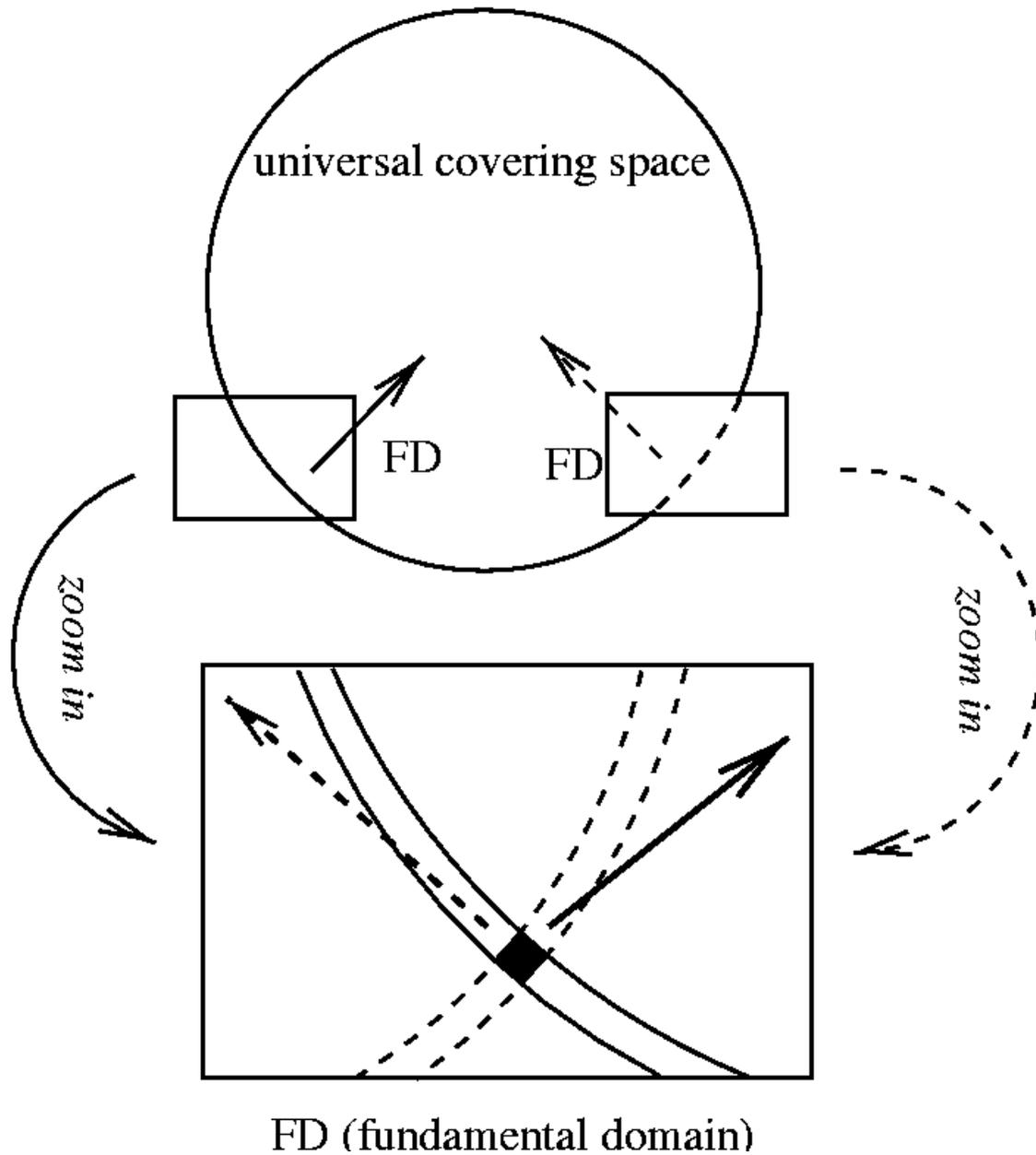
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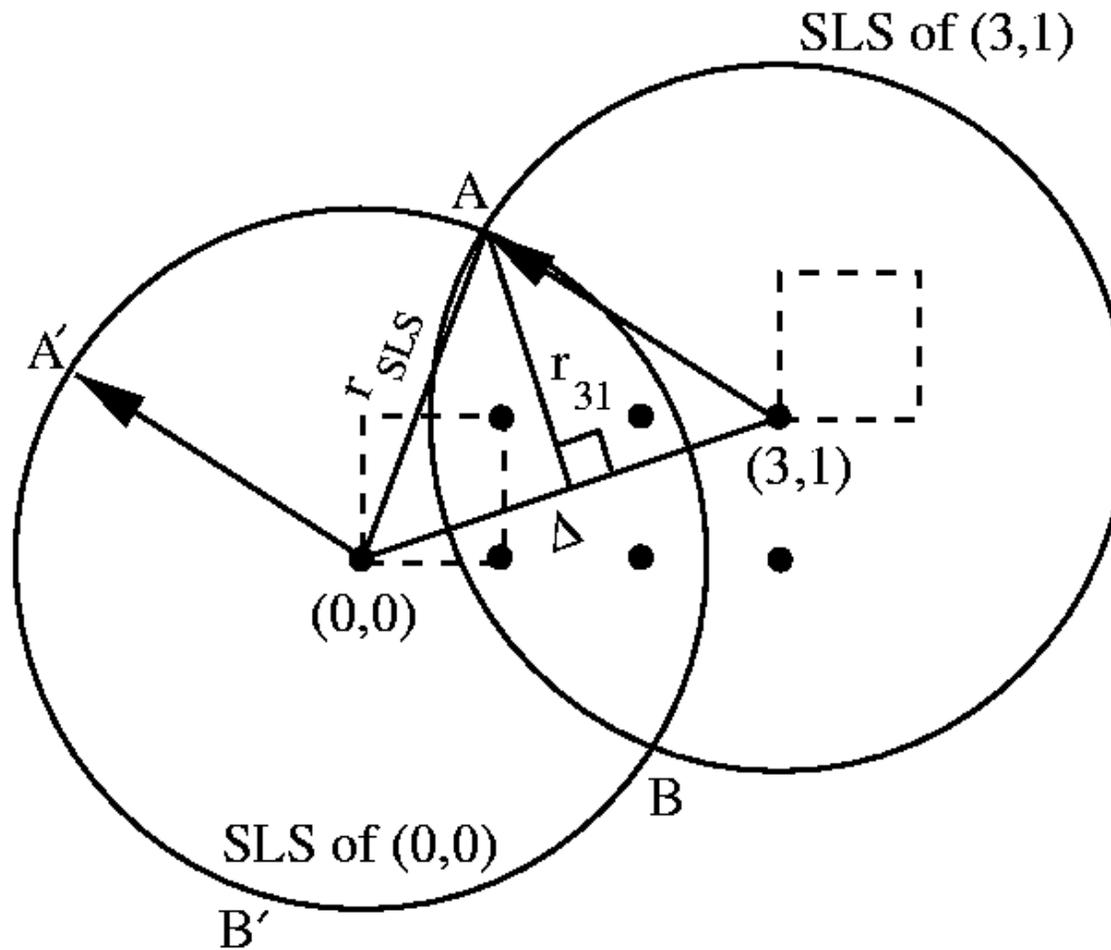
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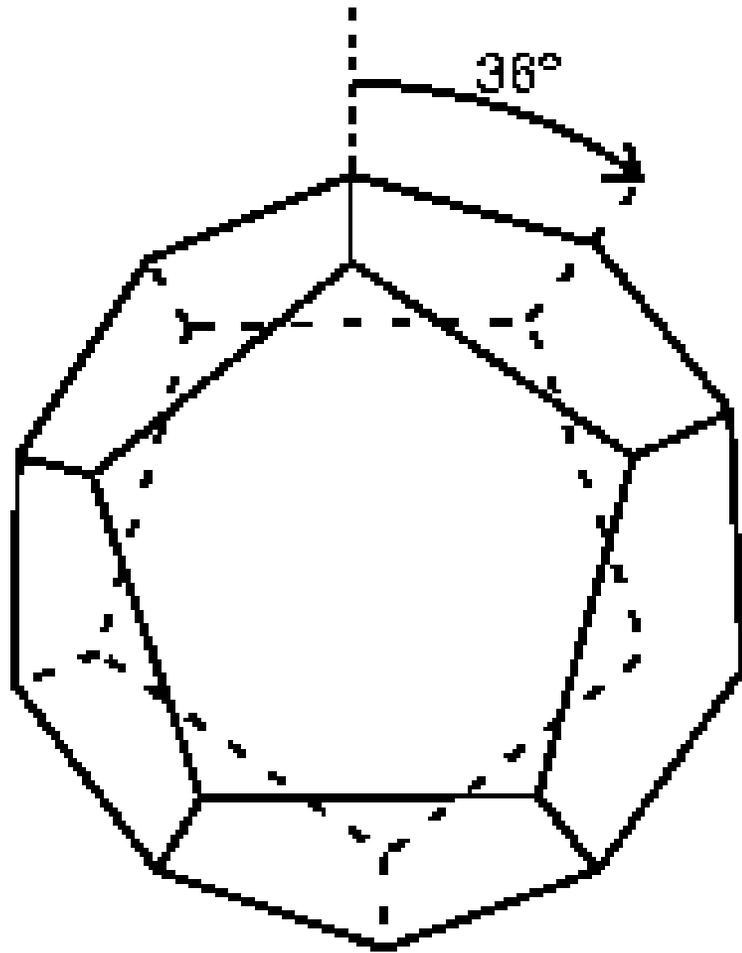
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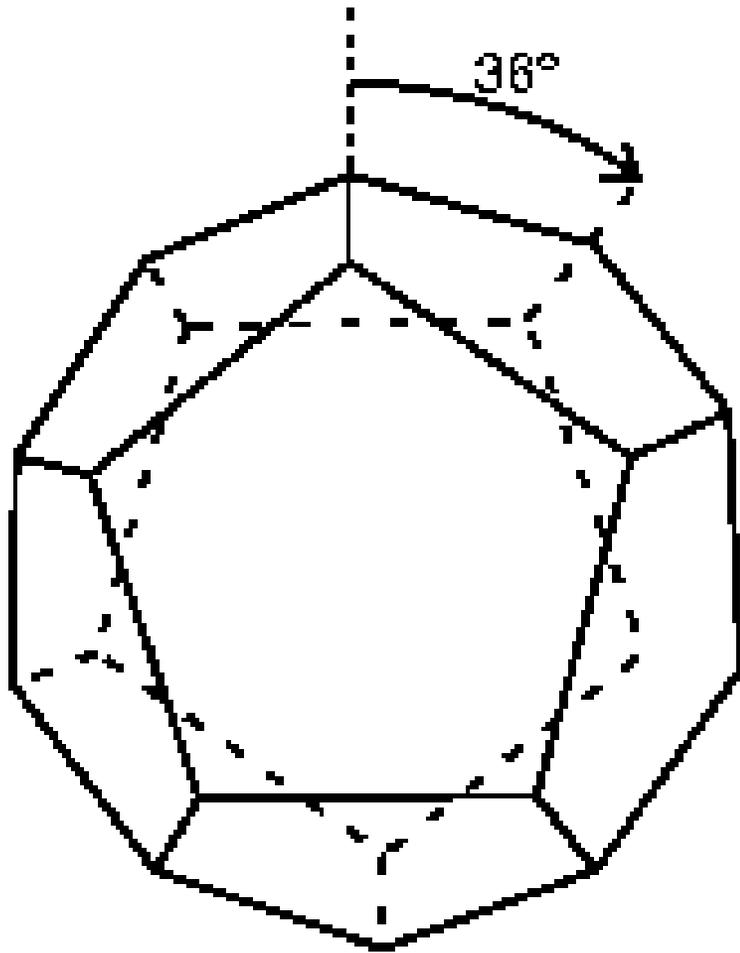


# The Poincaré Dodecahedral 3-Manifold



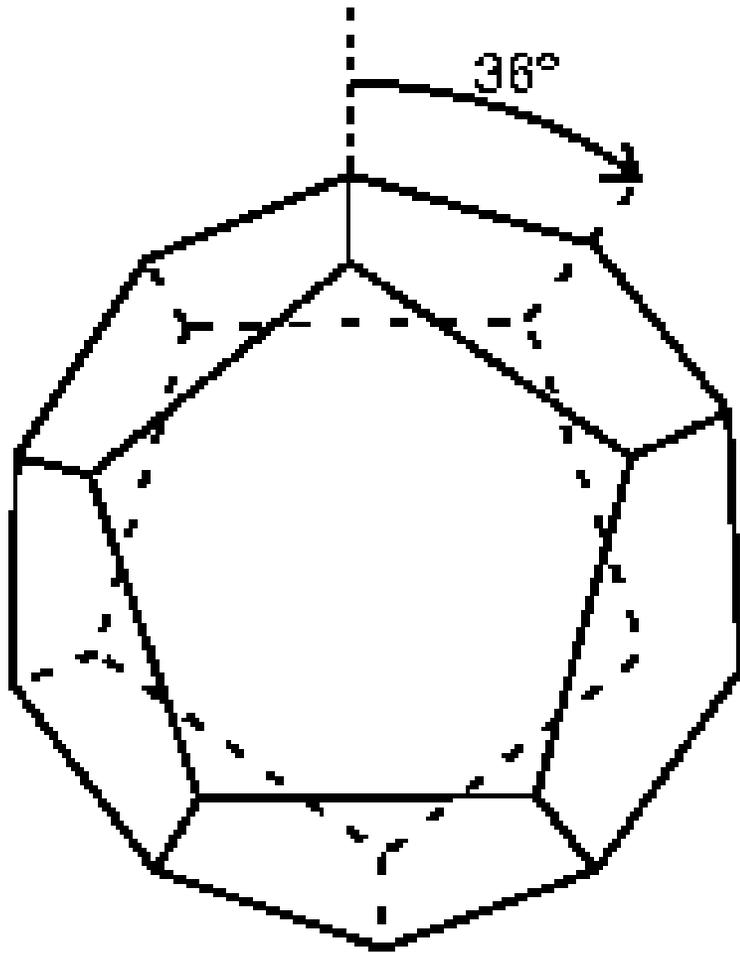
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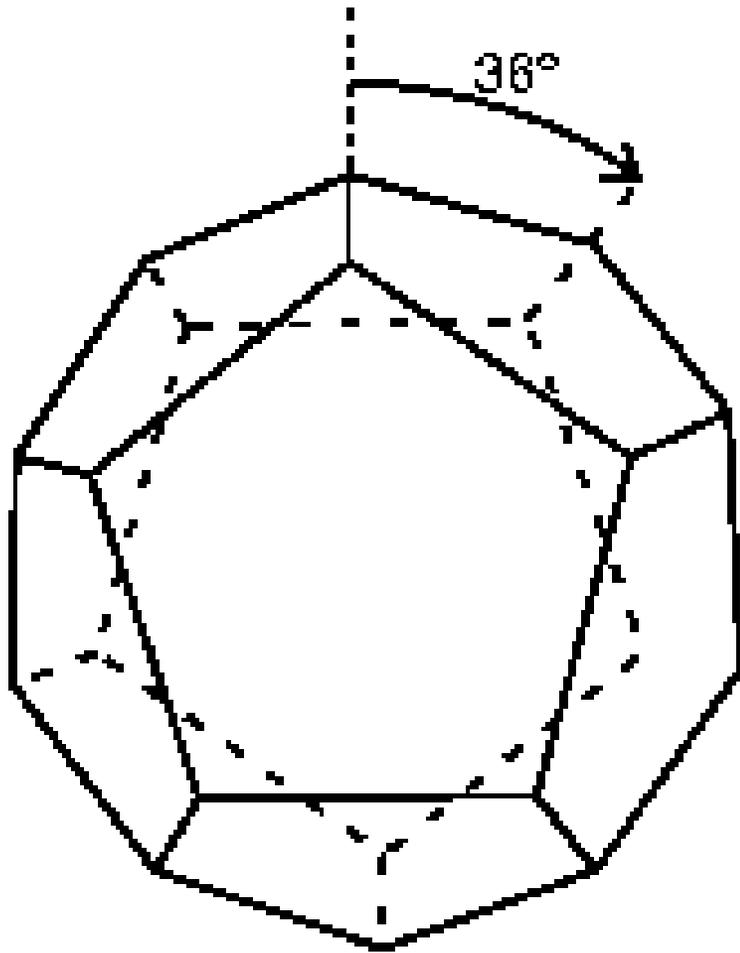
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- [Luminet et al. \(2003\)](#):  $S^3/I^*$  favoured by WMAP statistics

# Optimal cross-correlation method

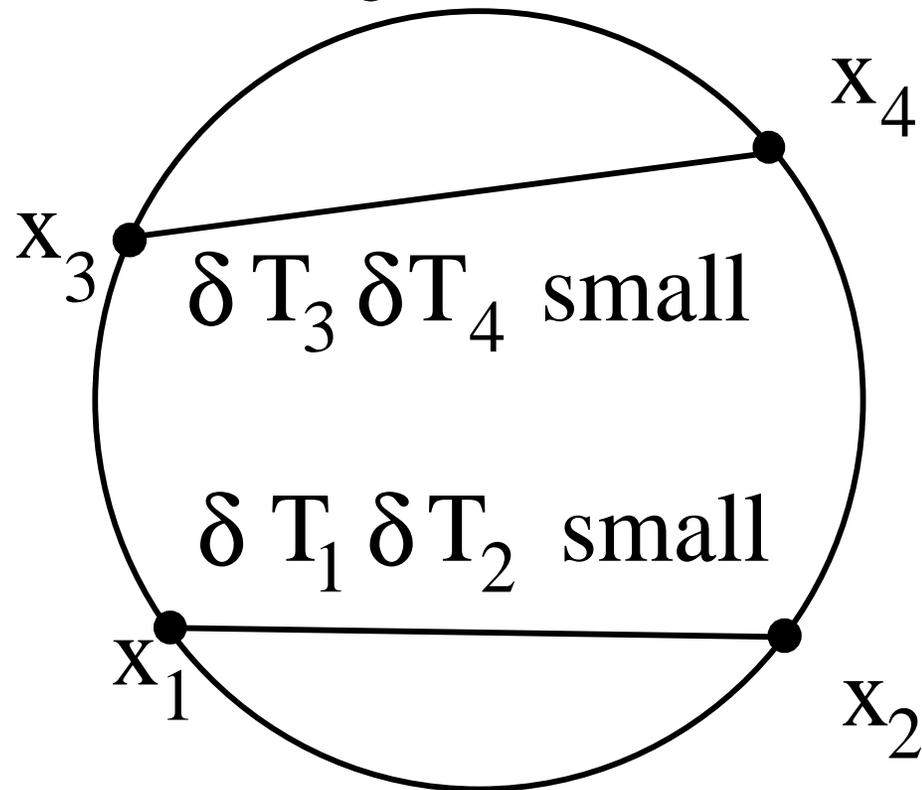
- extension to identified circles principle:

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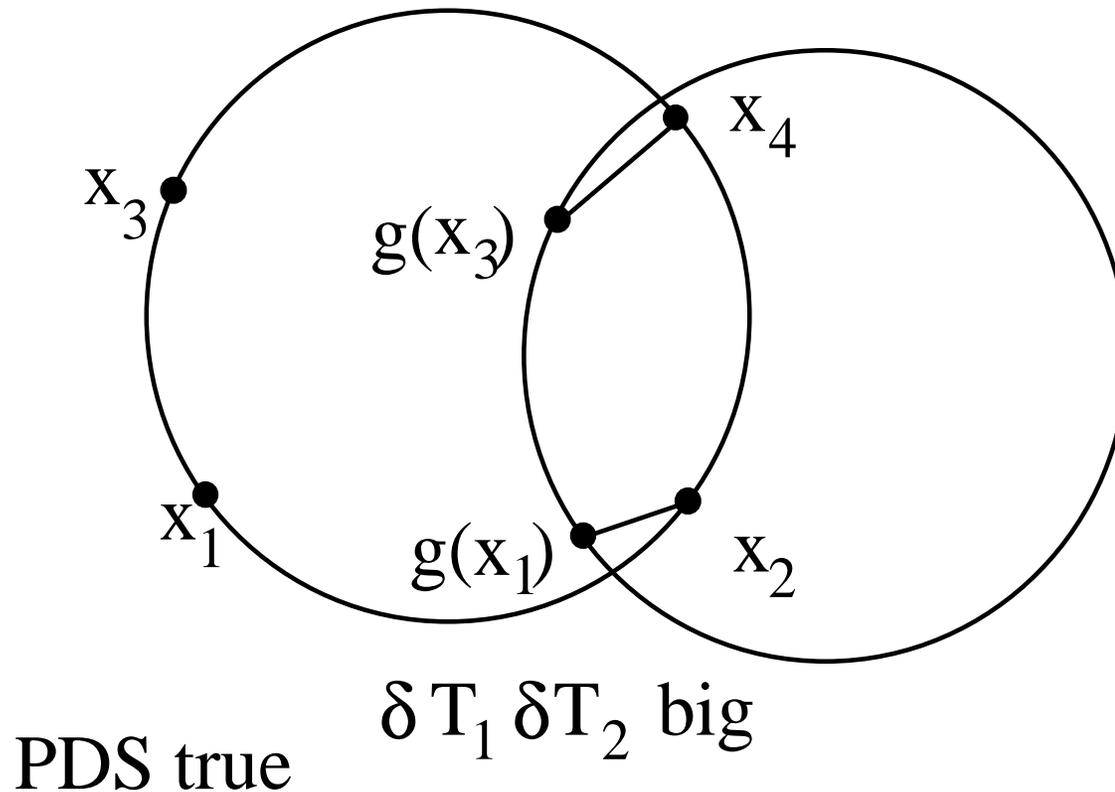


PDS false

$S^3/I^*$ :

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 $\delta T_3 \delta T_4$  big



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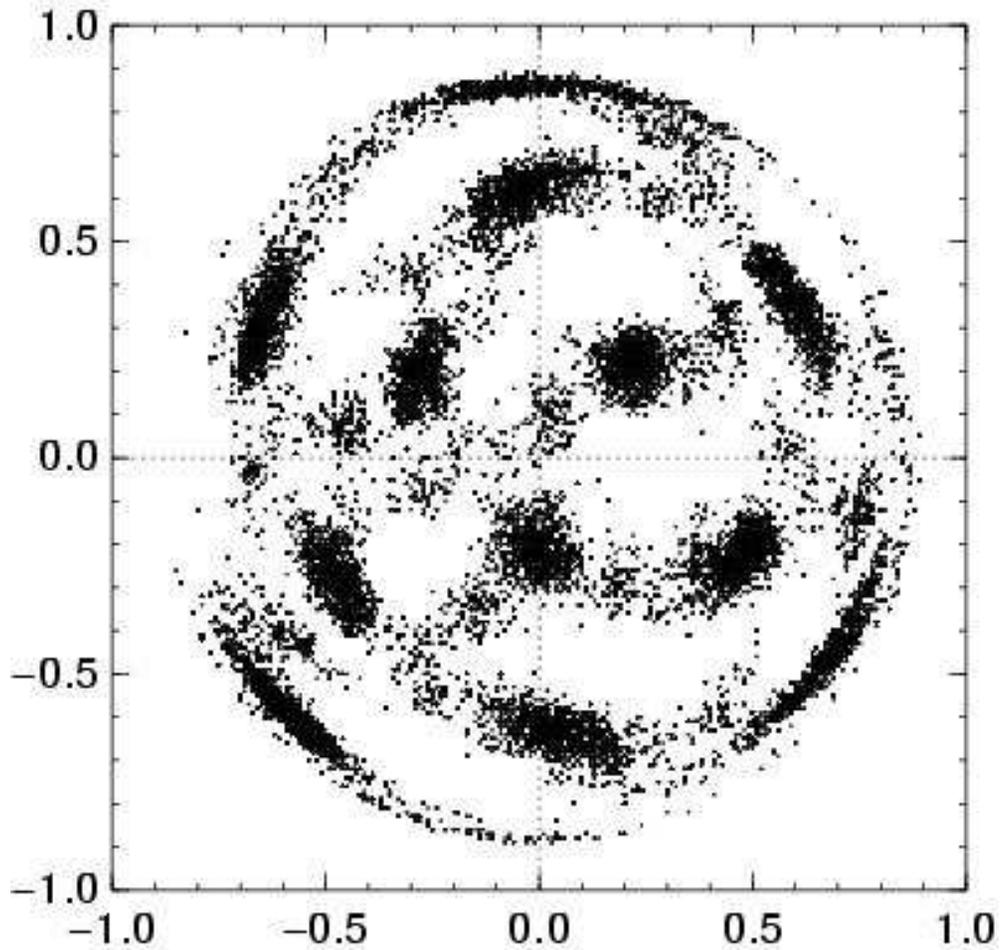
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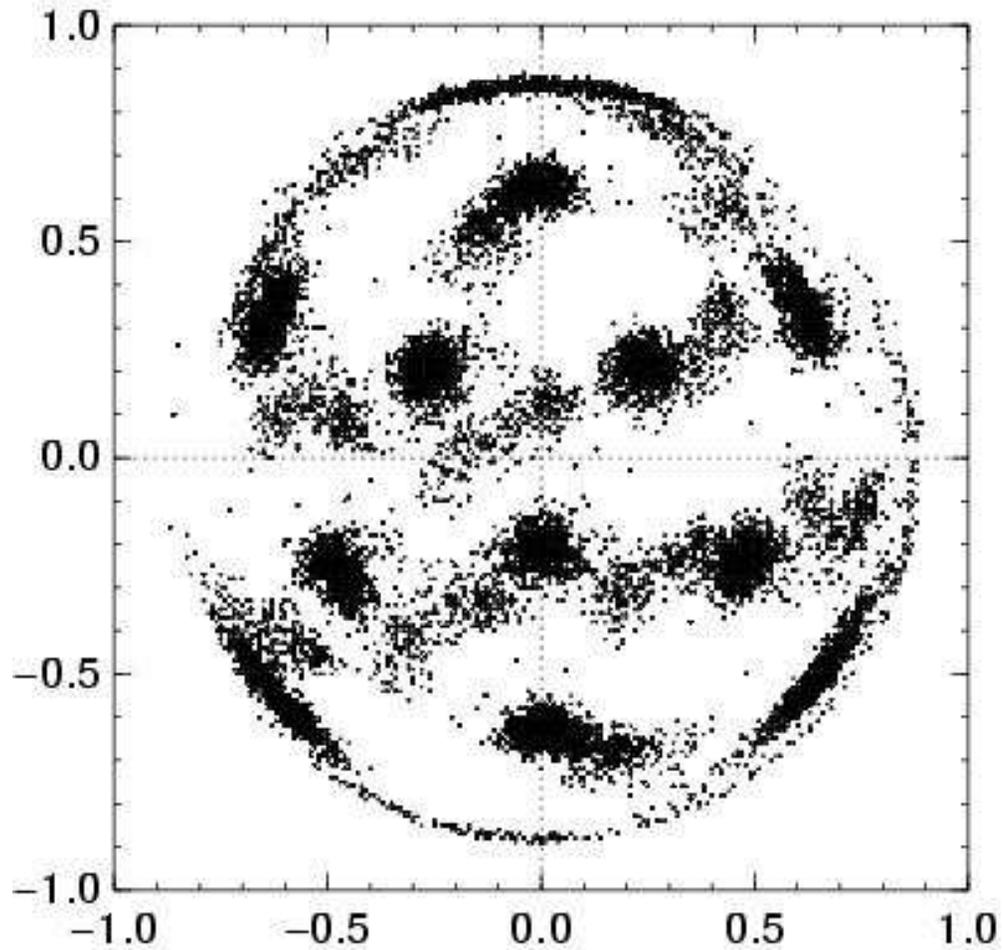
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- allow arbitrary  $\phi$  so that accidental correlations are likely to give an invalid value  $\phi \neq \pm 36^\circ$

# WMAP + Poincaré $S^3/I^*$



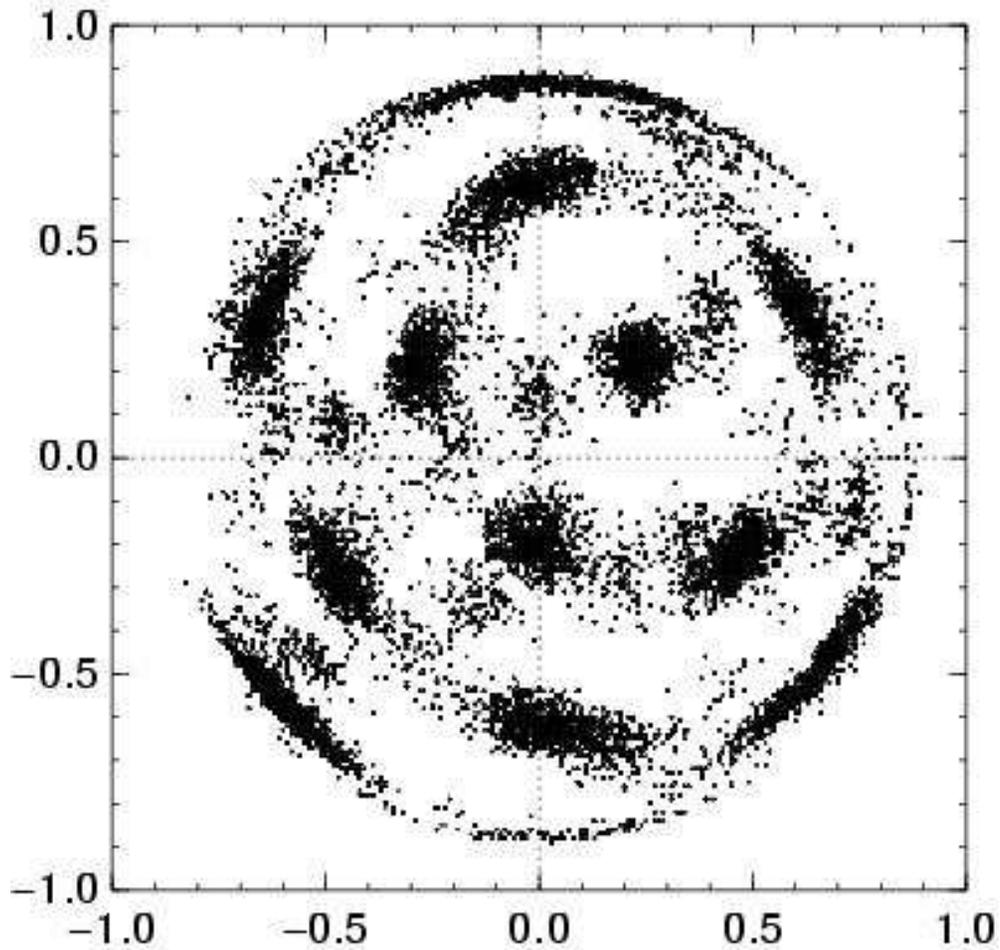
ILC + kp2

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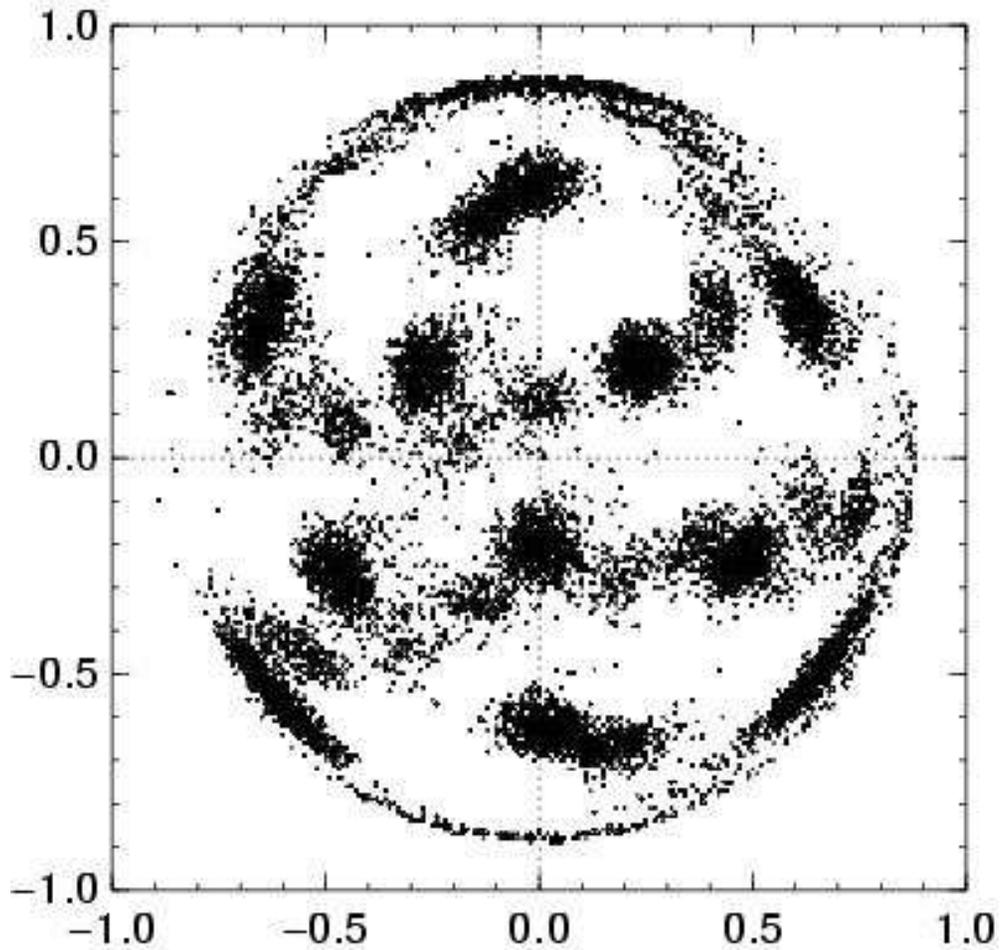
ILC + nomask

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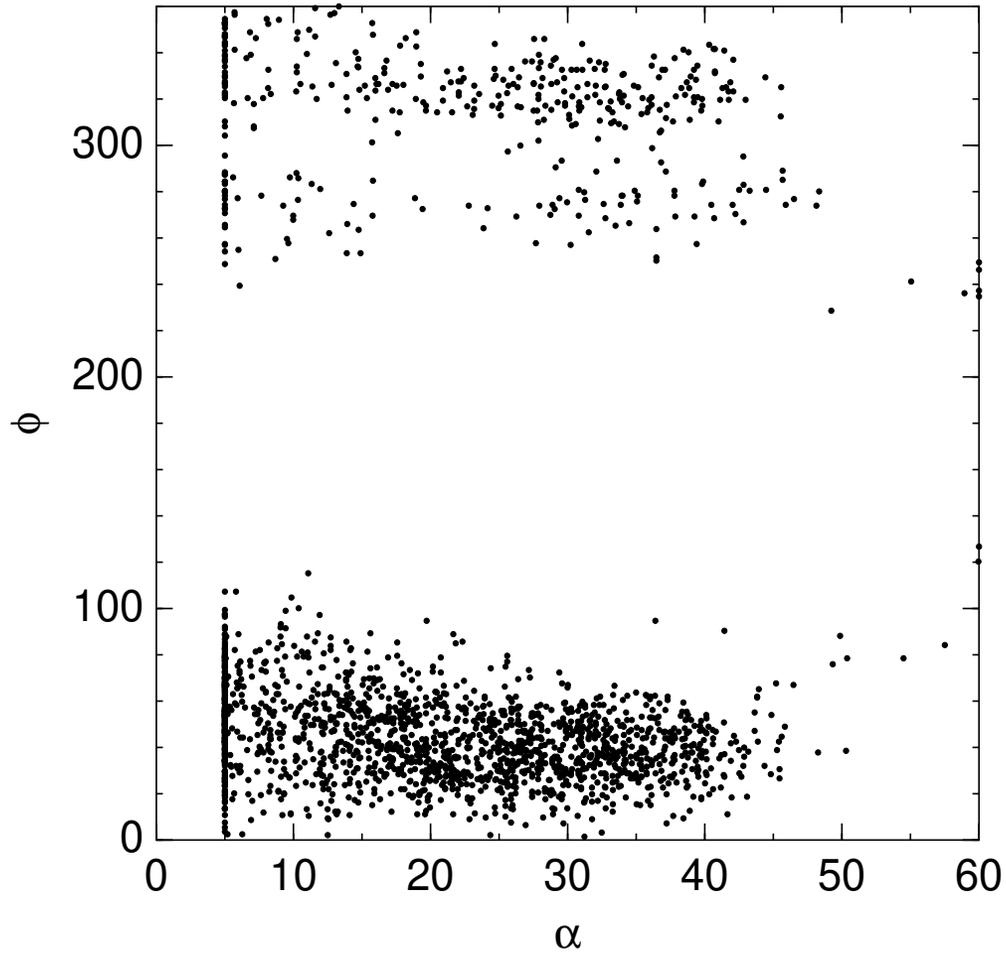
TOH + kp2

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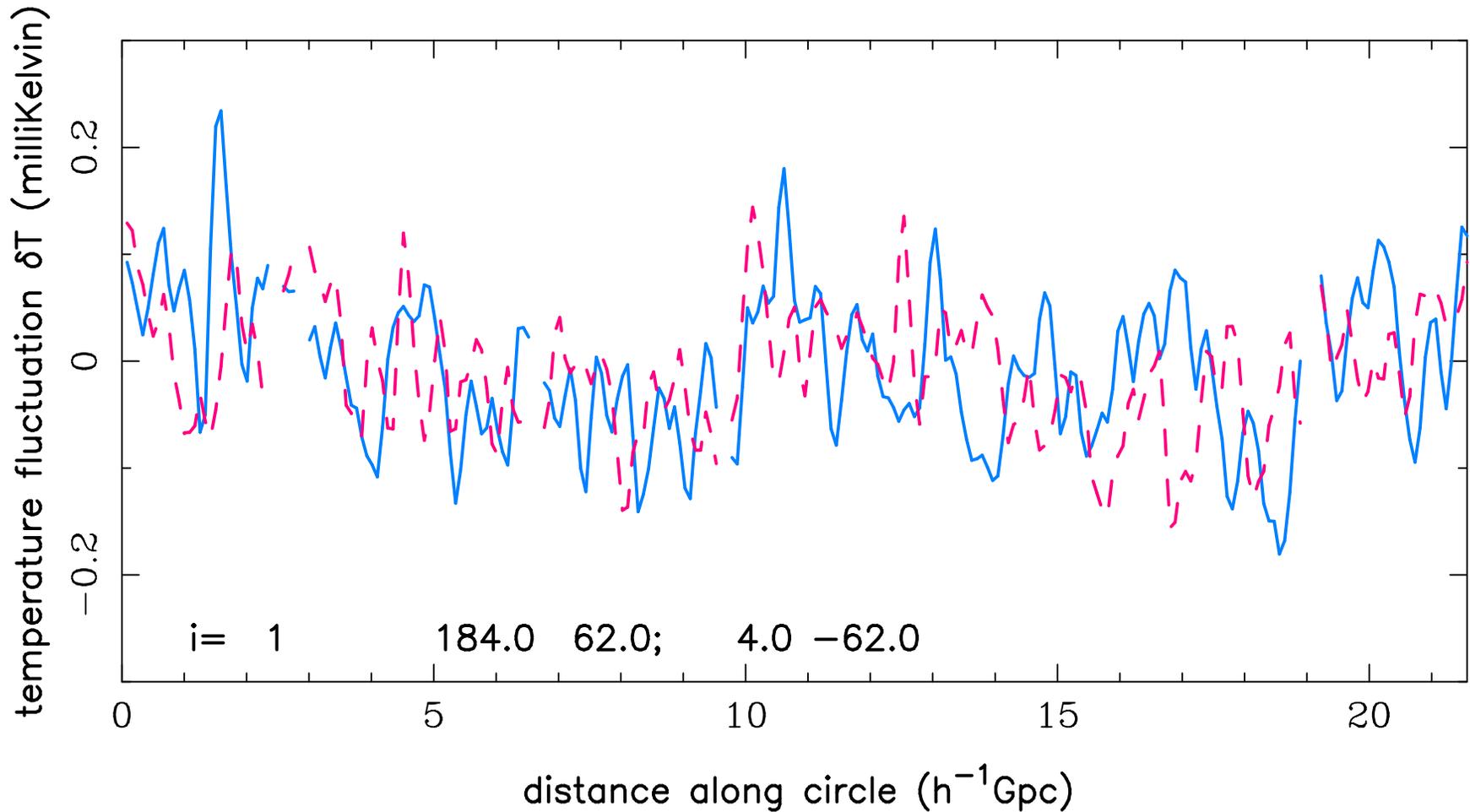


$\phi$  vs matched circle size  $\alpha$

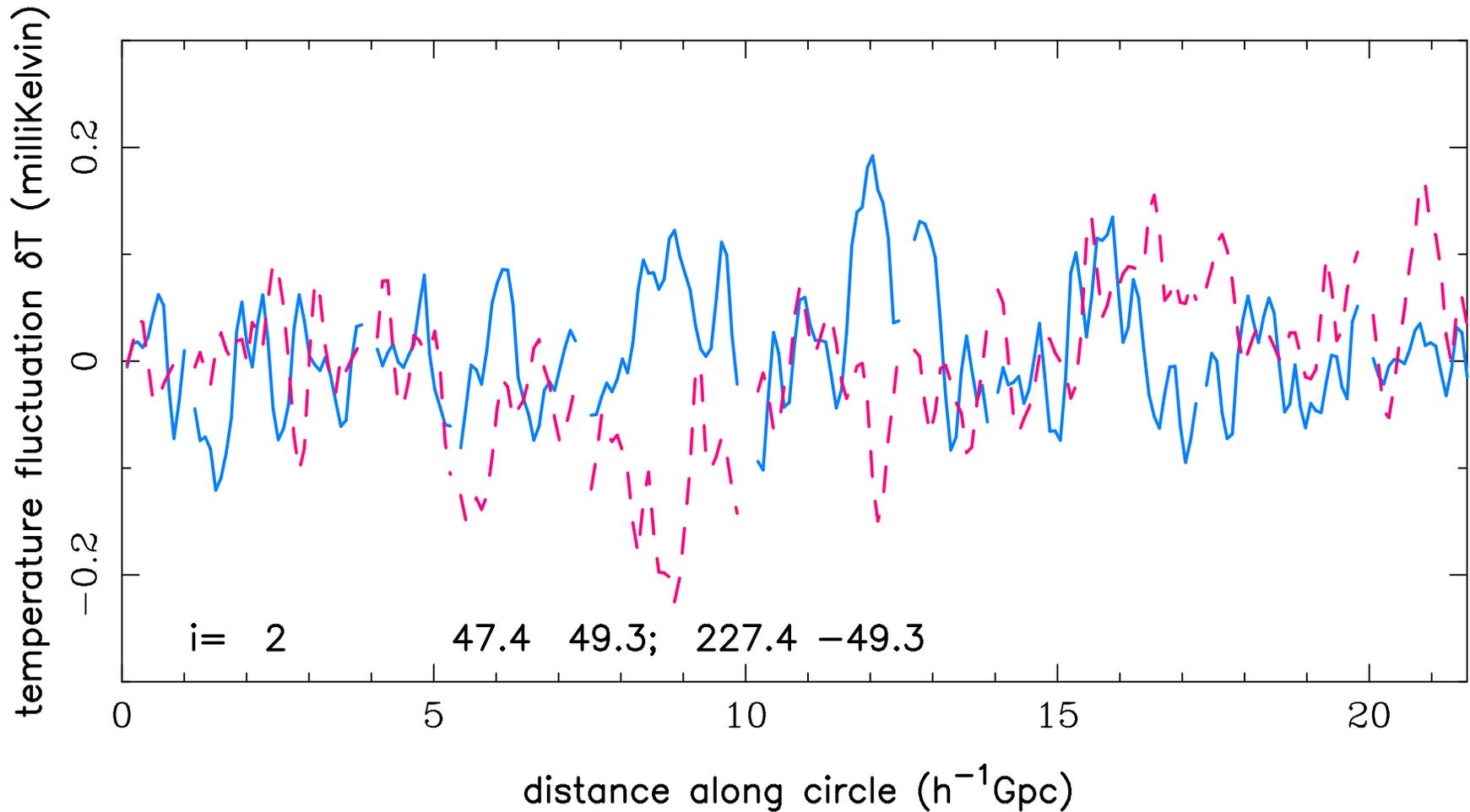
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$P_{\min}$	$n$	$\alpha^\circ$	$\sigma_{\langle\alpha\rangle}^\circ$	$\phi^\circ$	$\sigma_{\langle\phi\rangle}^\circ$
0.4	12589.0	20.6	0.6	<b>39.0</b>	<b>2.4</b>
0.5	6537.5	20.8	0.7	<b>38.7</b>	<b>2.2</b>
0.6	2961.0	22.1	0.5	<b>37.4</b>	<b>2.1</b>

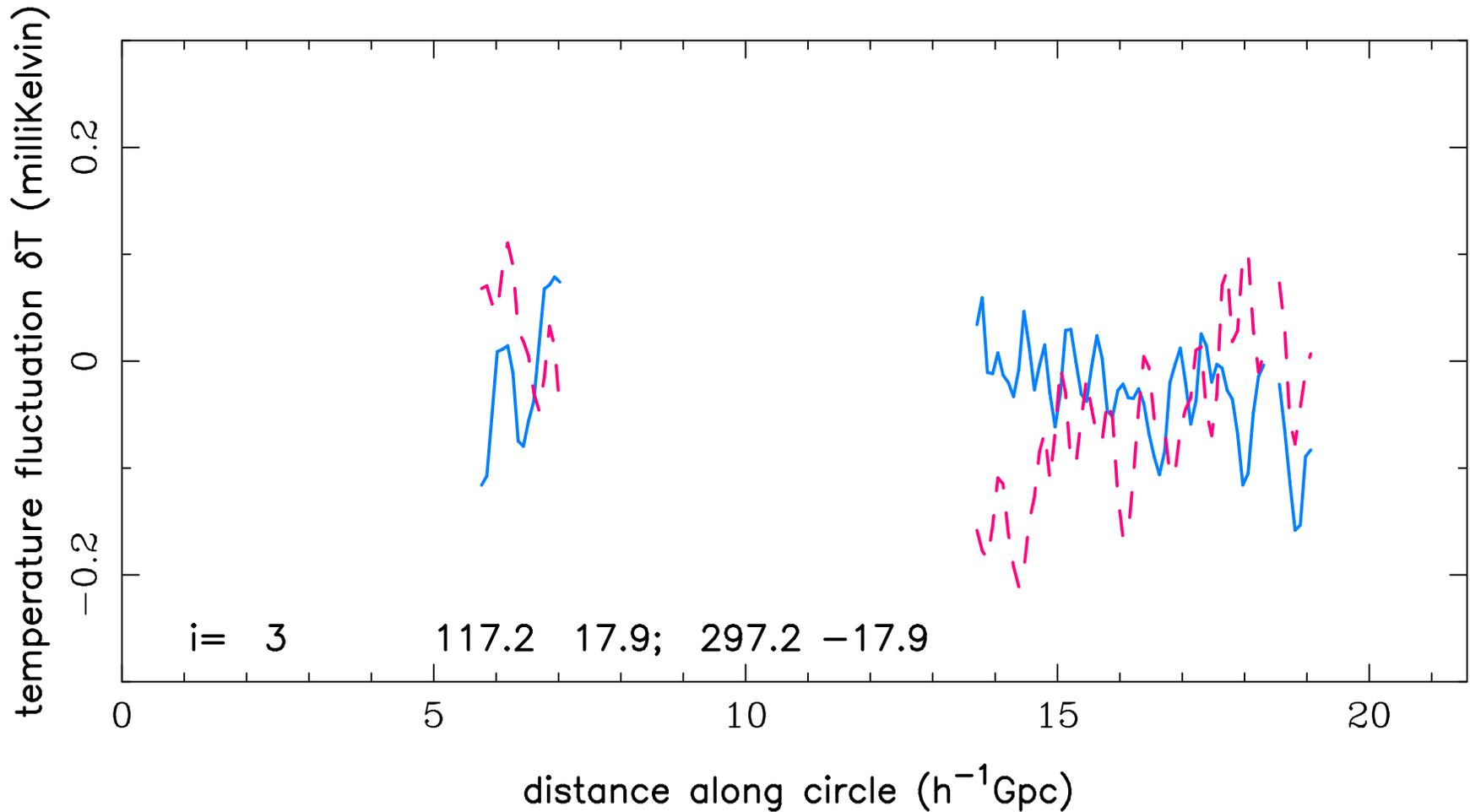
# RBSG08 matched circles



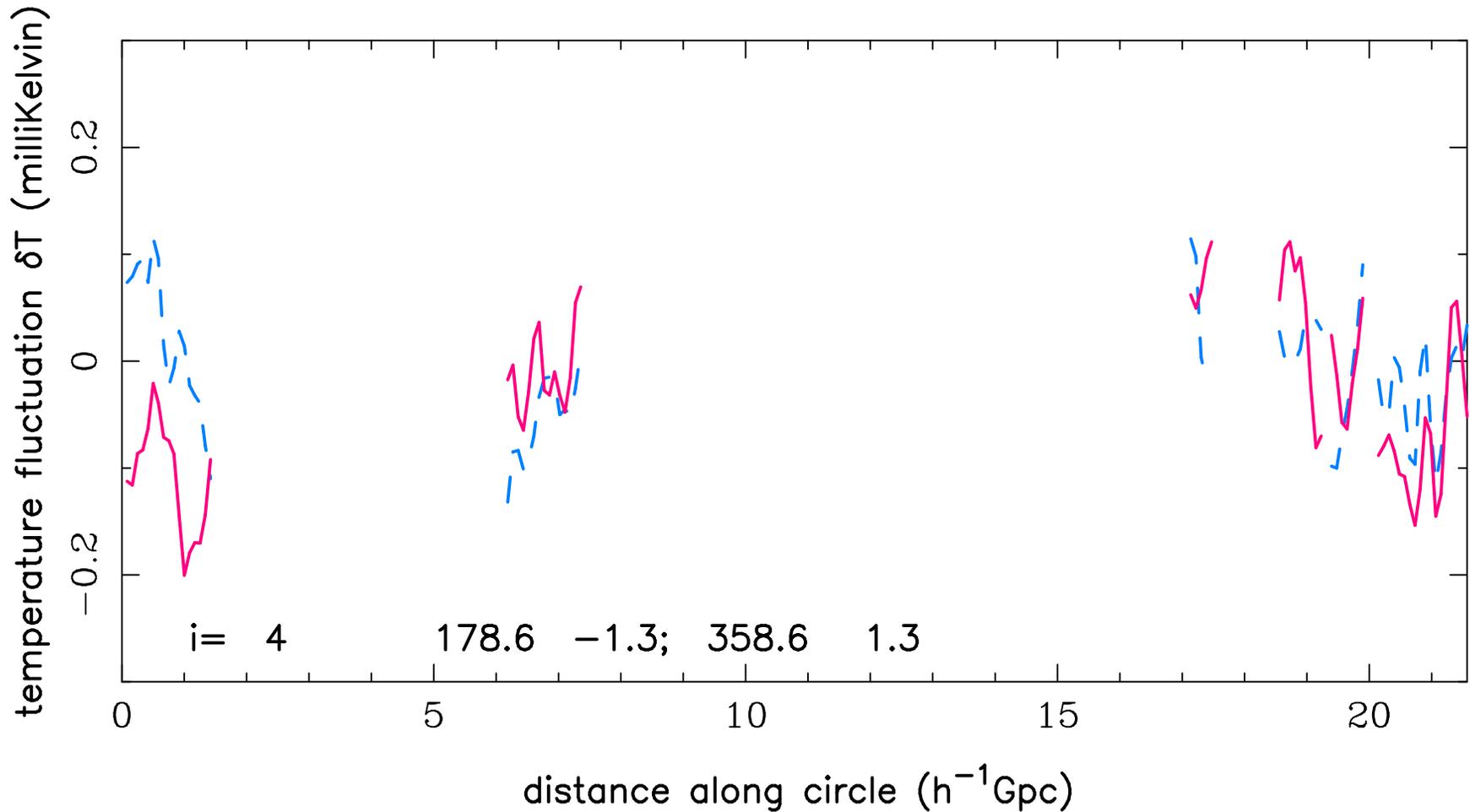
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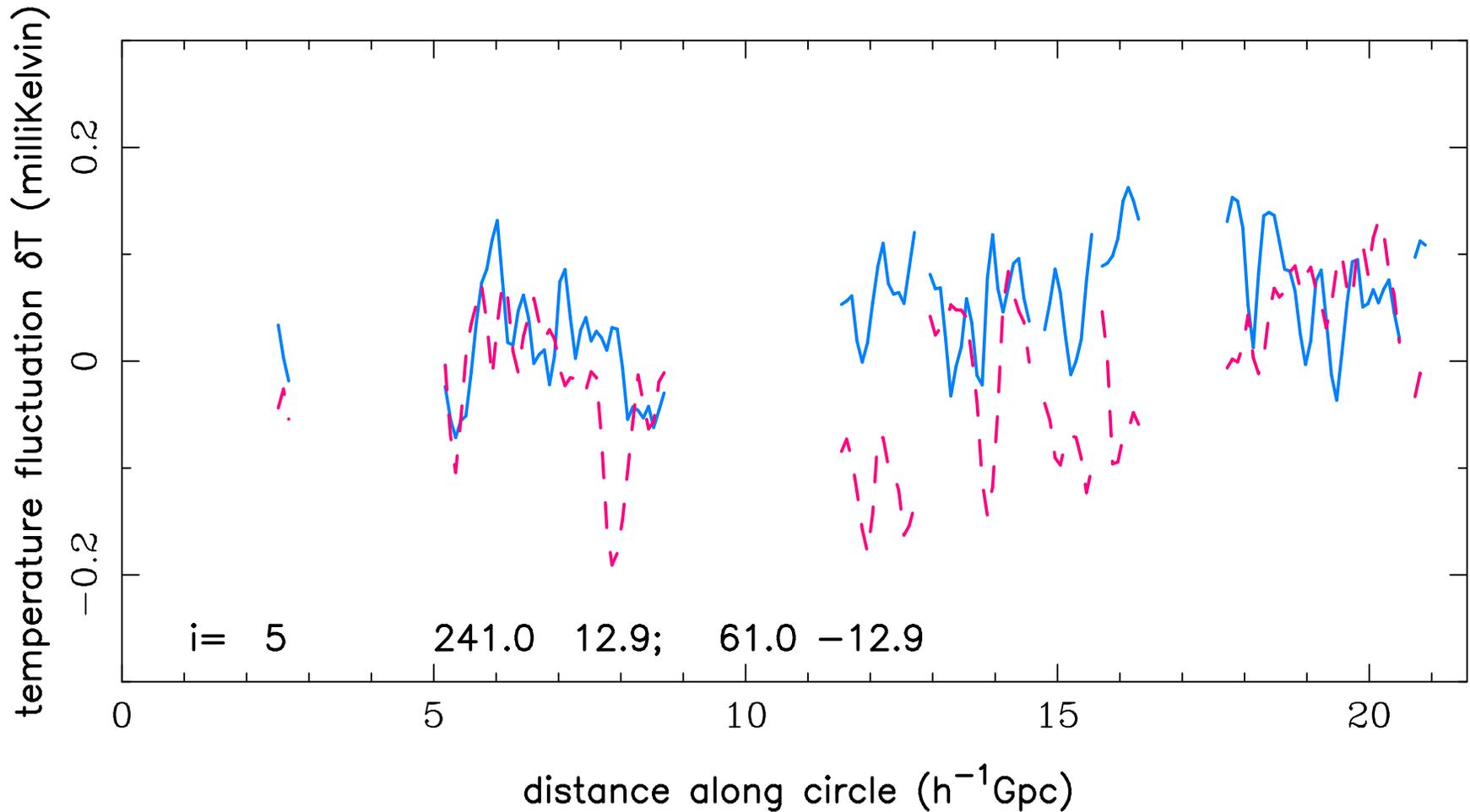
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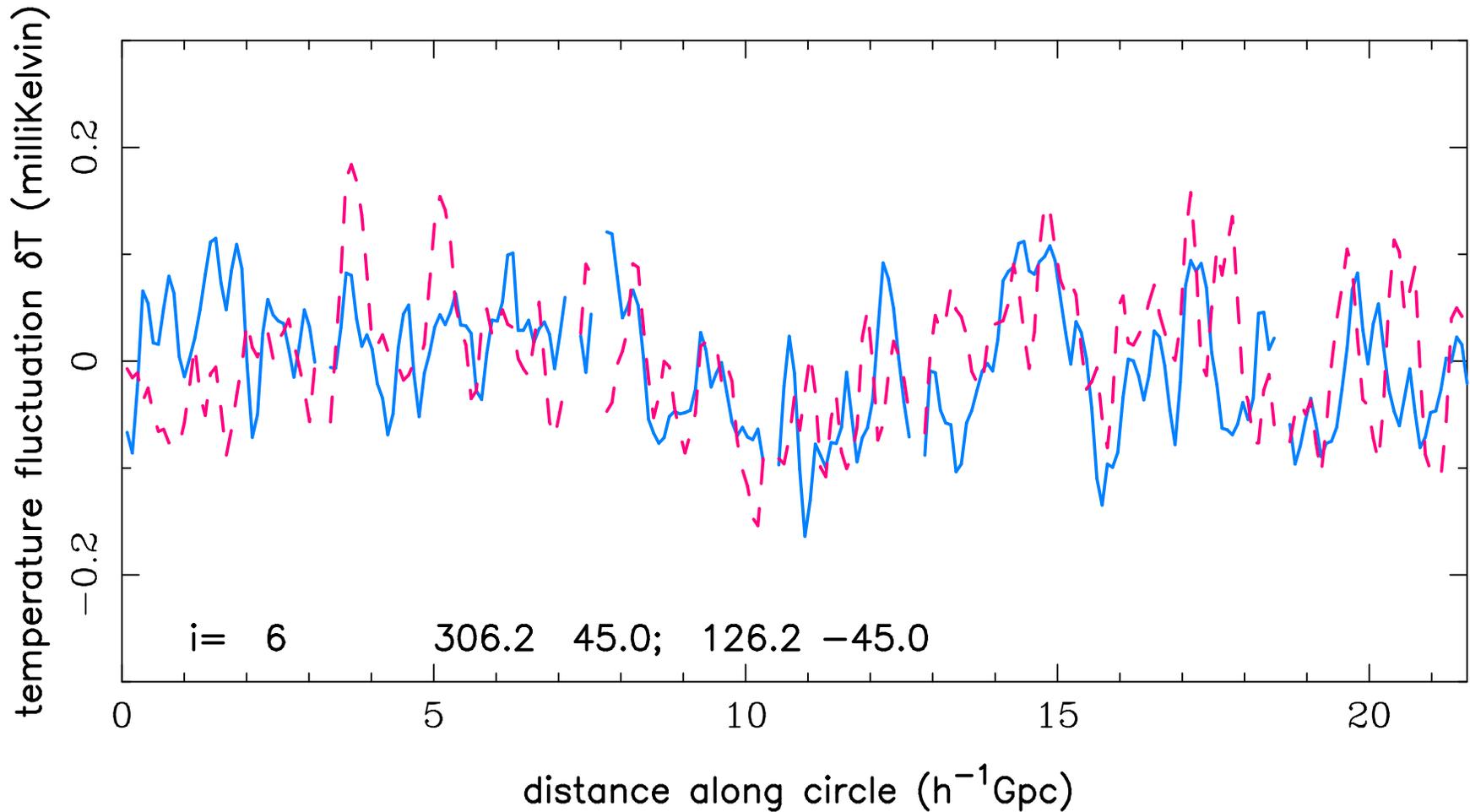
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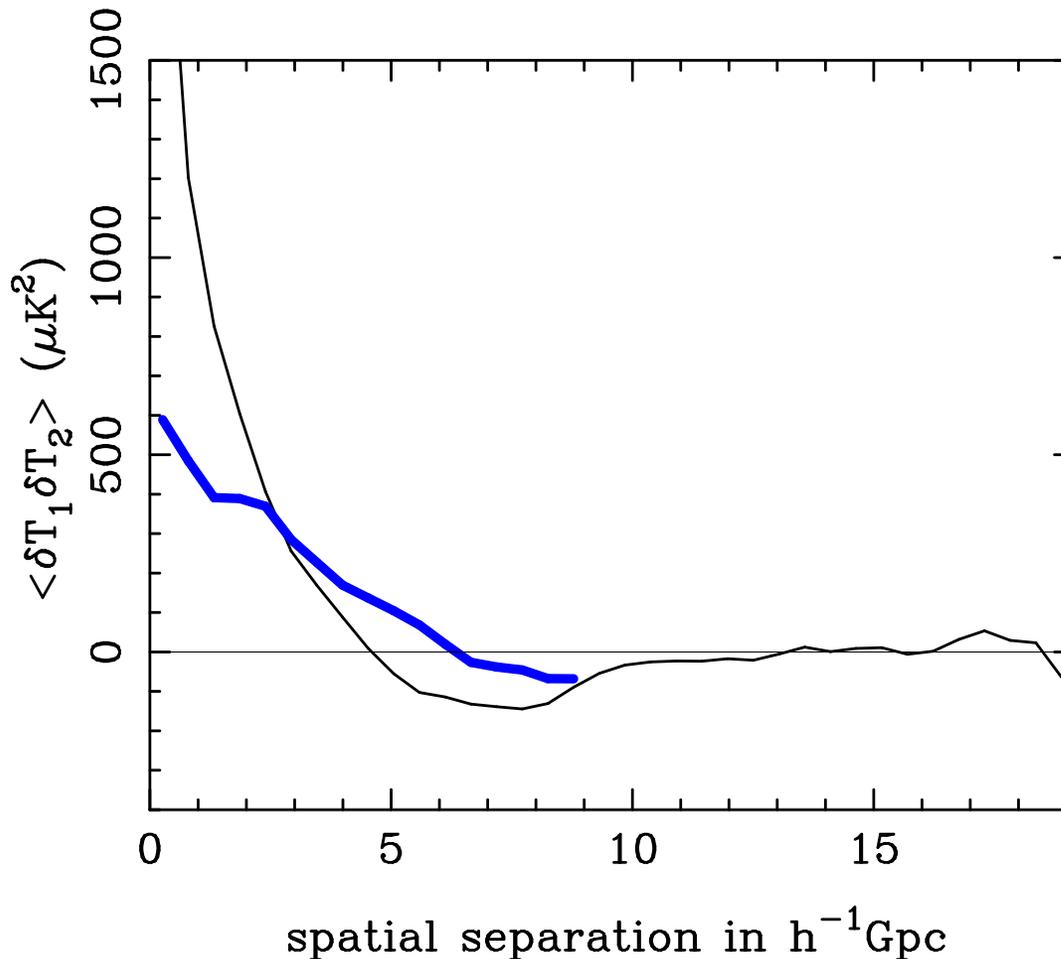


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- matched discs method, RK11 [arXiv:1106.0727](https://arxiv.org/abs/1106.0727):  
 $2r_{\text{inj}} = 18.2 \pm 0.5 h^{-1} \text{ Gpc}$

# WMAP + Poincaré $S^3/I^*$

- → favoured Poincaré dodecahedral space orientation/size, RBSG08 [arXiv:0801.0006](https://arxiv.org/abs/0801.0006)
- $\{(l, b)\}_{i=1,6} \approx \{(184^\circ, 62^\circ), (305^\circ, 44^\circ), (46^\circ, 49^\circ), (117^\circ, 20^\circ), (176^\circ, -4^\circ), (240^\circ, 13^\circ)\} (\pm \approx 2^\circ)$
- matched discs method, RK11 [arXiv:1106.0727](https://arxiv.org/abs/1106.0727):  
 $2r_{\text{inj}} = 18.2 \pm 0.5 h^{-1} \text{ Gpc}$
- Planck (2013): (i) perturbation statistics assumption method; + (ii) identified circles: small correlation signal from  $S^3/I^*$  and other well-proportioned spaces, but consistent with noise

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- cosmic topology with inhomogeneities: very much unexplored ...

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- patterns of time-integrated effects of topological acceleration should exist at  $\sim 10\text{--}1000h^{-1}$  Mpc
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- numerical simulations — Buliński 2015 PhD thesis NCU